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Existence of an equilibrium with limited stock market participation and power utilities [☆]

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Abstract

For constants $\gamma \in (0, 1)$ and $A \in (1, \infty)$, we prove existence and uniqueness of a solution to the singular and path-dependent Riccati-type ODE

$$\begin{cases} h'(y) = \frac{1+\gamma}{y}\big(\gamma - h(y)\big) + h(y)\frac{\gamma+\big((A-\gamma)\exp\{\int_y^1 \frac{1-h(q)}{1-q}dq\}-A\big)h(y)}{1-y}, & y \in (0, 1), \\ h(0) = \gamma, \quad h(1) = 1. \end{cases}$$

As an application, we use the ODE solution to prove existence of a Radner equilibrium with homogeneous power-utility investors in the limited participation model from Basak and Cuoco (1998).

Keywords: Singular ODE; Shooting technique; Incomplete Radner equilibrium

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1. Introduction

There is an extensive literature on nonlinear second-order differential equations of the type

$$\big(\varphi(y)G(w')w'\big)' + \varphi(y)F(y, w, w') = 0, \quad y \in (0, 1). \tag{1.1}$$

When G and $F(y, w, \cdot)$ depend only on $|w'|$ and $\varphi(y) = y^{n-1}$ with $y := |x|$, equation (1.1) becomes the radial version of the differential equation

$$\operatorname{div}\big(G(|Dw|)Dw\big) + F(|x|, w, |Dw|) = 0, \tag{1.2}$$

see, for example, [16], [18], and [19]. In this case, solutions to (1.1), or related differential inequalities, are often used to prove comparison and maximum principles for the corresponding differential equation (see [19]). Important examples are $G(\xi) = 1$ and $G(\xi) = |\xi|^{p-2}$, $p > 1$, which yield the Laplace equation

$$\Delta u + F(|x|, u, |Du|) = 0, \tag{1.3}$$

and the p-Laplace equation

$$\operatorname{div}(|Du|^{p-2}Du) + F(|x|, u, |Du|) = 0, \tag{1.4}$$

respectively. In (1.3) and (1.4), we use $u(x) := w(|x|)$ and $y := |x|$.

The differential equation (1.1) also arises in the study of singular self-similar solutions to the porous media equation with absorption

$$\frac{\partial u}{\partial t} = \Delta(u^m) - u^p,$$

where one looks for solutions of the type

$$u(x, t) = t^{-a} w(t^{-b}|x|),$$

see, for example, [3], [11], and [14].

In this paper, we are interested in a special form of (1.1), which arises from a system of coupled stochastic control problems in financial economics (discussed below and detailed in Section 3). For $\gamma \in (0, 1)$ and $A \in (1, \infty)$, we consider the second order nonlinear differential equation

$$\big(\varphi(y)w'\big)' - \gamma(1+\gamma)y^{\gamma}(1-y)^{1-\gamma} - \varphi(y)\big((1-y)e^{w} - A\big)(w')^2 = 0, \tag{1.5}$$

for $y \in (0, 1)$, where

$$\varphi(y) := y^{1+\gamma}(1-y)^{\gamma}, \quad y \in (0, 1).$$

To highlight the singularities at $y = 0$ and $y = 1$, we write the differential equation (1.5) as

$$w''(y) = \frac{\gamma(1+\gamma)}{y(1-y)} + \frac{(1+\gamma)(2y-1)}{y(1-y)} w'(y) + \big((1-y)e^{w(y)} - A\big) w'(y)^2. \tag{1.6}$$

For our application in Section 3 below, we are interested in a singular solution $w \in \mathcal{C}^2([0, 1))$, which satisfies the boundary condition

$$\lim_{y \uparrow 1} (1-y) w'(y) = 1. \tag{1.7}$$

We will see that a necessary and sufficient condition for the local existence of a smooth solution for y near 0 is that the boundary condition

$$w'(0) = \gamma \tag{1.8}$$

holds. The main challenge of the paper is to determine the range of initial values

$$w(0) = w_0 \tag{1.9}$$

for which equation (1.6) admits a global solution $w(y)$ for $y \in (0, 1)$ and construct a unique solution that satisfies (1.7). Following [3] and [14], we use a shooting argument. To be precise, we show that for all

$$0 \le w_0 < A - \gamma,$$

the differential equation (1.6) has a unique solution satisfying the boundary conditions (1.8), (1.9), and

$$\lim_{y \uparrow 1} (1-y) w'(y) = \frac{\gamma}{A} \in (0, 1).$$

Then, we show that for all w_0 sufficiently large, the solutions to the Cauchy problem (1.6), (1.8), and (1.9) explode for $y < 1$. Finally, we show that the initial value w_0 for which (1.7) holds is given by the supremum of the set of initial values $w_0 > 0$ for which there is a smooth solution to (1.6), (1.8), and (1.9) satisfying

$$\lim_{y \uparrow 1} (1-y) w'(y) \le \gamma.$$

The main challenges with respect to previous work are that: (i) the ODE (1.6) is singular at both $y = 0$ and $y = 1$, (ii) the ODE (1.6) is not variational, so standard techniques (see, e.g., [15]) cannot be applied, (iii) the ODE (1.6) is of Riccati type due to the presence of the term $(w')^2$, (iv) there is exponential growth in w, and (v) we are looking for singular solutions (see, e.g., [8] and [16]).

To state our main result, we rewrite (1.6) in terms of

$$h(y) := (1-y) w'(y), \quad y \in (0, 1).$$

Then, the ODE (1.6) becomes

$$\begin{cases} h'(y) = \frac{1+\gamma}{y}\big(\gamma - h(y)\big) + h(y)\frac{\gamma + ((A-\gamma)\exp\{\int_y^1 \frac{1-h(q)}{1-q}dq\} - A)h(y)}{1-y}, & y \in (0,1), \\ h(0) = \gamma, \quad h(1) = 1. \end{cases} \tag{1.10}$$

Our main mathematical result is:

Theorem 1.1. *For $\gamma \in (0,1)$ and $A \in (1,\infty)$, there exists a unique solution $h \in \mathcal{C}^1([0,1])$ of (1.10) satisfying $\gamma \le h \le 1$.*

Our motivation for studying (1.10) comes from the limited stock-market participation model in [2]. We use Theorem 1.1 to prove the existence of a Radner equilibrium when both investors have identical power utility functions with common relative risk-aversion coefficient $\gamma \in (0,1)$ and common time-preference parameter $\beta > 0$. Our analysis is significantly more involved than the log-log-utility model originally developed in [2]. Because of the log-utility assumption placed on the restricted investor, the model in [2] is explicitly solvable and no ODE is needed.

There are several existing model extensions of [2]. [10] proves existence of equilibrium bubbles (i.e., models where the equilibrium stock-price process is a strict local martingale) and shows that equilibrium uniqueness can fail in models with multiple stocks. In another extension, [17] proves the existence of an equilibrium when the unrestricted investor has a power-utility function. The existence proofs in [2], [10], and [17] all rely crucially on the restricted investor having a logarithmic utility function. This is because the optimal policy for an investor with a logarithmic utility function is available in closed form. [21] uses coupled BSDEs to prove equilibrium existence for investors with heterogeneous exponential utilities. Finally, our existence proof also puts some of the experimental numerics for homogeneous investors reported for power-power utilities in [4] on a solid mathematical foundation.

We use the mathematical setting from [2], which falls under the theory of incomplete Radner equilibria. Incompleteness stems from the restricted investor's inability to hold stocks. Utilities defined on the positive real line — such as log and power — restrict consumption processes to be nonnegative, which complicates the underlying mathematical structure. In continuous-time settings with noise generated by Brownian motions, incomplete Radner equilibria are originally discussed in [6] but no general existence result is available.[1]

2. ODE existence

2.1. Auxiliary ODE results

Lemma 2.1. *Let $\gamma \in (0,1)$. Then, we have:*

1. *The function*

$$\varphi(y) := y^{1+\gamma}(1-y)^{\gamma}, \quad y \in [0,1], \tag{2.1}$$

satisfies

$$\frac{\varphi'(y)}{\varphi(y)} = \frac{1+\gamma}{y} - \frac{\gamma}{1-y}, \quad y \in (0,1). \tag{2.2}$$

[1] The counterpart of complete equilibrium models is fully developed; see, e.g., Chapter 4 in [13] and [1].

2. *The function*

$$\phi(y) := \frac{1}{\varphi(y)} \int_0^y \frac{\varphi(t)}{t} dt, \quad y \in (0,1),$$

can be extended to $\phi \in \mathcal{C}^1([0,1))$ *with*

$$\phi(0) = \frac{1}{1+\gamma}, \quad \phi'(0) = \frac{\gamma}{(1+\gamma)(2+\gamma)}. \tag{2.3}$$

Proof. 1: Follows by computing derivatives.

2: By Taylor's formula centered at 0, we have

$$\begin{aligned} \tfrac{\varphi(t)}{t} &= t^\gamma (1-t)^\gamma \\ &= t^\gamma - \gamma t^{1+\gamma} + o(t^{1+\gamma}), \\ \tfrac{1}{(1-y)^\gamma} &= 1 + y\gamma + o(y). \end{aligned}$$

Therefore,

$$\begin{aligned} \phi(y) = \tfrac{1}{\varphi(y)} \int_0^y \tfrac{\varphi(t)}{t} dt &= \tfrac{1+y\gamma+o(y)}{y^{1+\gamma}} \int_0^y \left(t^\gamma - \gamma t^{1+\gamma} + o(t^{1+\gamma})\right) dt \\ &= \tfrac{(1+y\gamma+o(y))\left(\frac{1}{1+\gamma} y^{1+\gamma} - \frac{\gamma}{2+\gamma} y^{2+\gamma} + o(y^{2+\gamma})\right)}{y^{1+\gamma}} \\ &= \tfrac{1}{1+\gamma} + \gamma\left(\tfrac{1}{1+\gamma} - \tfrac{1}{2+\gamma}\right) y + o(y). \end{aligned}$$

This shows that $\phi(y)$ can be extended continuously to $y = 0$ with the boundary values in (2.3). For $y > 0$, by (2.2), the previous calculation, and Taylor's formula, we have

$$\begin{aligned} \phi'(y) &= -\frac{\varphi'(y)}{\varphi(y)^2} \int_0^y \frac{\varphi(t)}{t} dt + \frac{1}{\varphi(y)} \frac{\varphi(y)}{y} \\ &= -\left(\tfrac{1+\gamma}{y} - \tfrac{\gamma}{1-y}\right) \phi(y) + \tfrac{1}{y} \\ &= -\left(\tfrac{1+\gamma}{y} - \gamma - y\gamma + o(y)\right) \left(\tfrac{1}{1+\gamma} + \gamma\big(\tfrac{1}{1+\gamma} - \tfrac{1}{2+\gamma}\big) y + o(y)\right) + \tfrac{1}{y} \\ &= \tfrac{\gamma}{(1+\gamma)(2+\gamma)} + o(1) \\ &= \phi'(0) + o(1). \end{aligned}$$

Hence, ϕ's extension is in $\mathcal{C}^1([0,1))$. $\square$

Theorem 2.2. *Let* $a : [0, 1] \to \mathbb{R}$ *be a continuous function,* $\gamma \in (0, 1)$*, and define*

$$a_0(y) := \frac{\gamma(1+\gamma)}{y}, \quad a_1(y) := \frac{(2\gamma+1)y - (1+\gamma)}{y}, \quad y \in (0, 1]. \tag{2.4}$$

Then, the singular Riccati ODE

$$f'(y) = a_0(y) + \tfrac{a_1(y)}{1-y} f(y) + \tfrac{a(y)}{1-y} f(y)^2, \quad y > 0, \tag{2.5}$$

with the boundary condition $f(0) = \gamma$*, has a strictly positive local solution in* $\mathcal{C}^1([0, \delta])$ *in the sense that there exist* $\delta \in (0, \frac{1}{2}]$ *and a function* $f : [0, \delta] \to (0, \infty)$ *with* $f \in \mathcal{C}^1([0, \delta])$ *satisfying* (2.5) *for* $y \in (0, \delta)$.

Proof. Step 1/6: We eliminate the linear term in (2.5) by multiplying both sides of (2.5) by φ from Lemma 2.1 to see

$$\big(\varphi(y) f(y)\big)' = a_0(y)\varphi(y) + \frac{a(y)}{1-y}\varphi(y) f(y)^2, \quad y > 0.$$

Integrating both sides and using $\varphi(0) = 0$ and $f(0) = \gamma$, yields

$$\varphi(y) f(y) = \int_0^y \varphi(t) \left(a_0(t) + \frac{a(t)}{1-t} f(t)^2\right) dt, \quad y > 0.$$

Hence, if a local solution f of (2.5) exists, it satisfies the integral equation

$$f(y) = \frac{1}{\varphi(y)} \int_0^y \varphi(t) \left(a_0(t) + \frac{a(t)}{1-t} f(t)^2\right) dt, \quad y > 0.$$

The next steps construct a fixed point for this integral equation for $y \in [0, \delta]$ for some $\delta \in (0, \frac{1}{2}]$.

Step 2/6: For $0 < \delta \le \frac{1}{2}$ to be chosen later and an arbitrary $R > 0$, we consider the closed convex set

$$\mathcal{X} := \{f \in \mathcal{C}([0, \delta]) : \ \|f - \gamma\|_\infty \le R\}, \tag{2.6}$$

where $\|f\|_\infty := \max_{y \in [0,\delta]} |f(y)|$. For $f \in \mathcal{X}$ and $0 < y \le \delta$, we define T by

$$T(f)(y) := \frac{1}{\varphi(y)} \int_0^y \varphi(t) \left(a_0(t) + \tfrac{a(t)}{1-t} f(t)^2\right) dt. \tag{2.7}$$

We claim that $\lim_{y \downarrow 0} T(f)(y)$ exists and

$$\lim_{y \downarrow 0} T(f)(y) = \gamma. \tag{2.8}$$

To see (2.8), we use L'Hospital's rule, (2.2), and (2.7) to get

$$\begin{aligned}
&\lim_{y\downarrow 0}\frac{\varphi(y)}{\varphi'(y)}\left[a_0(y)+\frac{a(y)}{1-y}f(y)^2\right]\\
&=\lim_{y\downarrow 0}\frac{\gamma(1+\gamma)+y\frac{a(y)}{1-y}f(y)^2}{1+\gamma-y\frac{\gamma}{1-y}}\\
&=\gamma.
\end{aligned}$$

We extend T continuously to $y=0$ by defining $T(f)(0):=\gamma$ so that $T:\mathcal{X}\to\mathcal{C}([0,\delta])$.

Step 3/6: We claim that if $\delta>0$ is sufficiently small, then

$$T(\mathcal{X})\subseteq\mathcal{X}. \tag{2.9}$$

Consider the continuous function

$$\psi(y,z):=\tfrac{a(y)}{1-y}z^2,\qquad y\in[0,\tfrac{1}{2}],\qquad z\in[\gamma-R,\gamma+R]. \tag{2.10}$$

Because ψ is continuous, there exists a constant $M>0$ such that

$$|\psi(y,z)|\le M,\quad \forall (y,z)\in[0,\tfrac{1}{2}]\times[\gamma-R,\gamma+R]. \tag{2.11}$$

The constant M depends on the function $a(y)$ and R but M does not depend on δ.
Provided $\delta\in(0,\frac{1}{2})$ satisfies

$$\delta\le\delta_1:=\frac{R(2+\gamma)}{2M2^\gamma},$$

for $f\in\mathcal{X}$ and $y\in[0,\delta]$, we have the following estimate

$$\begin{aligned}
&\frac{1}{\varphi(y)}\int_0^y\varphi(t)\left|\tfrac{a(t)}{1-t}f(t)^2\right|dt\\
&=\frac{1}{\varphi(y)}\int_0^y\varphi(t)\left|\psi\big(t,f(t)\big)\right|dt\\
&\le\frac{M}{y^{1+\gamma}(1-y)^\gamma}\int_0^y t^{1+\gamma}(1-t)^\gamma\,dt \qquad (2.12)\\
&\le\frac{M2^\gamma}{y^{1+\gamma}}\int_0^y t^{1+\gamma}\,dt\\
&=\frac{M2^\gamma}{(2+\gamma)}y
\end{aligned}$$

$$\le \frac{R}{2}.$$

On the other hand, L'Hospital's rule, (2.1), and (2.4) give

$$\lim_{y\downarrow 0} \frac{1}{\varphi(y)} \int_0^y \varphi(t) a_0(t)\,dt = \gamma.$$

Hence, there exists $\delta_2 > 0$ such that

$$\forall y \in (0, \delta_2) : \left| \frac{1}{\varphi(y)} \int_0^y \varphi(t) a_0(t)\,dt - \gamma \right| \le \frac{R}{2}. \tag{2.13}$$

Taking $\delta \le \min\{\delta_1, \delta_2, \frac{1}{2}\}$, it follows from the estimates (2.12) and (2.13) that

$$|T(f)(y) - \gamma| \le R, \quad \forall y \in [0, \delta], \quad \forall f \in \mathcal{X}.$$

Therefore, the inclusion (2.9) holds.

Step 4/6: This step proves that the operator T from step 2 is equi-continuous (even better, T turns out to be an equi-Lipschitz operator). For $0 < y_1 < y_2 \le \delta$ and $f \in \mathcal{X}$, the formula for $a_0(y)$ in (2.4), (2.7), and (2.10) give

$$\begin{aligned}
&|T(f)(y_2) - T(f)(y_1)| \\
&\le (1+\gamma)\gamma \left| \frac{1}{\varphi(y_2)} \int_0^{y_2} \frac{\varphi(t)}{t} dt - \frac{1}{\varphi(y_1)} \int_0^{y_1} \frac{\varphi(t)}{t} dt \right| \\
&\quad + \left| \frac{1}{\varphi(y_2)} \int_0^{y_2} \varphi(t)\psi\big(t, f(t)\big)\,dt - \frac{1}{\varphi(y_1)} \int_0^{y_1} \varphi(t)\psi\big(t, f(t)\big)\,dt \right| \\
&\le (1+\gamma)\gamma \left| \frac{1}{\varphi(y_2)} \int_0^{y_2} \frac{\varphi(t)}{t} dt - \frac{1}{\varphi(y_1)} \int_0^{y_1} \frac{\varphi(t)}{t} dt \right| \\
&\quad + \frac{1}{\varphi(y_2)} \int_{y_1}^{y_2} \varphi(t)|\psi\big(t, f(t)\big)|\,dt + \left| \frac{1}{\varphi(y_2)} - \frac{1}{\varphi(y_1)} \right| \int_0^{y_1} \varphi(t)|\psi\big(t, f(t)\big)|\,dt \\
&=: I + II + III.
\end{aligned} \tag{2.14}$$

We estimate each of the three terms separately. Lemma 2.1 ensures $\phi \in \mathcal{C}^1([0, \delta])$, and so, the Mean-Value Theorem gives

$$I \le (1+\gamma)\gamma(y_2 - y_1)\|\phi'\|_\infty.$$

The bound in (2.11) and $y_2 \le \delta \le \frac{1}{2}$ give

$$
\begin{aligned}
II &\le M \frac{1}{y_2^{1+\gamma}(1-y_2)^\gamma} \int_{y_1}^{y_2} t^{1+\gamma}(1-t)^\gamma \, dt \\
&\le M \frac{2^\gamma}{y_2^{1+\gamma}} \int_{y_1}^{y_2} t^{1+\gamma} \, dt \\
&\le M 2^\gamma (y_2 - y_1).
\end{aligned}
$$

Similar, the bound in (2.11) and $0 < y_1 < y_2 \le \delta \le \frac{1}{2}$ give

$$
\begin{aligned}
III &\le \frac{|\varphi(y_1) - \varphi(y_2)|}{\varphi(y_2)\varphi(y_1)} M \int_0^{y_1} \varphi(t) \, dt \\
&\le 2^{2\gamma} \frac{|\varphi(y_1) - \varphi(y_2)|}{y_2^{1+\gamma} y_1^{1+\gamma}} M \int_0^{y_1} t^{1+\gamma} \, dt \\
&\le 2^{2\gamma} \frac{|y_1 - y_2|}{y_2^{1+\gamma}} |\varphi'(z)| M y_1 \\
&\le 2^{2\gamma} \frac{|y_1 - y_2|}{y_2^{\gamma}} |\varphi'(z)| M,
\end{aligned}
$$

where the constant $z \in (y_1, y_2)$ is produced by the Mean-Value Theorem. For arbitrary $t \in (y_1, y_2)$, we have

$$
\begin{aligned}
|\varphi'(t)| &= |(1+\gamma)t^\gamma(1-t)^\gamma - \gamma t^{1+\gamma}(1-t)^{\gamma-1}| \\
&\le (1+\gamma) y_2^\gamma + \gamma y_2^{1+\gamma} 2^{1-\gamma}.
\end{aligned}
$$

Hence,

$$
III \le 2^{2\gamma}\big(1 + \gamma + \gamma 2^{\gamma-1}\big) M |y_1 - y_2|.
$$

Combining the estimates for I, II, and III gives a uniform (in f) Lipschitz constant L such that

$$
|T(f)(y_2) - T(f)(y_1)| \le L|y_2 - y_1|, \quad \forall 0 < y_1 < y_2 \le \delta, \quad \forall f \in \mathcal{X}.
$$

Using (2.8), it follows that this inequality holds also for $y_1 = 0$. Because L is independent of $f \in \mathcal{X}$, T is equi-continuous.

Step 5/6: The family $\{T(f) : f \in \mathcal{X}\}$ is equi-bounded (step 3) and equi-continuous (step 4). Hence, by the Ascoli–Arzelá theorem, T is compact. Then, the Schauder fixed point theorem produces a function $f \in \mathcal{X}$ such that $f(y) = T(f)(y)$ for all $y \in [0, \delta]$. In particular, $f(0) = T(f)(0) = \gamma$, and

$$f(y) = \frac{1}{\varphi(y)} \int_0^y \varphi(t) \left(a_0(t) + \frac{a(t)}{1-t} f(t)^2 \right) dt, \quad y \in [0, \delta]. \tag{2.15}$$

Since the right-hand side is continuously differentiable for $y > 0$, it follows that f is continuously differentiable for $y \in (0, \delta)$. We need to show that f is continuously differentiable at $y = 0$. Let ϕ be the function in Lemma 2.1. Then,

$$f(y) = (1+\gamma)\gamma\phi(y) + \frac{1}{\varphi(y)} \int_0^y \varphi(t) \frac{a(t)}{1-t} f(t)^2 \, dt, \quad y \in (0, \delta].$$

Because $\phi \in \mathcal{C}^1([0, \frac{1}{2}])$, it remains to show that the second term on the right-hand side is also in $\mathcal{C}^1([0, \delta])$. To this end, we define

$$\omega(y) := \frac{1}{\varphi(y)} \int_0^y \varphi(t) \frac{a(t)}{1-t} f(t)^2 \, dt, \quad y \in (0, \delta].$$

Then,

$$\omega'(y) = -\frac{\varphi'(y)}{\varphi(y)^2} \int_0^y \varphi(t) \frac{a(t)}{1-t} f(t)^2 dt + \tfrac{a(y)}{1-y} f(y)^2, \quad y \in (0, \delta),$$

which is continuous. L'Hospital's rule and the definition of $\varphi(y)$ in (2.1) ensure that the integral term in $\omega'(y)$ has a finite limit as $y \downarrow 0$. The continuity of $a(y)$ and $f(y)$ at $y = 0$ ensures that the second term in $\omega'(y)$ also has a finite limit as $y \downarrow 0$. Therefore, $\omega \in \mathcal{C}^1([0, \delta])$.

Step 6/6: The local solution $f(y)$ for $y \in [0, \delta]$ is necessarily strictly positive. This is because $f(0) = \gamma > 0$ and should there be a point $y_0 \in (0, \delta)$ with $f(y_0) = 0$, then $f'(y_0) = \frac{\gamma(1+\gamma)}{y_0} > 0$, and so f can never become zero. □

Remark 2.1. Because the solution in Theorem 2.2 belongs to $\mathcal{C}^1([0, \delta])$, we know that $f(y) = \gamma + \int_0^y f'(t)dt$ for all $y \in [0, \delta]$. However, because the coefficient functions in (2.5) have singularities, we cannot use (2.5) to write $\int_0^y f'(t)dt$ as a sum of three individual integrals. For example, the definition of $a_0(y)$ in (2.4) implies that the first term satisfies $\int_0^y a_0(t)dt = \infty$ for all $y > 0$.

Theorem 2.3 *(Comparison). Let* $\delta \in (0, 1]$ *and let* $a, b : [0, \delta] \mapsto \mathbb{R}$ *be continuous functions with* $a(y) \le b(y)$ *for all* $y \in [0, \delta]$*. Assume* $f, g : [0, \delta] \mapsto \mathbb{R}$ *solve the Cauchy problems*

$$\begin{cases} f'(y) = a_0(y) + \frac{a_1(y)}{1-y} f(y) + \frac{a(y)}{1-y} f(y)^2, & y \in (0, \delta), \\ f(0) = \gamma, \end{cases}$$

and

$$\begin{cases} g'(y) = a_0(y) + \frac{a_1(y)}{1-y} g(y) + \frac{b(y)}{1-y} g(y)^2, & y \in (0, \delta), \\ g(0) = \gamma. \end{cases}$$

Then, $f(y) \le g(y)$ for all $y \in [0, \delta]$. Furthermore, if $a(y) < b(y)$ for all $y \in [0, \delta]$, then $f(y) < g(y)$ for all $y \in (0, \delta]$.

Proof. We argue by contradiction and assume there exists $y_1 \in (0, \delta]$ such that $f(y_1) > g(y_1)$. Then, the function $F(y) := f(y) - g(y)$ satisfies $F(0) = 0$ and $F(y_1) > 0$. We define

$$y_0 := \sup\{y \in [0, y_1] : F(y) = 0\} \in [0, y_1].$$

Because F is continuous, we have $y_0 \in [0, y_1)$, $F(y_0) = 0$, and $F(y) > 0$ for all $y \in (y_0, y_1]$. For $y \in (y_0, y_1)$, we have

$$\begin{aligned} F'(y) &= \frac{a_1(y)}{1-y} F(y) + \frac{a(y)}{1-y} f(y)^2 - \frac{b(y)}{1-y} g(y)^2 \\ &\le \frac{2\gamma + 1}{1-y} F(y) + \frac{b(y)}{1-y} \big(f(y)^2 - g(y)^2 \big) \\ &= \frac{1}{1-y} F(y) \Big(2\gamma + 1 + b(y) \big(f(y) + g(y) \big) \Big) \\ &\le \frac{M}{1-y} F(y), \quad M := \max_{y \in [y_0, \delta]} \Big(2\gamma + 1 + b(y) \big(f(y) + g(y) \big) \Big). \end{aligned}$$

Because $F(y_0) = 0$, Gronwall's inequality yields the contradiction

$$F(y) \le F(y_0) \exp\Big\{ \int_{y_0}^{y} \frac{M}{1-q} dq \Big\} = 0, \quad y \in (y_0, y_1).$$

When $a < b$, the integral representation (2.15) and $f \le g$ give

$$\begin{aligned} f(y) &= \frac{1}{\varphi(y)} \int_0^y \varphi(t) \Big(a_0(t) + \tfrac{a(t)}{1-t} f(t)^2 \Big) dt \\ &< \frac{1}{\varphi(y)} \int_0^y \varphi(t) \Big(a_0(t) + \tfrac{b(t)}{1-t} f(t)^2 \Big) dt \\ &\le \frac{1}{\varphi(y)} \int_0^y \varphi(t) \Big(a_0(t) + \tfrac{b(t)}{1-t} g(t)^2 \Big) dt \end{aligned}$$

$$= g(y), \quad y \in (0, \delta],$$

where we have used $f > 0$. $\square$

Proposition A.1 in Appendix A contains additional properties of the solution of the ODE defined in (2.16) below. For example, we show $f \notin \mathcal{C}^1([0,1])$.

Theorem 2.4. *Let $\gamma \in (0,1)$ and $a_3 \in (-\infty, -\gamma]$. Then, there exists a function $f \in \mathcal{C}([0,1]) \cap \mathcal{C}^1([0,1))$ with $f(y) \in (0,1]$ for $y \in [0,1]$ that satisfies*

$$f(y) = \frac{1}{y^{1+\gamma}(1-y)^\gamma} \int_0^y \Big(\gamma(1+\gamma)\big(x(1-x)\big)^\gamma + a_3 \frac{x^{1+\gamma}}{(1-x)^{1-\gamma}} f(x)^2\Big) dx, \tag{2.16}$$

for $y \in (0,1)$. Furthermore, the function f satisfies the ODE

$$f'(y) = a_0(y) + \tfrac{a_1(y)}{1-y} f(y) + \tfrac{a_3}{1-y} f(y)^2, \quad y \in (0,1), \tag{2.17}$$

and the boundary conditions

$$f(0) = \gamma, \quad f(1) = -\frac{\gamma}{a_3} \le 1. \tag{2.18}$$

Finally, when $a_3 \in (-\infty, -\gamma)$, we have $f(y) \in [0,1)$ for $y \in [0,1]$ and when $a_3 \in [-1, -\gamma]$, we have $f(y) \ge \gamma$ for $y \in [0,1]$.

Proof. Step 1/4 (Global existence): We can rewrite (2.17) as

$$f'(y) = a_0(y) + \tfrac{f(y)}{1-y}\big(a_1(y) + a_3 f(y)\big), \quad y \in (0,1). \tag{2.19}$$

Theorem 2.2 produces local existence and so let f be a local solution of (2.17), which is strictly positive. To see that $f(y)$ exists and is uniformly bounded for $y \in [0,1]$, we argue by contradiction and split the argument into various cases. First, we rule out finite oscillations at some interior point $y_\infty \in (0,1)$ in the sense that

$$0 \le m < M < \infty, \quad m := \liminf_{y \uparrow y_\infty} f(x), \quad M := \limsup_{y \uparrow y_\infty} f(x), \tag{2.20}$$

is impossible. Assume such $y_\infty \in (0,1)$ exists. Then, there exists $\delta \in (0, \frac{y_\infty}{2})$ such that

$$\forall y \in [y_\infty - \delta, y_\infty]: \quad m - 1 \le f(y) \le M + 1.$$

Therefore, the right-hand-side of (2.17) is uniformly bounded for $y \in [y_\infty - \delta, y_\infty]$. Because $f'(y)$ is uniformly bounded for $y \in [y_\infty - \delta, y_\infty]$, the solution $f(y)$ of (2.17) is Lipschitz continuous. This is a contradiction because Lipschitz continuous functions cannot oscillate.

Second, we rule out that f is unbounded. If f were to oscillate to infinity at some point $y_\infty \in (0,1]$ in the sense

$$\liminf_{y\uparrow y_\infty} f(x) < \limsup_{y\uparrow y_\infty} f(x) = \infty,$$

then we could find an increasing sequence $(y_n)_{n\in\mathbb{N}}$ of local maxima such that $y_n \uparrow y_\infty$, $f'(y_n) = 0$, and $f(y_n) \to \infty$. Because $a_3 < 0$ and $a_1(y_\infty) \in \mathbb{R}$, we see from (2.19) that $\lim_{n\to\infty} f'(y_n) = -\infty$, which is a contradiction. Alternatively, if we have continuous explosion in the sense

$$\lim_{y\uparrow y_\infty} f(y) = \infty,$$

we get from (2.19) that $\lim_{y\uparrow y_\infty} f'(y) = -\infty$, which is again a contradiction because it implies that f is decreasing near y_∞ while $\lim_{y\uparrow y_\infty} f(y) = \infty$. All in all, $f(y)$ is uniformly bounded for $y \in [0, y_\infty)$.

Third, it remains to be shown that $\lim_{y\uparrow 1} f(y)$ exists and we do this by considering oscillations at $y = 1$. To this end, first assume there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset (0, 1)$ monotonically increasing to $y = 1$ such that $f'(y_n) = 0$ for all $n \in \mathbb{N}$. Because f is bounded, we can extract a subsequence if necessary to ensure that $\ell := \lim_{n\to\infty} f(y_n)$ exists in $[0, \infty)$. The ODE in (2.19) and $a_1(1) = \gamma$ yield

$$\begin{aligned} 0 &= \lim_{n\to\infty} \Big(a_0(y_n) + \frac{f(y_n)}{1-y_n}\big(a_1(y_n) + a_3 f(y_n)\big)\Big) \\ &= (1+\gamma)\gamma + \lim_{n\to\infty} \frac{f(y_n)}{1-y_n}\Big(\gamma + a_3 f(y_n)\Big), \end{aligned} \tag{2.21}$$

which forces $\ell \in \{0, -\frac{\gamma}{a_3}\}$. To see $\ell = -\frac{\gamma}{a_3}$, we argue by contradiction and assume $\ell = 0$. In that case, (2.21) yields

$$\lim_{n\to\infty} \frac{\ell - f(y_n)}{1-y_n} = -\lim_{n\to\infty} \frac{f(y_n)}{1-y_n} = 1+\gamma > 0, \tag{2.22}$$

which is impossible because $f(y)$ is nonnegative for all $y \in [0, 1)$. So should $f(y)$ oscillate as $y \uparrow 1$, the sequence of corresponding max and min values $f(y_n)$, $n \in \mathbb{N}$, converges to $-\frac{\gamma}{a_3}$. Because $f(y)$ is uniformly bounded for $y \in [0, 1)$, the only alternative is that $f(y)$ is monotone as $y \uparrow 1$ in which case $\lim_{y\uparrow 1} f(y)$ obviously exists. All in all, this shows that $f \in \mathcal{C}([0, 1]) \cap \mathcal{C}^1([0, 1))$.

Step 2/4 (Boundary values): By multiplying $(1-y)$ in (2.17), we get

$$(1-y)f'(y) = (1-y)a_0(y) + f(y)\big(a_1(y) + a_3 f(y)\big), \quad y \in (0, 1), \tag{2.23}$$

Because $f(1) = \lim_{y\uparrow 1} f(y)$, we see from (2.23) that

$$\lim_{y\uparrow 1}(1-y)f'(y) = f(1)\big(\gamma + a_3 f(1)\big) =: c.$$

To see (2.18), we argue by contradiction. First, we assume $c > 0$. For $\epsilon \in (0, c)$, we let $y_\epsilon \in (0, 1)$ be such that

$$\forall y \in (y_\epsilon, 1) : c - \epsilon \le (1-y) f'(y) \le c + \epsilon.$$

Integrating gives the lower bound

$$\forall y \in (y_\epsilon, 1) : f(y) - f(y_\epsilon) = \int_{y_\epsilon}^{y} f'(q) dq \ge (c - \epsilon) \int_{y_\epsilon}^{y} \frac{1}{1-q} dq.$$

By using the Monotone Convergence Theorem and $f(1) < \infty$, this gives the contradiction

$$\infty > f(1) - f(y_\epsilon) \ge (c - \epsilon) \lim_{y \uparrow 1} \int_{y_\epsilon}^{y} \frac{1}{1-q} dq = (c - \epsilon) \int_{y_\epsilon}^{1} \frac{1}{1-q} dq = \infty.$$

A similar argument rules out $c < 0$. Therefore, $c = 0$ and so $f(1) \in \{0, -\frac{a_3}{\gamma}\}$. To rule out $f(1) = 0$, we again argue by contradiction and assume $f(1) = 0$. The property

$$\lim_{y \uparrow 1} \big(a_1(y) + a_3 f(y)\big) = a_1(1) + a_3 f(1) = \gamma > 0,$$

allows us to find $y_0 \in (0, 1)$ such that

$$\forall y \in (y_0, 1) : a_1(y) + a_3 f(y) > 0. \tag{2.24}$$

To see that $f(y) > 0$ for $y \in [0, 1)$, we use $a_0(y) > 0$ for all $y \in (0, 1]$ to see that if $f(y_1) = 0$ with $f'(y_1) \le 0$ for some $y_1 \in (0, 1)$, we get the contradiction

$$0 \ge f'(y_1) = a_0(y_1) = \frac{\gamma(1+\gamma)}{y_1} > 0.$$

Because $f(y) > 0$ for $y \in [0, 1)$, it follows from (2.23) and (2.24) that $f'(y) > 0$ for $y \in (y_0, 1)$, which gives the contradiction

$$0 = f(1) = \lim_{y \uparrow 1} f(y) \ge f(y_0) > 0.$$

All in all, we have $f(1) = -\frac{a_3}{\gamma}$ as claimed in (2.18).

Step 3/4 (Global upper bound): We claim that $f(y) \le 1$ for all $y \in [0, 1)$. Because f is continuous on $[0, 1]$, there exists $y_1 \in [0, 1]$ such that

$$f(y_1) = \max_{y \in [0,1]} f(y).$$

If $y_1 = 0$, then $f(y) \le f(0) = \gamma < 1$ for all $y \in [0, 1]$, and the claim is proved. Similarly, if $y_1 = 1$, then $f(y) \le f(1) = -\frac{\gamma}{a_3} \le 1$ for all $y \in [0, 1]$. The only remaining case left to consider is $y_1 \in (0, 1)$ with $f'(y_1) = 0$. In this case, the unique positive solution of (2.19) with $f'(y_1) = 0$ is

$$f(y_1) = \frac{2a_0(y_1)(1-y_1)}{-a_1(y_1) + \sqrt{a_1(y_1)^2 - 4a_3a_0(y_1)(1-y_1)}}. \tag{2.25}$$

and we claim that $f(y_1) \le 1$. This claim is proven by proving the inequality

$$2a_0(y)(1-y) + a_1(y) \le \sqrt{(a_1(y))^2 - 4a_3a_0(y)(1-y)}, \quad y \in (0,1), \tag{2.26}$$

Assume there is $y \in (0,1)$ such that (otherwise there is nothing to prove)

$$0 \le 2a_0(y)(1-y) + a_1(y) = \frac{\gamma + (2\gamma^2 - 1)(1-y)}{y}.$$

Squaring both sides of (2.26) produces the requirement

$$0 \ge y\left(a_3 - \gamma^2 + \gamma + 1\right) + \gamma^2 - 1 = y(a_3 + \gamma) - (1-\gamma^2)(1-y). \tag{2.27}$$

The inequality (2.27) holds because $a_3 \le -\gamma$.

When $a_3 \in (-\infty, -\gamma)$, we have $f(1) < 1$. Then, for any maximizer $y_1 \in (0,1)$ with $f'(y_1) = 0$, it suffices to prove that the inequality in (2.26) is strict. This follows because the right-hand-side of (2.27) is strictly negative for all $y \in (0,1)$.

Step 4/4 (Global lower bound): Let $a_3 \in [-1, -\gamma]$. To show $f \ge \gamma$, we argue by contradiction and assume there is a point $y_1 \in [0,1]$ such that $f(y_1) < \gamma$ and $f'(y_1) = 0$. Because $f(0) = \gamma$ and $f(1) = -\frac{\gamma}{a_3} \ge \gamma$, it must be that $y_1 \in (0,1)$. The unique positive solution of $f'(y_1) = 0$ is as in (2.25). The contradictory property $f(y_1) \ge \gamma$ is equivalent to

$$2a_0(y_1)(1-y_1) + \gamma a_1(y_1) \ge \gamma\sqrt{a_1(y_1)^2 - 4a_3a_0(y_1)(1-y_1)}. \tag{2.28}$$

The left-hand-side of (2.28) equals

$$2a_0(y_1)(1-y_1) + \gamma a_1(y_1) = \frac{\gamma(\gamma + 1 - y_1)}{y_1} > 0.$$

Squaring both sides of (2.28) produces the equivalent property $a_3 + 1 \ge 0$, which holds by assumption. □

Remark 2.2. For given constants $y_0 \in (0,1)$ and $f_0 \in (0,1)$, we can argue as in the above proof of Theorem 2.4 and construct $f \in \mathcal{C}([y_0,1]) \cap \mathcal{C}^1([y_0,1))$ with $f(y) \in [0,1]$ for $y \in [y_0,1]$ that satisfies the ODE

$$f'(y) = a_0(y) + \tfrac{a_1(y)}{1-y} f(y) + \tfrac{a_3}{1-y} f(y)^2, \quad y \in (y_0, 1), \tag{2.29}$$

and the boundary conditions

$$f(y_0) = f_0, \quad f(1) = -\frac{\gamma}{a_3}. \tag{2.30}$$

Indeed, since the right-hand-side of (2.29) is locally Lipschitz for $y \in [y_0, y_0 + \delta_0]$ with $y_0 \in (0, 1-\delta_0)$, standard existence theory for ODEs gives a local solution $f(y)$ for $y \in [y_0, y_0+\delta]$ for some $\delta \in (0, \delta_0)$. By reasoning as in Step 1, we see that $f(y)$ exists for $y \in [y_0, 1)$ and is uniformly bounded. Step 2 still holds. Step 3 holds by using $f(y_0) = f_0 < 1$ in place of $f(0) = \gamma < 1$.

Finally, assuming that $f_0 \in [\gamma, 1)$ and $a_3 \in [-1, -\gamma]$, Step 4 still produces $f(y) \geq \gamma$ for all $y \in [y_0, 1]$. $\diamond$

2.2. *Local ODE existence for all ξ*

For constants σ_D^2, ξ, A, and a continuous function $h : [0, 1] \mapsto \mathbb{R}$, we define the coefficient functions (consistent with (2.4))

$$\begin{aligned} a_0(y) &:= \frac{\gamma(1+\gamma)}{y}, \quad y \in (0,1], \\ a_1(y) &:= \frac{(2\gamma+1)y - (1+\gamma)}{y}, \quad y \in (0,1], \\ a_2(h, y) &:= \frac{\xi}{\sigma_D^2} \exp\left\{ \int_0^y \frac{h(q)-1}{1-q} dq \right\} - A, \quad y \in (0,1). \end{aligned} \tag{2.31}$$

Theorem 2.5. *Let $\gamma \in (0, 1)$, $\sigma_D^2 > 0$, and $A > 1$. For $\xi_0 > 0$, there exists $\delta_0 \in (0, \frac{1}{2}]$ such that*

1. *For all $\xi > 0$ with $|\xi - \xi_0| \leq 1$, the integral equation*

$$h(y) = \frac{1}{y^{1+\gamma}(1-y)^\gamma} \int_0^y \left(a_0(x)(1-x)^\gamma + \frac{a_2(h,x)}{(1-x)^{1-\gamma}} h(x)^2 \right) x^{1+\gamma} dx, \quad y \in (0,1), \tag{2.32}$$

has a local unique solution $h_\xi \in \mathcal{C}^1([0, \delta_0])$ and h_ξ satisfies the boundary value problem

$$\begin{cases} h'(y) = a_0(y) + \frac{a_1(y)}{1-y} h(y) + \frac{a_2(h,y)}{1-y} h(y)^2, \quad y \in (0, \delta_0), \\ h(0) = \gamma. \end{cases} \tag{2.33}$$

2. *For all $\xi_i > 0$ with $|\xi_i - \xi_0| \leq 1$ for $i \in \{1, 2\}$, comparison holds for (2.33) in the sense that $0 < \xi_1 < \xi_2$ implies $h_{\xi_1}(y) < h_{\xi_2}(y)$ for all $y \in (0, \delta_0]$,*

Proof. 1. The proof of local existence only requires minor modifications to the proof of Theorem 2.2. For $0 < \delta \leq \frac{1}{2}$ and $R > 0$ with $R + \gamma \leq 1$, let $f \in \mathcal{X}$ with $\mathcal{X}$ defined in (2.6). We modify the operator T in (2.7) by defining

$$T(f)(y) := \frac{1}{\varphi(y)} \int_0^y \varphi(t) \left(a_0(t) + \frac{a_2(f,t)}{1-t} f(t)^2 \right) dt, \quad y \in (0, \delta]. \tag{2.34}$$

To replace the estimates in (2.12) and (2.14), we first need the bound

$$
\begin{aligned}
|a_2(f,t)| &= \Big| \tfrac{\xi}{\sigma_D^2}(1-t)\exp\Big\{ \int_0^t \frac{f(q)}{1-q}dq \Big\} - A \Big| \\
&\le \tfrac{1+\xi_0}{\sigma_D^2}(1-t)e^{(R+\gamma)\int_0^t \frac{1}{1-q}dq} + A \\
&\le \tfrac{1+\xi_0}{\sigma_D^2}(1-t)^{1-R-\gamma} + A \\
&\le \tfrac{1+\xi_0}{\sigma_D^2} + A, \quad t \in [0,\delta].
\end{aligned} \tag{2.35}
$$

Therefore,

$$
|a_2(f,t)|f(t)^2 \le \big(\tfrac{1+\xi_0}{\sigma_D^2} + A\big)(R+\gamma)^2 \le \tfrac{1+\xi_0}{\sigma_D^2} + A, \quad t \in [0,\delta]. \tag{2.36}
$$

The estimate in (2.36) implies that the analogous of (2.12) and (2.14) hold with M replaced by $\frac{1+\xi_0}{\sigma_D^2} + A$. With these modifications, the rest of the local existence proof is identical.

We start by proving uniqueness for $y \in [0, y_1]$ for some $y_1 \in (0, \delta_0]$. To this end, we let h and $\tilde{h}$ be two solutions of (2.32) and set $g := h - \tilde{h}$. Because $h(0) = \tilde{h}(0) = \gamma$, it suffices to prove $g(y) = 0$ for $y \in [0, y_1]$. Moreover, because $\gamma < 1$, by taking $y_1 \in (0,1)$ small enough, we can assume $h(y) \le 1$ and $\tilde{h}(y) \le 1$ for $y \in [0, y_1]$. For $y \in (0, y_1]$, we have

$$
\begin{aligned}
|g(y)| &\le \frac{1}{y^{1+\gamma}(1-y)^\gamma} \int_0^y \frac{x^{1+\gamma}}{(1-x)^{1-\gamma}} \Big| a_2(h,x)h(x)^2 - a_2(\tilde{h},x)\tilde{h}(x)^2 \Big| dx \\
&\le \frac{1}{y^{1+\gamma}(1-y)^\gamma} \int_0^y \frac{x^{1+\gamma}}{(1-x)^{1-\gamma}} \Big| a_2(h,x) - a_2(\tilde{h},x) \Big| h(x)^2 dx \\
&+ \frac{1}{y^{1+\gamma}(1-y)^\gamma} \int_0^y \frac{x^{1+\gamma}}{(1-x)^{1-\gamma}} |a_2(\tilde{h},x)| \Big| h(x)^2 - \tilde{h}(x)^2 \Big| dx.
\end{aligned} \tag{2.37}
$$

We consider the last two integrals in (2.37) separately. The Mean-Value Theorem gives the inequality

$$
|e^a - e^b| \le e^{|a|+|b|}|a-b|, \quad a, b \in \mathbb{R}.
$$

Therefore, the first integral in (2.37) is bounded by

$$
\frac{1}{y^{1+\gamma}(1-y)^\gamma} \int_0^y \frac{x^{1+\gamma}}{(1-x)^{1-\gamma}} \frac{\xi(1-x)}{\sigma_D^2} \Big| e^{\int_0^x \frac{h(q)}{1-q}dq} - e^{\int_0^x \frac{\tilde{h}(q)}{1-q}dq} \Big| dx
$$

$$
\begin{aligned}
&\leq \frac{\xi}{\sigma_D^2(1-y_1)^\gamma}\int_0^y e^{\int_0^x \frac{h(q)+\tilde{h}(q)}{1-q}dq}\int_0^x \frac{|g(q)|}{1-q}dqdx \\
&\leq \frac{\xi}{\sigma_D^2(1-y_1)^\gamma}\int_0^y \frac{1}{(1-x)^2}\int_0^y \frac{|g(q)|}{1-q}dqdx \qquad (2.38)\\
&\leq \frac{\xi}{\sigma_D^2(1-y_1)^{\gamma+2+1}}\int_0^y |g(q)|dq.
\end{aligned}
$$

For the second integral in (2.37), the estimate in (2.35) ensures that $a_2(\tilde{h}, x)$ is uniformly bounded by some constant $c_1 > 0$. Therefore,

$$
\begin{aligned}
&\frac{1}{y^{1+\gamma}(1-y)^\gamma}\int_0^y \frac{x^{1+\gamma}}{(1-x)^{1-\gamma}}|a_2(\tilde{h},x)|\big|h(x)^2-\tilde{h}(x)^2\big|dx \\
&\leq \frac{c_1}{(1-y_1)}\int_0^y \big(h(x)+\tilde{h}(x)\big)\big|h(x)-\tilde{h}(x)\big|dx \qquad (2.39)\\
&\leq \frac{2c_1}{(1-y_1)}\int_0^y \big|g(x)\big|dx.
\end{aligned}
$$

By combining (2.37) with the two estimates (2.38) and (2.39) we get

$$
|g(y)| \leq c_2 \int_0^y \big|g(x)\big|dx, \qquad y \in [0, y_1],
$$

for some constant $c_2 > 0$. Because $g(0) = 0$, Gronwall's inequality produces $g(y) = 0$ for all $y \in [0, y_1]$.

Next, we prove uniqueness for $y \in [y_1, \delta_0]$. The function $[y_1, \delta_0] \ni y \mapsto \int_0^y \frac{h(q)}{1-q}dq$ transforms the ODE (2.33) into a second-order ODE. Because $y_1 > 0$, there is no singularity and standard uniqueness arguments apply.

2. To argue by contradiction, we assume there exists $y_0 \in (0, \delta_0]$ such that $h_1(y_0) \geq h_2(y_0)$. First, because $\xi_1 < \xi_2$, there exists $y_1 \in (0, 1)$ such that

$$
\forall y \in [0, y_1] : \xi_1(1-y)\exp\left\{\int_0^y \frac{h_1(q)}{1-q}dq\right\} < \xi_2(1-y)\exp\left\{\int_0^y \frac{h_2(q)}{1-q}dq\right\}.
$$

The strict comparison in Theorem 2.3 gives $h_1(y) < h_2(y)$ for $y \in [0, y_1]$. Second, we define

$$
y_2 := \sup\{y \in [0, \delta_0] : \forall q \in (0, y]\, h_1(q) < h_2(q)\} \in [y_1, y_0].
$$

The continuity of h_1 and h_2 gives $h_1(y_2) = h_2(y_2)$. Because $\xi_1 < \xi_2$, we have

$$\forall y \in [0, y_2] : \xi_1(1-y)\exp\left\{\int_0^y \frac{h_1(q)}{1-q}dq\right\} < \xi_2(1-y)\exp\left\{\int_0^y \frac{h_2(q)}{1-q}dq\right\}.$$

The strict comparison in Theorem 2.3 gives the contradiction $h_1(y) < h_2(y)$ for $y \in (0, y_2]$. $\square$

2.3. Global ODE existence for ξ small

Theorem 2.6. *Let $\gamma \in (0,1)$, $\sigma_D^2 > 0$, and $A > 1$. For a constant ξ satisfying*

$$0 \leq \frac{\xi}{\sigma_D^2} < A - \gamma, \tag{2.40}$$

the integral equation (2.32) has a global solution $h_\xi \in \mathcal{C}([0,1]) \cap \mathcal{C}^1([0,1))$ with $0 < h_\xi \leq 1$. Furthermore, h_ξ satisfies the boundary value problem (2.33) globally, as well as the boundary condition

$$h(1) = c_0, \quad c_0 := \frac{\gamma}{A} \in (0, \gamma). \tag{2.41}$$

Proof. Let f be as in Theorem 2.4 for the constant

$$a_3 := \frac{\xi}{\sigma_D^2} - A < -\gamma < 0. \tag{2.42}$$

Step 1/3 (Global Existence): Let $[0, \delta_1)$ be the maximal interval of existence for $h = h_\xi$ produced by Theorem 2.5(1). First, we claim $h(y) \leq 1$ for $y \in [0, \delta_1)$. To see this, we argue by contradiction and assume there exists $y_1 \in [0, \delta_1)$ such that $h(y_1) > 1$. Because $h(0) = \gamma < 1$, it must be that $y_1 \in (0, \delta_1)$. We define

$$y_0 := \sup\{y \in [0, y_1) : h(y) \leq 1\} \in (0, y_1).$$

The continuity of h gives

$$\forall y \in [0, y_0] : h(y) \leq 1 \quad \text{and} \quad h(y_0) = 1.$$

Therefore, we have

$$a_2(h_\xi, y) = \frac{\xi}{\sigma_D^2}\exp\left\{\int_0^y \frac{h(q)-1}{1-q}dq\right\} - A \leq \frac{\xi}{\sigma_D^2} - A = a_3, \quad y \in [0, y_0]. \tag{2.43}$$

Theorem 2.4 gives an ODE solution f with $f < 1$. The comparison result in Theorem 2.3 gives the contradiction

$$1 = h(y_0) \leq f(y_0) < 1.$$

Because $h(y) \le 1$ for all $y \in [0, \delta_1)$, (2.43) holds and we can again use Theorems 2.3 and 2.4 to see

$$h(y) \le f(y) < 1, \quad y \in [0, \delta_1).$$

Therefore, $\delta_1 = 1$.

Step 2/3 (Boundary values): We start by writing

$$(1-y)\exp\left\{\int_0^y \frac{h(x)}{1-x}\,dx\right\} = \exp\left\{\int_0^y \frac{h(x)-1}{1-x}\,dx\right\}.$$

Since $h \le 1$, the integral $\int_0^1 \frac{h(x)-1}{1-x}\,dx$ exists in $[-\infty, 0]$, and so the following limit exists

$$L := \lim_{y\uparrow 1}(1-y)\exp\left\{\int_0^y \frac{h(x)}{1-x}\,dx\right\} = \exp\left\{\int_0^1 \frac{h(x)-1}{1-x}\,dx\right\} \in [0,1]. \tag{2.44}$$

As in Step 1 in the proof of Theorem 2.4, assume first that there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset (0,1)$ monotonically increasing to $y = 1$ such that $h'(y_n) = 0$ for all $n \in \mathbb{N}$. Because $h \le 1$, we can extract a subsequence, if necessary, to ensure that $\ell := \lim_{n\to\infty} h(y_n) \in [0,1]$ exists. The ODE in (2.33) yields

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(a_0(y_n) + \frac{h(y_n)\big(a_1(y_n) + a_2(h, y_n)h(y_n)\big)}{1-y_n}\right) \\ &= (1+\gamma)\gamma + \lim_{n\to\infty}\frac{h(y_n)\big(a_1(y_n) + a_2(h, y_n)h(y_n)\big)}{1-y_n}. \end{aligned}$$

This implies that

$$\begin{aligned} 0 &= \lim_{n\to\infty} h(y_n)\big(a_1(y_n) + a_2(h, y_n)h(y_n)\big) \\ &= \ell\left(\gamma + \Big(\frac{\xi}{\sigma_D^2}L - A\Big)\ell\right). \end{aligned}$$

Hence, either $\ell = 0$ or $\ell = \ell_1$ where

$$\ell_1 := \frac{\gamma}{A - \frac{\xi}{\sigma_D^2}L}. \tag{2.45}$$

We argue by contradiction to rule out $\ell = 0$. We assume $\ell = 0$ to get

$$\begin{aligned} 0 &= \lim_{n\to\infty}\Big(a_0(y_n)+\frac{h(y_n)}{1-y_n}\big(a_1(y_n)+a_2(h,y_n)h(y_n)\big)\Big) \\ &= (1+\gamma)\gamma+\lim_{n\to\infty}\frac{h(y_n)}{1-y_n}\gamma. \end{aligned}$$

This is a contradiction because h is positive.

Based on the above, should h oscillate as $y\uparrow 1$, the sequence of corresponding max and min values $h(y_n)$ would converge to ℓ_1 given in (2.45). Therefore, $\ell := \lim_{y\uparrow 1} h(y)$ exists and equals ℓ_1. Because $h\le 1$, the alternative is that h is monotone near $y=1$, in which case the limit $\ell := \lim_{y\uparrow 1} h(y)$ exists. To show $\ell=\ell_1$ also when h is monotone near $y=1$, we follow the argument in Step 2 in the proof of Theorem 2.4 but replace (2.23) with

$$(1-y)h'(y)=(1-y)a_0(y)+h(y)a_1(y)+a_2(h,y)h(y)^2,\qquad y\in(0,1).$$

Taking limits yields

$$\lim_{y\uparrow 1}(1-y)h'(y)=\ell\gamma+\Big(\frac{\xi}{\sigma_D^2}L-A\Big)\ell^2.$$

As in the proof of Theorem 2.4, we get $\ell\gamma+\Big(\frac{\xi}{\sigma_D^2}L-A\Big)\ell^2=0$ and so either $\ell=0$ or $\ell=\ell_1$ for ℓ_1 in (2.45). We rule out $\ell=0$ as in Step 2 of the proof of Theorem 2.4. Therefore, when $h(y)$ is monotone near $y=1$, we also have $\ell=\ell_1$.

Let us show that $\ell=c_0$ where c_0 is from (2.41). Since $\frac{\xi}{\sigma_D^2}<A-\gamma$ and $L\in[0,1]$, we have that ℓ_1 from (2.45) satisfies

$$\ell_1=\frac{\gamma}{A-\frac{\xi}{\sigma_D^2}L}<1. \tag{2.46}$$

By taking $\varepsilon>0$ so small that $\eta:=\ell_1+\varepsilon<1$, we see there exists $y_1\in(0,1)$ such that

$$\forall y\in[y_1,1]: h(y)\le\eta<1.$$

It follows that $h(y)\le\eta$ for all $y\in[y_1,1]$, while $h(y)\le 1$ for all $y\in[0,y_1]$. Hence, for all $y\in[y_1,1]$,

$$\begin{aligned} 0 &\le (1-y)\exp\Big\{\int_0^y \frac{h(q)}{1-q}dq\Big\} \\ &\le (1-y)\exp\Big\{\int_0^{y_1}\frac{1}{1-q}dq\Big\}\exp\Big\{\int_{y_1}^{y}\frac{\eta}{1-q}dq\Big\} \\ &= \frac{1-y}{1-y_1}\Big(\frac{1-y_1}{1-y}\Big)^{\eta}, \end{aligned}$$

which converges to zero as $y\uparrow 1$. Therefore, $L=0$ and so $\ell=c_0$.

To see that $h' \in \mathcal{C}([0,1))$, we can repeat the argument from Step 2 of the proof of Theorem 2.2.

Step 3/3: To see $h(y) > 0$ for all $y \in [0,1]$, the boundary values (2.33) imply that it suffices to show that there is no point $y \in (0,1)$ such that

$$h(y) = h'(y) = 0. \tag{2.47}$$

However, the ODE (2.33) makes (2.47) impossible because $a_0(y) > 0$ for all $y \in [0,1]$. $\square$

2.4. *Explosion for ξ big*

Lemma 2.7. *Let $\gamma \in (0,1)$, $\sigma_D^2 > 0$, and $A > 1$. For $y_0 \in (0,1)$, there exists $\xi(y_0) \in (0,\infty)$ such that*

$$\forall \xi \geq \xi(y_0) \, : \, h_\xi(y_0) = \infty.$$

Proof. Let $y_0 \in (0,1)$ be arbitrary and assume to the contrary that for all $\xi > 0$, the local solution $h_\xi(y)$ of (2.33) exists for $y \in [0, y_0]$. We consider $\xi > 0$ such that

$$\frac{\xi}{\sigma_D^2} > \frac{A}{1-y_0}. \tag{2.48}$$

By (2.32), the ODE for $g_\xi(y) := h_\xi(y)(1-y)^\gamma y^{1+\gamma}$ gives

$$\begin{aligned} g_\xi'(y) &= a_0(y)(1-y)^\gamma y^{1+\gamma} + \tfrac{a_2(h_\xi, y)}{((1-y)y)^{1+\gamma}} g_\xi(y)^2 \\ &= \gamma(1-\gamma)(1-y)^\gamma y^\gamma + \frac{\frac{\xi}{\sigma_D^2}\exp\{\int_0^y \frac{h_\xi(q)-1}{1-q}dq\} - A}{((1-y)y)^{1+\gamma}} g_\xi(y)^2 \\ &\geq \gamma(1-\gamma)(1-y)^\gamma y^\gamma + \frac{\frac{\xi}{\sigma_D^2}(1-y) - A}{((1-y)y)^{1+\gamma}} g_\xi(y)^2 \\ &\geq \gamma(1-\gamma)(1-y)^\gamma y^\gamma + \frac{\frac{\xi}{\sigma_D^2}(1-y_0) - A}{((1-y)y)^{1+\gamma}} g_\xi(y)^2, \quad y \in (0, y_0), \end{aligned} \tag{2.49}$$

where the first inequality uses $h_\xi > 0$. Integrating (2.49) and using (2.48) and $g_\xi(0) = 0$ give

$$g_\xi(\tfrac{y_0}{2}) \geq \gamma(1-\gamma) \int_0^{\frac{y_0}{2}} (1-q)^\gamma q^\gamma dq > 0. \tag{2.50}$$

To create a contradiction, we define the constant

$$c_\xi := \frac{\frac{\xi}{\sigma_D^2}(1-y_0) - A}{((1-y_0)\frac{y_0}{2})^{1+\gamma}} > 0.$$

Because c_ξ is affine in ξ, we can increase $\xi > 0$ if needed such that ξ satisfies both (2.48) and

$$2 \le c_\xi y_0 \gamma(1-\gamma) \int_0^{\frac{y_0}{2}} (1-q)^\gamma q^\gamma dq. \tag{2.51}$$

The Riccati equation

$$\begin{cases} f'(y) = c_\xi f(y)^2, & y > \frac{y_0}{2}, \\ f(\frac{y_0}{2}) = g_\xi(\frac{y_0}{2}), \end{cases}$$

has the explicit solution

$$f(y) = \frac{g_\xi(\frac{y_0}{2})}{1 - c_\xi g_\xi(\frac{y_0}{2})(y - \frac{y_0}{2})}, \quad y \in [\tfrac{y_0}{2}, \tfrac{1}{c_\xi g_\xi(\frac{y_0}{2})} + \tfrac{y_0}{2}).$$

The solution f explodes at

$$y = \frac{1}{c_\xi g_\xi(\frac{y_0}{2})} + \frac{y_0}{2} \le y_0, \tag{2.52}$$

where the inequality follows from (2.50) and (2.51). For $y \in [\frac{y_0}{2}, y_0]$, (2.49) gives

$$g_\xi'(y) \ge \frac{\frac{\xi}{\sigma_D^2}(1-y_0) - A}{((1-y)y)^{1+\gamma}} g_\xi(y)^2 \ge c_\xi g_\xi(y)^2. \tag{2.53}$$

Then, comparison gives us that $g_\xi(y) \ge f(y)$ for $y \ge \frac{y_0}{2}$. Therefore, g_ξ also explodes at or before y in (2.52). $\square$

2.5. Lipschitz estimates

For $\xi > 0$, we denote by h_ξ a local solution of the boundary value problem (2.33) and define the constant

$$y_\xi := \inf\{y > 0 : h_\xi(y) = 1\} \wedge 1 \in (0, 1], \tag{2.54}$$

where $\inf \emptyset := \infty$ and $a \wedge b := \min\{a, b\}$ for $a, b \in [0, \infty]$. We also define the non-increasing function

$$F_\xi(y) := \frac{\xi}{\sigma_D^2} \exp\Big\{ \int_0^y \frac{h_\xi(q) - 1}{1-q} \Big\}, \quad y \in [0, y_\xi]. \tag{2.55}$$

Lemma 2.8. *Let $\gamma \in (0, 1)$, $\sigma_D^2 > 0$, $A > 1$, $y_0 \in (0, 1)$, and $\bar{\xi} > 0$. Then, there exist constants $M_1 > 0$ and $M_2 > 0$ such that*

$$|h_{\xi_1}(y) - h_{\xi_2}(y)| \le M_1 y |\xi_1 - \xi_2|, \quad y \in [0, y_{\xi_1} \wedge y_{\xi_2} \wedge y_0], \quad \xi_1, \xi_2 \in [0, \bar{\xi}], \tag{2.56}$$

$$|F_{\xi_1}(y) - F_{\xi_2}(y)| \le M_2 |\xi_1 - \xi_2|, \quad y \in [0, y_{\xi_1} \wedge y_{\xi_2} \wedge y_0], \quad \xi_1, \xi_2 \in [0, \bar{\xi}]. \tag{2.57}$$

Proof. For $\xi_1, \xi_2 \in [0, \bar{\xi}]$, we let h_1 and h_2 be the corresponding local ODE solutions of (2.33)

$$h_i'(y) + \frac{1+\gamma}{y}\big(h_i(y) - \gamma\big) = \frac{\gamma}{1-y}h_i(y) + \frac{a_2(\xi_i, h_i, y)}{1-y}h_i(y)^2, \quad y \le y_{\xi_i},$$

where we have augmented the a_2 function from (2.31) by writing

$$a_2(\xi_i, h_i, y) := \frac{\xi_i}{\sigma_D^2}\exp\Big\{\int_0^y \frac{h_i(q)-1}{1-q}\,dq\Big\} - A, \quad y \le y_{\xi_i}.$$

Subtracting and multiplying by $y^{1+\gamma}$ yield

$$\begin{aligned}
&y^{1+\gamma}\big(h_1'(y) - h_2'(y)\big) + y^{\gamma}(1+\gamma)\big(h_1(y) - h_2(y)\big)\\
&= y^{1+\gamma}\frac{\gamma}{1-y}\big(h_1(y) - h_2(y)\big)\\
&+ y^{1+\gamma}\frac{a_2(\xi_1, h_1, y) - a_2(\xi_2, h_2, y)}{1-y}h_1(y)^2 + y^{1+\gamma}\frac{a_2(\xi_2, h_2, y)}{1-y}\big(h_1(y)^2 - h_2(y)^2\big),
\end{aligned}$$

for $y \le y_{\xi_1} \wedge y_{\xi_2}$. Computing the y derivative gives

$$\begin{aligned}
&\Big(y^{1+\gamma}\big(h_1(y) - h_2(y)\big)\Big)' = y^{1+\gamma}\frac{\gamma}{1-y}\big(h_1(y) - h_2(y)\big)\\
&+ y^{1+\gamma}\frac{a_2(\xi_1, h_1, y) - a_2(\xi_2, h_2, y)}{1-y}h_1(y)^2 + y^{1+\gamma}\frac{a_2(\xi_2, h_2, y)}{1-y}\big(h_1(y)^2 - h_2(y)^2\big).
\end{aligned}$$

Because $h_i \in \mathcal{C}([0, y_{\xi_i}])$ and $h_i(0) = \gamma$, we can integrate from 0 to y with $y \in (0, y_{\xi_1} \wedge y_{\xi_2}]$ to get

$$\begin{aligned}
&y^{1+\gamma}\big(h_1(y) - h_2(y)\big)\\
&= \gamma\int_0^y \frac{t^{1+\gamma}}{1-t}\big(h_1(t) - h_2(t)\big)dt\\
&+ \int_0^y t^{1+\gamma}\frac{a_2(\xi_1, h_1, t) - a_2(\xi_2, h_2, t)}{1-t}h_1(t)^2\,dt \qquad (2.58)\\
&+ \int_0^y a_2(\xi_2, h_2, t)\frac{t^{1+\gamma}}{1-t}\big(h_1(t)^2 - h_2(t)^2\big)dt\\
&= \int_0^y \frac{t^{1+\gamma}}{1-t}\Big(\gamma + a_2(\xi_2, h_2, t)\big(h_1(t) + h_2(t)\big)\Big)\big(h_1(t) - h_2(t)\big)dt
\end{aligned}$$

$$+\int_0^y t^{1+\gamma}\frac{a_2(\xi_1,h_1,t)-a_2(\xi_2,h_2,t)}{1-t}h_1(t)^2\,dt.$$

The bound $h_i(t)\le 1$ for $t\in[0,y_{\xi_1}\wedge y_{\xi_2}]$ gives

$$\left|\frac{a_2(\xi_2,h_2,t)}{1-t}\big(h_1(t)+h_2(t)\big)\right|\le 2\frac{\frac{\xi_2}{\sigma_D^2}+A}{1-y_0}\le 2\frac{\frac{\bar{\xi}}{\sigma_D^2}+A}{1-y_0}=:M_0,\quad t\in[0,y_{\xi_1}\wedge y_{\xi_2}\wedge y_0].$$

Because

$$\begin{aligned}&a_2(\xi_1,h_1,t)-a_2(\xi_2,h_2,t)\\&=\frac{(1-t)(\xi_1-\xi_2)}{\sigma_D^2}e^{\int_0^t\frac{h_1(q)}{1-q}\,dq}+\frac{(1-t)\xi_2}{\sigma_D^2}\left(e^{\int_0^t\frac{h_1(q)}{1-q}\,dq}-e^{\int_0^t\frac{h_2(q)}{1-q}\,dq}\right),\end{aligned}$$

the Mean-Value Theorem applied to the function

$$[0,1]\ni\theta\mapsto\exp\left\{\int_0^t\frac{\theta h_1(q)+(1-\theta)h_2(q)}{1-q}\,dq\right\},$$

gives $\theta(t)\in[0,1]$ such that

$$\begin{aligned}&\frac{|a_2(\xi_1,h_1,t)-a_2(\xi_2,h_2,t)|}{1-t}h_1(t)^2\\&\le\frac{|\xi_1-\xi_2|}{\sigma_D^2}+\frac{\xi_2}{\sigma_D^2}\left|e^{\int_0^t\frac{\theta(t)h_1(q)+(1-\theta(t))h_2(q)}{1-q}\,dq}\int_0^t\frac{h_1(q)-h_2(q)}{1-q}\,dq\right|\\&\le\frac{|\xi_1-\xi_2|}{\sigma_D^2}+\frac{\xi_2}{\sigma_D^2(1-t)^2}\int_0^t|h_1(q)-h_2(q)|\,dq,\quad t\in[0,y_{\xi_1}\wedge y_{\xi_2}\wedge y_0].\end{aligned}\tag{2.59}$$

Therefore, for $y\in[0,y_{\xi_1}\wedge y_{\xi_2}\wedge y_0]$, we have by (2.58)

$$\begin{aligned}y^{1+\gamma}|h_1(y)-h_2(y)|\le\left(\frac{\gamma}{1-y_0}+M_0\right)&\int_0^y t^{1+\gamma}|h_1(t)-h_2(t)|\,dt\\&+\frac{1}{\sigma_D^2}|\xi_1-\xi_2|\int_0^y t^{1+\gamma}\,dt\\&+\frac{\xi_2}{\sigma_D^2(1-y_0)^2}\int_0^y t^{1+\gamma}\,dt\int_0^y|h_1(q)-h_2(q)|\,dq.\end{aligned}$$

In turn, for $y \in [0, y_{\xi_1} \wedge y_{\xi_2} \wedge y_0]$, we have

$$|h_1(y) - h_2(y)| \le \left(\frac{\gamma}{1-y_0} + M_0 + \frac{\xi_2 y}{\sigma_D^2 (1-y_0)^2} \right) \int_0^y |h_1(q) - h_2(q)|\, dq + \frac{1}{\sigma_D^2} |\xi_1 - \xi_2| y.$$

For $y \in [0, y_{\xi_1} \wedge y_{\xi_2} \wedge y_0]$, Gronwall's inequality gives (2.56) because

$$\begin{aligned} |h_1(y) - h_2(y)| &\le \frac{|\xi_1 - \xi_2|}{\sigma_D^2} y e^{\left(\frac{\gamma}{1-y_0} + M_0 + \frac{\xi_2}{\sigma_D^2 (1-y_0)^2} \right) y} \\ &\le M_1 y |\xi_1 - \xi_2|, \quad M_1 := \frac{1}{\sigma_D^2} e^{\left(\frac{\gamma}{1-y_0} + M_0 + \frac{\bar{\xi}}{\sigma_D^2 (1-y_0)^2} \right) y_0}. \end{aligned} \tag{2.60}$$

Finally, to see (2.57), we use (2.59) to see for $y \in [0, y_{\xi_1} \wedge y_{\xi_2} \wedge y_0]$

$$\begin{aligned} |F_{\xi_1}(y) - F_{\xi_2}(y)| &\le \frac{|\xi_1 - \xi_2|}{\sigma_D^2} (1-y) + \frac{\xi_2}{\sigma_D^2} \int_0^y |h_1(q) - h_2(q)|\, dq \\ &\le \frac{|\xi_1 - \xi_2|}{\sigma_D^2} + \frac{\bar{\xi}}{\sigma_D^2} M_1 |\xi_1 - \xi_2| \int_0^y q\, dq \\ &\le M_2 |\xi_1 - \xi_2|, \quad M_2 := \frac{1 + \frac{1}{2} \bar{\xi} M_1}{\sigma_D^2}. \quad \square \end{aligned}$$

The uniqueness claim in Theorem 1.1 follows from the following result.

Corollary 2.9. *Let $\gamma \in (0, 1)$, $\sigma_D^2 > 0$, and $A > 1$. For all $\xi > 0$, the boundary value problem (2.33) has a unique local solution.*

Proof. Existence follows from Theorem 2.5. To prove uniqueness, we let $h_i : I_i \to (0, \infty)$, $i \in \{1, 2\}$, be two solutions where $I_i \subseteq [0, 1]$ is their maximal interval of existence. We let y_i be the right-end point of I_i. Because $h_i(0) = \gamma < 1$, there exists $y_0 \in (0, y_1 \wedge y_2)$ such that $h_i(y) < 1$ for all $y \in [0, y_0]$. By applying (2.56) with $\xi_1 := \xi_2 := \bar{\xi} := \xi$, we see that $h_1(y) = h_2(y)$ for all $y \in [0, y_0]$. Because $y_0 > 0$, there is no singularity and standard uniqueness arguments imply $I_1 = I_2$ and $h_1(y) = h_2(y)$ for all $y \in I_1$. $\square$

2.6. Global ODE existence for critical ξ

The existence claim in Theorem 1.1 follows from the following result after we note that the ODE (2.33) coincides with the ODE in (1.10) because the integral in (2.61) below ensures

$$(A-\gamma)\exp\Big\{\int\limits_y^1 \frac{1-h_{\xi_0}(q)}{1-q}dq\Big\} = \frac{\xi_0}{\sigma_D^2}\exp\Big\{\int\limits_0^y \frac{h_{\xi_0}(q)-1}{1-q}dq\Big\}, \quad y\in[0,1].$$

Lemma 2.10. *Let $\gamma\in(0,1)$, $\sigma_D^2>0$, $A>1$, and define the set*

$$\Xi := \big\{\xi>0 : h_\xi \in \mathcal{C}([0,1])\cap\mathcal{C}^1([0,1)) \text{ solves (2.33) with } h_\xi(0)=\gamma \text{ and } h_\xi(1)\le\gamma\big\}.$$

Then, the following properties hold:

1. *Ξ is a non-empty and bounded subset of* $(0,\infty)$.
2. *For $\xi\in\Xi$, $h_\xi(y)<1$ for all $y\in[0,1]$.*
3. $\xi_0 := \sup\Xi \notin \Xi$.
4. *$h_{\xi_0}\in\mathcal{C}([0,1])\cap\mathcal{C}^1([0,1))$ solves (2.33) with $h_{\xi_0}(1)=1$.*
5. *$\int_0^1 \frac{1-h_{\xi_0}(q)}{1-q}dq\in(0,\infty)$ and the integral satisfies*

$$\lim_{y\uparrow 1}\exp\Big\{\int\limits_0^y \frac{h_{\xi_0}(q)-1}{1-q}dq\Big\} = \frac{\sigma_D^2(A-\gamma)}{\xi_0}\in(0,1). \tag{2.61}$$

6. *$h_{\xi_0}(y)\ge\gamma$ for all $y\in[0,1]$.*
7. *$h_{\xi_0}\in\mathcal{C}^1([0,1])$ with $h'_{\xi_0}(1)=\frac{1-\gamma^2}{A-1}>0$.*

Proof. 1. The set Ξ is not empty by Theorem 2.6. The set Ξ is bounded from above by Lemma 2.7.
2. Let $h=h_\xi$ for some $\xi\in\Xi$. We argue by contradiction and assume there is $\hat y\in[0,1]$ such that

$$h(\hat y) = \max_{y\in[0,1]} h(y)\ge 1.$$

Because $h(0)=\gamma$ and $h(1)\le\gamma$, we have $\hat y\in(0,1)$ and so

$$h(\hat y)>\gamma,\quad h'(\hat y)=0,\quad h''(\hat y)\le 0.$$

Inserting $h'(\hat y)=0$ into the ODE (2.33) produces the relation

$$\frac{\xi}{\sigma_D^2}\exp\Big\{\int\limits_0^{\hat y}\frac{h(q)-1}{1-q}dq\Big\} = \frac{h(\hat y)\big(A\hat y h(\hat y)+1+\gamma-(2\gamma+1)\hat y\big)-\gamma(1+\gamma)(1-\hat y)}{\hat y h(\hat y)^2}.$$

Furthermore, inserting this expression into the second derivative of the ODE (2.33) yields

$$\begin{aligned}
0 &\ge \hat y^2(1-\hat y)^2 h''(\hat y)\\
&= -\gamma(1+\gamma)(1-\hat y)^2 + \Big(1+\gamma+\hat y\big(\gamma(3+\gamma)\hat y+\hat y-(1+\gamma)(2+\gamma)\big)\Big)h(\hat y)\\
&+ \hat y\big(1+\gamma-\hat y(1+A+2\gamma)\big)h(\hat y)^2 + A\hat y^2 h(\hat y)^3.
\end{aligned}$$

First, because $A > 1$ and $h(\hat{y}) \geq 1$, we have

$$A\hat{y}^2\big(h(\hat{y})^3 - h(\hat{y})^2\big) \geq \hat{y}^2\big(h(\hat{y})^3 - h(\hat{y})^2\big).$$

Therefore,

$$\begin{aligned}
0 &\geq -\gamma(1+\gamma)(1-\hat{y})^2 + \Big(1+\gamma+\hat{y}\big(\gamma(3+\gamma)\hat{y}+\hat{y}-(1+\gamma)(2+\gamma)\big)\Big)h(\hat{y}) \\
&\quad + \hat{y}\big(\gamma+1-y(2\gamma+2)\big)h(\hat{y})^2 + \hat{y}^2h(\hat{y})^3 \\
&= \big(h(\hat{y})-\gamma\big)\Big(1+\gamma+(1+\gamma)\big(h(\hat{y})-2\big)\hat{y}+\big(h(\hat{y})-1\big)\big(h(\hat{y})-1-\gamma\big)\hat{y}^2\Big).
\end{aligned}$$

Second, dividing by $\big(h(\hat{y})-\gamma\big) > 0$ implies the contradiction

$$\begin{aligned}
0 &\geq 1+\gamma+(1+\gamma)\big(h(\hat{y})-2\big)\hat{y}+\big(h(\hat{y})-1\big)\big(h(\hat{y})-1-\gamma\big)\hat{y}^2 \\
&= (1+\gamma)(1-\hat{y})+(1+\gamma)\big(h(\hat{y})-1\big)\hat{y}+\big(h(\hat{y})-1\big)^2\hat{y}^2-\gamma\big(h(\hat{y})-1\big)\hat{y}^2 \\
&> 0.
\end{aligned} \tag{2.62}$$

In (2.62), the strict inequality follows from

$$(1+\gamma)(1-\hat{y}) > 0, \quad \big(h(\hat{y})-1\big)^2\hat{y}^2 \geq 0,$$

and the sum of the remaining two terms in (2.62) satisfies the bound

$$\big(h(\hat{y})-1\big)\big((1+\gamma)\hat{y}-\gamma\hat{y}^2\big) = \big(h(\hat{y})-1\big)\big(\hat{y}+\gamma\hat{y}(1-\hat{y})\big) \geq 0.$$

3. We argue by contradiction and assume $\xi_0 \in \Xi$.
Step 1/4: Given $\epsilon \in (0, 1-\gamma)$, there exists $y_0 \in (0,1)$ such that

$$\forall y \in (y_0, 1): \quad h_{\xi_0}(y) < \gamma+\epsilon < 1.$$

The previous step ensures $h_{\xi_0} < 1$, and so the non-negative function from (2.55), i.e.,

$$F_{\xi_0}(y) := \frac{\xi_0}{\sigma_D^2}\exp\Big\{\int_0^y \frac{h_{\xi_0}(q)-1}{1-q}\Big\}, \quad y \in [0,1),$$

is strictly decreasing. Therefore, the limit $F_{\xi_0}(1) := \lim_{y\uparrow 1} F_{\xi_0}(y)$ exists and is zero because

$$\begin{aligned}
F_{\xi_0}(1) &= \lim_{y\uparrow 1} F_{\xi_0}(y_0)\exp\Big\{\int_{y_0}^y \frac{h_{\xi_0}(q)-1}{1-q}dq\Big\} \\
&\leq \lim_{y\uparrow 1} F_{\xi_0}(y_0)\exp\Big\{\int_{y_0}^y \frac{\gamma+\epsilon-1}{1-q}dq\Big\}
\end{aligned}$$

$$= \lim_{y\uparrow 1} F_{\xi_0}(y_0)\left(\frac{1-y_0}{1-y}\right)^{\gamma+\epsilon-1}$$
$$= 0.$$

Because $A > 1$, by increasing y_0 if necessary, we can therefore also assume y_0 satisfies

$$\forall y \in (y_0, 1): \quad F_{\xi_0}(y) < \frac{A-1}{2}. \tag{2.63}$$

Step 2/4: For $\xi \in (\xi_0, \xi_0 + 1)$, Theorem 2.5 gives a local solution $h_\xi(y)$ for $y \in [0, y_\xi]$ where y_ξ is defined in (2.54). Lemma 2.8 with $y_0 \in (0, 1)$ from the previous step and $\bar{\xi} := \xi_0 + 1$ gives a constant $M_1 > 0$ such that

$$|h_\xi(y) - h_{\xi_0}(y)| \leq M_1|\xi - \xi_0|, \quad y \leq y_\xi \wedge y_0, \quad \xi \in (\xi_0, \xi_0 + 1).$$

Because $h_{\xi_0}(y) < 1$ for all $y \in [0, 1]$, we can consider $\xi \in (\xi_0, \xi_0 + 1)$ such that

$$M_1|\xi - \xi_0| + \sup_{y\in[0,1]} h_{\xi_0}(y) < 1. \tag{2.64}$$

For such ξ and $y \in [0, y_\xi \wedge y_0]$, we have

$$\begin{aligned} h_\xi(y) &\leq |h_\xi(y) - h_{\xi_0}(y)| + h_{\xi_0}(y) \\ &\leq M_1|\xi - \xi_0| + h_{\xi_0}(y) \\ &< 1. \end{aligned}$$

Therefore, whenever $\xi \in (\xi_0, \xi_0 + 1)$ satisfies (2.64), we have $y_\xi > y_0$.

Step 3/4: In addition to (2.64), this step creates an additional restriction on $\xi \in (\xi_0, \xi_0 + 1)$. First, let F_ξ be defined as in (2.55). Lemma 2.8 with $y_0 \in (0, 1)$ from the previous step and $\bar{\xi} := \xi_0 + 1$ gives a constant $M_2 > 0$ such that

$$|F_\xi(y) - F_{\xi_0}(y)| \leq M_2|\xi - \xi_0|, \quad y \leq y_\xi \wedge y_0, \quad \xi \in (\xi_0, \xi_0 + 1).$$

For ξ satisfying (2.64), the previous step ensures $y_\xi > y_0$ and so the bound in (2.63) gives

$$\begin{aligned} F_\xi(y_0) &\leq |F_\xi(y_0) - F_{\xi_0}(y_0)| + F_{\xi_0}(y_0) \\ &\leq M_2|\xi - \xi_0| + \frac{A-1}{2}. \end{aligned}$$

This shows that we can lower $\xi \in (\xi_0, \xi_0 + 1)$ such that, in addition to (2.64), we also have

$$F_\xi(y_0) < A - 1. \tag{2.65}$$

Step 4/4: Finally, this step creates a contradiction using comparison of ODE solutions. Let $\xi \in (\xi_0, \xi_0 + 1)$ and $y_0 \in (0, 1)$ be as in the previous step. Using (2.65), the function in (2.55) satisfies

$$\forall y \in [y_0, y_\xi]: \quad F_\xi(y) = F_\xi(y_0)\exp\left\{\int\limits_{y_0}^{y} \frac{h_\xi(q)-1}{1-q}\right\} \le F_\xi(y_0) < A-1.$$

This bound gives the ODE inequality

$$\begin{aligned} h'_\xi(y) &= a_0(y) + h_\xi(y)\frac{a_1(y)+\big(F_\xi(y)-A\big)h_\xi(y)}{1-y} \\ &\le a_0(y) + h_\xi(y)\frac{a_1(y)-h_\xi(y)}{1-y}, \quad y \in (y_0, y_\xi). \end{aligned}$$

Using $a_3 := -1$ in Theorem 2.4 yields an ODE solution f of

$$\begin{cases} f'(y) = a_0(y) + f(y)\frac{a_1(y)-f(y)}{1-y}, & y \in (y_0, 1), \\ f(y_0) = h_\xi(y_0). \end{cases}$$

The standard comparison principle gives $h_\xi(y) \le f(y)$ for $y \in [y_0, y_\xi]$. Theorem 2.4 and Remark 2.2 give $f(1) = \gamma$ and $f(y) < 1$ for $y \in [y_0, 1]$ and so we have $y_\xi = 1$. All in all, we have the contradiction $\xi > \xi_0$ and $\xi \in \Xi$ because

$$h_\xi(1) \le f(1) = \gamma.$$

4. Let $\xi_n \uparrow \xi_0 := \sup \Xi$ and let h_{ξ_n} denote the corresponding solution of the ODE (2.33). Because of h_ξ's monotonicity in ξ, we can define

$$h_{\xi_0}(y) := \lim_{n\to\infty} h_{\xi_n}(y), \quad y \in [0, 1),$$

which satisfies $0 \le h_{\xi_0}(y) \le 1$ for $y \in [0, 1)$. By taking limits in the ODE (2.33), we see that the ODE

$$h'_{\xi_0}(y) = a_0(y) + h_{\xi_0}(y)\frac{a_1(y)+\big(F_{\xi_0}(y)-A\big)h_{\xi_0}(y)}{1-y}, \quad y \in (0, 1),$$

holds. We can argue as in the first part of Step 2 in the proof of Theorem 2.6 to show that $h_{\xi_0}(y)$ cannot oscillate as $y \uparrow 1$ and so $h_{\xi_0}(1) := \lim_{y\uparrow 1} h_{\xi_0}(y)$ exists. Because $\xi_0 \notin \Xi$, it must be that $h_{\xi_0}(1) > \gamma$.

To see $h_{\xi_0}(1) = 1$, the bound $h_{\xi_0} \le 1$ allows us to argue as in the second part of Step 2 in the proof of Theorem 2.6. Here, it is shown that $h_{\xi_0}(1) = \ell_1$ where ℓ_1 is defined in (2.45). The argument following (2.46) shows that if $\ell_1 < 1$ holds, we have the contradiction

$$h_{\xi_0}(1) = c_0 < \gamma < h_{\xi_0}(1),$$

where c_0 is defined in (2.41). Because $h_{\xi_0}(1) \le 1$, the only alternative is $\ell_1 = 1$.

5. This proof is similar to Step 2 in the proof of Theorem 2.6. Because $0 \le h_{\xi_0} \le 1$, the limit $\lim_{y\uparrow 1} \exp\left\{\int_0^y \frac{h_{\xi_0}(q)-1}{1-q} dq\right\}$ exists in $[0, 1]$, which allows us to use L'Hospital's rule below. The integral representation (2.32) of h_{ξ_0} yields

$$\frac{1}{y^{1+\gamma}}\int_0^y x^{1+\gamma}(1-x)^\gamma\left(a_0(x)+\frac{a_2(h_{\xi_0},x)}{1-x}h_\xi(x)^2\right)dx = h_{\xi_0}(y)(1-y)^\gamma.$$

Because $h_{\xi_0}(y)$ is continuous at $y=1$ with a finite limit, the right-hand-side converges to 0 as $y\uparrow 1$. L'Hospital's rule gives us

$$\begin{aligned}
h_{\xi_0}(1) &= \lim_{y\uparrow 1}\frac{1}{y^{1+\gamma}(1-y)^\gamma}\int_0^y x^{1+\gamma}(1-x)^\gamma\left(a_0(x)+\frac{a_2(h_{\xi_0},x)}{1-x}h_{\xi_0}(x)^2\right)dx\\
&= \lim_{y\uparrow 1}\frac{y^{1+\gamma}(1-y)^\gamma\left(a_0(y)+\frac{a_2(h_{\xi_0},y)}{1-y}h_{\xi_0}(y)^2\right)}{(1+\gamma)y^\gamma(1-y)^\gamma-\gamma y^{1+\gamma}(1-y)^{\gamma-1}}\\
&= \lim_{y\uparrow 1}\frac{y\big((1-y)a_0(y)+a_2(h_{\xi_0},y)h_{\xi_0}(y)^2\big)}{(1+\gamma)(1-y)-\gamma y}\\
&= \frac{1}{\gamma}\left(A-\frac{\xi_0}{\sigma_D^2}\lim_{y\uparrow 1}\exp\left\{\int_0^y\frac{h_{\xi_0}(q)-1}{1-q}dq\right\}\right)h_{\xi_0}(1)^2.
\end{aligned}$$

Because $h_{\xi_0}(1)=1$, this implies the limit in (2.61).

Because $A>\gamma$, the limit in (2.61) is strictly positive. The upper bound $h_{\xi_0}\le 1$ gives

$$\exp\left\{\int_0^1\frac{h_{\xi_0}(x)-1}{1-x}\,dx\right\}\le 1. \tag{2.66}$$

Because $h_{\xi_0}(0)=\gamma\in(0,1)$ and $h_{\xi_0}(y)$ is continuous at $y=0$, the inequality in (2.66) is strict.
6. Because $h_{\xi_0}\le 1$, the limit in (2.61) gives the monotonicity property

$$\begin{aligned}
\exp\left\{\int_0^y\frac{h_{\xi_0}(q)-1}{1-q}dq\right\} &\ge \exp\left\{\int_0^1\frac{h_{\xi_0}(q)-1}{1-q}dq\right\}\\
&= \frac{\sigma_D^2(A-\gamma)}{\xi_0},\quad y\in[0,1].
\end{aligned}\tag{2.67}$$

Therefore,

$$a_2(h_{\xi_0},y)=\frac{\xi_0}{\sigma_D^2}\exp\left\{\int_0^y\frac{h_{\xi_0}(q)-1}{1-q}dq\right\}-A\ge-\gamma.$$

Using $a_3:=-\gamma$ in Theorem 2.4 produces an ODE solution f of

$$\begin{cases} f'(y)=a_0(y)+f(y)\frac{a_1(y)-\gamma f(y)}{1-y}, & y\in(0,1),\\ f(0)=\gamma, \end{cases}$$

Because $1+a_3>0$, Theorem 2.4 also gives f with $f\geq\gamma$. The comparison result in Theorem 2.3 gives the lower bound $h_{\xi_0}\geq f\geq\gamma$.
7. We define

$$g(y):=\frac{1-h_{\xi_0}(y)}{1-y},\qquad y\in(0,1).$$

Step 1/4: To get an a priori estimate, this step assumes $h'_{\xi_0}(y)$ is continuous at $y=1$. The ODE in (2.33) and (2.61) imply the representation

$$\begin{aligned}&h'_{\xi_0}(y)\\&=\frac{1+\gamma}{y}\big(\gamma-h_{\xi_0}(y)\big)+h_{\xi_0}(y)^2(A-\gamma)\frac{\exp\Big\{\int_y^1 g(q)dq\Big\}-1}{1-y}+\gamma h_{\xi_0}(y)g(y).\end{aligned}\tag{2.68}$$

Because $h'_{\xi_0}(y)$ is continuous at $y=1$, L'Hospital's rule gives

$$\begin{aligned}g(1)&=(1+\gamma)(\gamma-1)+(A-\gamma)\lim_{y\uparrow 1}\frac{\exp\Big\{\int_y^1 g(q)dq\Big\}-1}{1-y}+\gamma g(1)\\&=\gamma^2-1+(A-\gamma)g(1)+\gamma g(1).\end{aligned}$$

We can solve to see $g(1)=\frac{1-\gamma^2}{A-1}>0$. The next steps prove that $h'_{\xi_0}(y)$ is indeed continuous at $y=1$.

Step 2/4: This step rules out that $g(y)$ can monotonically explode to ∞ as $y\uparrow 1$. We argue by contradiction. The Mean-Value Theorem and $g'(y)>0$ for y near 1 yields $\theta(y)\in(y,1)$ such that

$$\begin{aligned}\exp\Big\{\int_y^1 g(q)dq\Big\}-1&=\exp\Big\{\int_y^1 g(q)dq\Big\}g\big(\theta(y)\big)(1-y)\\&\geq\exp\Big\{\int_y^1 g(q)dq\Big\}g(y)(1-y),\end{aligned}$$

for y near 1. The ODE in (2.68) gives

$$\begin{aligned}&h'_{\xi_0}(y)\\&\geq\frac{1+\gamma}{y}\big(\gamma-h_{\xi_0}(y)\big)+\Big(h_{\xi_0}(y)^2(A-\gamma)\exp\Big\{\int_y^1 g(q)dq\Big\}+\gamma h_{\xi_0}(y)\Big)g(y),\end{aligned}\tag{2.69}$$

for y near 1. By differentiation, we see

$$g'(y) = \frac{1}{1-y}\big(g(y) - h'_{\xi_0}(y)\big) \tag{2.70}$$

and so $g'(y) > 0$ implies that $g(y) \geq h'_{\xi_0}(y)$ for y near 1. The inequality (2.69) gives

$$\begin{aligned} &g(y) \\ &\geq \frac{1+\gamma}{y}\big(\gamma - h_{\xi_0}(y)\big) + \Big(h_{\xi_0}(y)^2(A-\gamma)\exp\Big\{\int_y^1 g(q)dq\Big\} + \gamma h_{\xi_0}(y)\Big)g(y), \end{aligned} \tag{2.71}$$

for y near 1. Because $g \geq 0$, we have $\exp\Big\{\int_y^1 g(q)dq\Big\} \geq 1$ and so the inequality (2.71) gives

$$\frac{1+\gamma}{y}\big(h_{\xi_0}(y) - \gamma\big) \geq \big(h_{\xi_0}(y)^2(A-\gamma) + \gamma h_{\xi_0}(y) - 1\big)g(y),$$

for y near 1. By taking limits as $y \uparrow 1$ and using $A > 1$ give the contradiction

$$(1+\gamma)(1-\gamma) = \lim_{y\uparrow 1}\frac{1+\gamma}{y}\Big(h_{\xi_0}(y) - \gamma\Big) \geq (A-1)\lim_{y\uparrow 1} g(y) = \infty.$$

Step 3/4: This step considers oscillations. The ODE (2.68) and (2.70) give

$$\begin{aligned} -(1-y)g'(y) &= \frac{1+\gamma}{y}\big(\gamma - h_{\xi_0}(y)\big) + h_{\xi_0}(y)^2(A-\gamma)\frac{\exp\Big\{\int_y^1 g(q)dq\Big\} - 1}{1-y} \\ &\quad + \big(\gamma h_{\xi_0}(y) - 1\big)g(y), \quad y \in (0,1). \end{aligned} \tag{2.72}$$

Differentiating (2.72) at a point $y \in (0,1)$ with $g'(y) = 0$ gives

$$\begin{aligned} &(1-y)^2 y g''(y) \\ &= (1+\gamma)\big(\gamma - h_{\xi_0}(y)\big) - h_{\xi_0}(y)^2(A-\gamma)\left(\exp\Big\{\int_y^1 g(q)dq\Big\} - 1\right) \\ &+ \left(h_{\xi_0}(y)\left(y(A-\gamma)\exp\Big\{\int_y^1 g(q)dq\Big\}\big(h_{\xi_0}(y) - 2\big) + 2Ay - \gamma\right)\right. \\ &\left. + \gamma - (\gamma+3)y + 2 - \gamma(1-y)yg(y)\right)g(y), \quad y \in (0,1). \end{aligned} \tag{2.73}$$

First, assume that $(y_n)_{n\in\mathbb{N}}$ is a sequence of local maxima for g with $y_n \uparrow 1$ such that

$$\forall n \in \mathbb{N}: \quad g'(y_n) = 0, \quad g''(y_n) \leq 0.$$

We can extract a subsequence $(y_{n_k})_{k\in\mathbb{N}} \subseteq (y_n)_{n\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} g(y_{n_k}) = \limsup_{n\to\infty} g(y_n) \in [0,\infty].$$

Because $h_{\xi_0}(1) = 1$, we have

$$\lim_{k\to\infty} (1-y_{n_k})y_{n_k} g(y_{n_k}) = \lim_{k\to\infty} y_{n_k}\big(1 - h_{\xi_0}(y_{n_k})\big) = 0,$$

and so replacing y with y_{n_k} in (2.73) and taking limits give

$$0 \geq \lim_{k\to\infty} (1-y_{n_k})^2 y_{n_k} g''(y_{n_k}) = \gamma^2 - 1 + (A-1)\lim_{k\to\infty} g(y_{n_k}).$$

This gives the upper bound for g's local maxima

$$\limsup_{n\to\infty} g(y_n) \leq \frac{1-\gamma^2}{A-1}. \tag{2.74}$$

Second, assume that $(y_n)_{n\in\mathbb{N}}$ is a sequence of local minima for g with $y_n \uparrow 1$ such that

$$\forall n \in \mathbb{N}: \quad g'(y_n) = 0, \quad g''(y_n) \geq 0.$$

Then, similarly to (2.74), we have the lower bound for g's local minima

$$\liminf_{n\to\infty} g(y_n) \geq \frac{1-\gamma^2}{A-1}. \tag{2.75}$$

Finally, combining the two bounds (2.74) and (2.75) shows that should $g(y)$ oscillate as $y \uparrow 1$, the function g will still have limit $\lim_{y\uparrow 1} g(y) = g(1) = \frac{1-\gamma^2}{A-1}$ as in Step 1.

Step 4/4: This step shows that $h_{\xi_0}(y)$ is continuously differentiable at $y = 1$. Step 2 and Step 3 ensure that $h'_{\xi_0}(1)$ exists and that $g(y)$ is continuous at $y = 1$. It is g's continuity at $y = 1$ which allows us to use L'Hospital's rule to see

$$\begin{aligned}
&\lim_{y\uparrow 1} \frac{\gamma + \big((A-\gamma)\exp\big\{\int_y^1 g(q)dq\big\} - A\big)h_{\xi_0}(y)}{1-y} \\
&= \lim_{y\uparrow 1} \frac{(A-\gamma)\big(\exp\{\int_y^1 g(q)dq\} - 1\big)h_{\xi_0}(y) + \gamma\big(1 - h_{\xi_0}(y)\big)}{1-y} \\
&= (A-\gamma)\lim_{y\uparrow 1} \frac{\exp\{\int_y^1 g(q)dq\} - 1}{1-y} + \gamma \lim_{y\uparrow 1} \frac{1 - h_{\xi_0}(y)}{1-y} \\
&= (A-\gamma)g(1) + \gamma g(1) \\
&= Ag(1).
\end{aligned}$$

Then, the ODE (2.68) and $h_{\xi_0}(1) = 1$ give

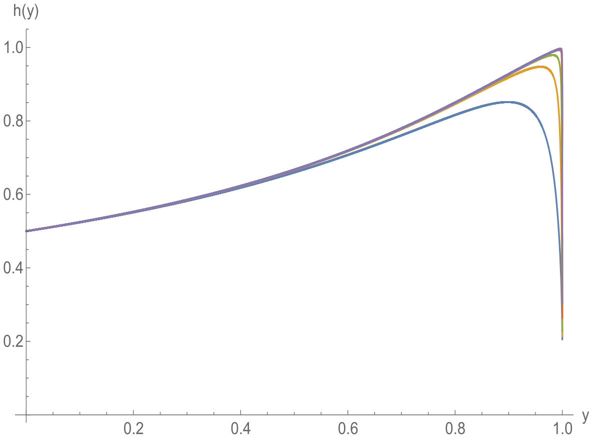

Fig. 1. ODE solutions h_{ξ_n} for $\xi_n \uparrow \sup \Xi$. The parameters are $\gamma := 0.5, \sigma_D := 0.2, A := 2$, and $\xi_n \in \{0.15, 0.152, 0.1522, 0.15223, 0.152232\}$.

$$\lim_{y\uparrow 1} h'_{\xi_0}(y) = (1+\gamma)(\gamma-1) + Ag(1) = \gamma^2 - 1 + Ag(1). \tag{2.76}$$

Another application of L'Hospital's rule gives

$$g(1) = \lim_{y\uparrow 1} \frac{1-h_{\xi_0}(y)}{1-y} = \lim_{y\uparrow 1} h'_{\xi_0}(y) = \gamma^2 - 1 + Ag(1). \tag{2.77}$$

By solving (2.77), we get $g(1) = \frac{1-\gamma^2}{A-1} > 0$ and so (2.76) gives $\lim_{y\uparrow 1} h'_{\xi_0}(y) = \frac{1-\gamma^2}{A-1}$. $\square$

Fig. 1 presents a numerical illustration.

3. Application to incomplete Radner equilibrium theory

On a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, we let $(B_t)_{t\geq 0}$ be a Brownian motion generating the raw filtration $(\mathcal{F}_t^0)_{t\geq 0}$ with $\mathcal{F}_t^0 := \sigma(B_s)_{s\in[0,t]}$ and assume $\mathbb{F} = \vee_{t\geq 0}\mathcal{F}_t^0$. We define the augmented filtration as $\mathcal{F}_t := \mathcal{F}_t^0 \vee \mathbb{F}$'s $\mathbb{P}$-nullsets for $t \geq 0$. For $t \geq 0$, we write $\mathbb{E}_t[\cdot]$ as short-hand notation for the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$.

Throughout this section, let h be the solution in Theorem 1.1.

3.1. Autonomous state process Y

We define the drift function μ_Y and volatility function σ_Y as

$$\begin{aligned} \mu_Y(y) &:= \frac{\sigma_D^2(1-y)\big(1+\gamma+2\gamma y h(y) - 2y(1+\gamma)\big)}{2yh(y)^2}, \quad y \in (0,1], \\ \sigma_Y(y) &:= \sigma_D \frac{1-y}{h(y)}, \quad y \in [0,1]. \end{aligned} \tag{3.1}$$

The next subsections use an autonomous state process $Y = (Y_t)_{t\geq 0}$ valued in $[0,1]$ with dynamics

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dB_t, \quad Y_0 \in (0,1). \tag{3.2}$$

It is $\mu_Y(y)$'s singularity at $y=0$ and the singularities of both $\mu_Y(y)$ and $\sigma_Y(y)$ at $y=1$ that are problematic in the following analysis. The boundary behavior of the ratio

$$\frac{\mu_Y(y)}{\sigma_Y(y)^2} = \frac{1+\gamma+2\gamma y h(y) - 2y(1+\gamma)}{2y(1-y)}, \quad y \in (0,1), \tag{3.3}$$

as $y \downarrow 0$ and $y \uparrow 1$ classifies Y's boundary behavior. We have the limits

$$\begin{aligned} \lim_{y\downarrow 0} y\frac{\mu_Y(y)}{\sigma_Y(y)^2} &= \frac{1+\gamma}{2} > 0, \\ \lim_{y\uparrow 1}(1-y)\frac{\mu_Y(y)}{\sigma_Y(y)^2} &= -\frac{1}{2}(1-\gamma) < 0. \end{aligned} \tag{3.4}$$

Example 5.4 in [5] illustrates that the singularities $\frac{1}{y}$ as $y \downarrow 0$ and $\frac{1}{1-y}$ as $y \uparrow 1$ in (3.4) require a careful error analysis. In particular, only identifying the limits in (3.4) is insufficient to classify the boundary points $y \in \{0,1\}$ for the SDE (3.2). The next subsections establish the following result.

Theorem 3.1. *Let $\gamma \in (0,1)$, $\sigma_D^2 > 0$, and $A > 1$. For any initial value $Y_0 \in [0,1)$, there exists a strong solution Y, pathwise unique, of the SDE (3.2) for the coefficients defined in (3.1).*

3.1.1. Left-boundary point for Y

We start with the left-boundary point $y=0$.

Lemma 3.2. *Let $\gamma \in (0,1)$, $\sigma_D^2 > 0$, and $A > 1$. The functions in (3.1) satisfy*

$$y\frac{\mu_Y(y)}{\sigma_Y(y)^2} = b_0 + b_1(y)y, \quad b_0 := \frac{1+\gamma}{2}, \quad b_1(y) := \frac{2\gamma h(y) - 1 - \gamma}{2(1-y)}, \quad y \in (0,1). \tag{3.5}$$

The boundary point $y=0$ is inaccessible and entrance. Furthermore, the SDE (3.2) has a non-negative weak solution, unique in law, starting from $Y_0 = 0$.

Proof. Because b_1 is uniformly lower bounded on any interval $[0,a]$ for $a \in (0,1)$, we can define a constant $M \in \mathbb{R}$ as

$$M := \inf_{y\in[0,a]} b_1(y).$$

As in Eq. (2.12) in [5], we define the function

$$\rho(y) := \exp\left(2\int_y^a \frac{\mu_Y(z)}{\sigma_Y(z)^2}dz\right), \quad y \in (0,a]. \tag{3.6}$$

The decomposition in (3.5) gives

$$
\begin{aligned}
\rho(y) &= \exp\left(2\int_y^a \frac{\mu_Y(z)}{\sigma_Y(z)^2}dz\right)\\
&= \exp\left(2\int_y^a \Big(\frac{b_0}{z}+b_1(z)\Big)dz\right)\\
&\geq \exp\left(2\int_y^a \Big(\frac{b_0}{z}+M\Big)dz\right) \qquad (3.7)\\
&= \exp\left(2b_0\Big(\log(a)-\log(y)\Big)\right)\exp\big(2M(a-y)\big)\\
&= \exp\left(\log\Big(\big(\frac{a}{y}\big)^{2b_0}\Big)\right)\exp\big(2M(a-y)\big)\\
&= \Big(\frac{a}{y}\Big)^{2b_0}\exp\big(2M(a-y)\big).
\end{aligned}
$$

Following Eq. (2.13) in [5], we define the scale function

$$
s(y) := -\int_y^a \rho(z)dz, \qquad y\in(0,a]. \tag{3.8}
$$

Because $b_0 \geq \frac{1}{2}$, we get $s(0) := \lim_{y\downarrow 0} s(y) = -\infty$. Therefore, Theorem 8A in [9] ensures that $y=0$ is inaccessible.

Next, we first calculate the derivative

$$
\begin{aligned}
\rho'(y) &= \frac{\partial}{\partial y}\exp\left(2\int_y^a \Big(\frac{b_0}{z}+b_1(z)\Big)dz\right)\\
&= -2\rho(y)\Big(\frac{b_0}{y}+b_1(y)\Big).
\end{aligned}
$$

The lower bound (3.7) gives

$$
y\rho(y) \geq y\Big(\frac{a}{y}\Big)^{2b_0}\exp\big(2M(a-y)\big). \tag{3.9}
$$

Because $b_0 > \frac{1}{2}$, the lower bound (3.9) gives $\lim_{y\downarrow 0} y\rho(y) = \infty$.

Second, two applications of L'Hospital's rule yield

$$
\lim_{y\downarrow 0}\frac{|s(y)|}{\rho(y)} = \lim_{y\downarrow 0}\frac{1}{\rho(y)}\int_y^a \rho(z)dz
$$

$$
\begin{aligned}
&= -\lim_{y\downarrow 0} \frac{1}{\rho'(y)}\rho(y) \\
&= \lim_{y\downarrow 0} \frac{y}{2\big(b_0 + b_1(y)y\big)} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\lim_{y\downarrow 0} \frac{|s(y)|}{y\rho(y)} &= \lim_{y\downarrow 0} \frac{1}{y\rho(y)} \int_y^a \rho(z)dz \\
&= -\lim_{y\downarrow 0} \frac{1}{\rho(y) + y\rho'(y)}\rho(y) \\
&= -\lim_{y\downarrow 0} \frac{1}{1 - 2y\Big(\frac{b_0}{y} + b_1(y)\Big)} \\
&= \frac{1}{2b_0 - 1} \in (0, \infty),
\end{aligned}
$$

where the last line follows from $b_0 > \frac{1}{2}$. Because b_1 is bounded and $h \geq \gamma$, we have $\sigma_Y(y) \geq \frac{\sigma_D(1-y)}{\gamma}$ and

$$
\int_0^a \frac{1 + |\mu_Y(y)|}{\rho(y)\sigma_Y(y)^2}|s(y)|dy \leq \int_0^a \left(\frac{1}{\rho(y)\sigma_Y(y)^2} + \frac{\frac{b_0}{y} + |b_1(y)|}{\rho(y)}\right)|s(y)|dy < \infty.
$$

Theorem 8C in [9] gives us that $y = 0$ is not natural (equivalently, $y = 0$ is an entrance point) because

$$
\int_0^a \frac{1}{\rho(y)\sigma_Y(y)^2}|s(y)|dy < \infty.
$$

Finally, Theorem 2.16 in [5] shows that $Y_0 = 0$ produces a nonnegative weak solution, unique in law, of the SDE (3.2). □

3.1.2. Right-boundary point for Y

Lemma 3.3. *Let $\gamma \in (0, 1)$, $\sigma_D^2 > 0$, and $A > 1$. Then, the boundary point $y = 1$ is inaccessible, natural, and attracting.*

Proof. We let $a \in (0, 1)$ be arbitrary and consider the function[2]

$$
\rho(x) := \exp\left(-2\int_a^x \frac{\mu_Y(y)}{\sigma_Y(y)^2}dy\right)
$$

[2] The definition of ρ in (3.10) coincides with the definition of ρ in (3.6).

$$
\begin{aligned}
&= \exp\left(\int_a^x \left(\tfrac{(2y-1)(1+\gamma)-2\gamma y h(y)}{(1-y)y}\right) dy\right) \\
&= \exp\left(\int_a^x \left(\tfrac{(2y-1)(1+\gamma)-2\gamma y-2\gamma y(h(y)-1)}{(1-y)y}\right) dy\right) \qquad (3.10) \\
&= \exp\left(\int_a^x \left(\tfrac{2-\frac{1}{y}(1+\gamma)-2\gamma(h(y)-1)}{1-y}\right) dy\right), \quad x \in [a,1).
\end{aligned}
$$

Taylor's formula for $\frac{1}{y}$ centered at $y = 1$ produces $\frac{1}{y} = 1 + (1 - y) + o(1 - y)$. Therefore,

$$
\int_a^x \frac{1}{(1-y)y} dy = \int_a^x \left(\frac{1}{1-y} + 1 + \frac{o(1-y)}{1-y}\right) dy, \quad x \in [a,1].
$$

This allows us to rewrite (3.10) as

$$
\begin{aligned}
\rho(x) &= \exp\left(\int_a^x \left(\tfrac{1-\gamma-2\gamma(h(y)-1)}{1-y}\right) dy - (1+\gamma)\Big(x - a + \int_a^x \tfrac{o(1-y)}{1-y} dy\Big)\right) \\
&= \exp\left(\int_a^x \tfrac{1-\gamma}{1-y} dy + 2\gamma \int_a^x \tfrac{1-h(y)}{1-y} dy - (1+\gamma)\Big(x - a + \int_a^x \tfrac{o(1-y)}{1-y} dy\Big)\right) \qquad (3.11) \\
&= \left(\frac{1-a}{1-x}\right)^{1-\gamma} \exp\big(C(x)\big),
\end{aligned}
$$

where we have defined the function

$$
C(x) := 2\gamma \int_a^x \tfrac{1-h(y)}{1-y} dy - (1+\gamma)\Big(x - a + \int_a^x \tfrac{o(1-y)}{1-y} dy\Big), \quad x \in [a,1].
$$

Lemma 2.10(5) ensures that $C(x)$ is a continuous function for $x \in [a, 1]$ and so

$$
c_1 := \min_{x\in[a,1]} C(x) \in \mathbb{R}, \quad c_2 := \max_{x\in[a,1]} C(x) \in \mathbb{R}. \qquad (3.12)
$$

Because $\gamma \in (0, 1)$, we see from (3.11) that $\rho(x)$ is integrable at $x = 1$ and so the function[3]

$$
s(y) := \int_a^y \rho(x) dx, \quad y \in [a,1), \qquad (3.13)
$$

[3] The definition of s in (3.13) coincides with the definition of s in (3.8).

satisfies $s(1) := \lim_{y\uparrow 1} s(y) < \infty$.

Because Lemma 2.10(6) ensures $h \geq \gamma$, we have

$$\sigma_Y(y)^2 = \frac{\sigma_D^2(1-y)^2}{h(y)^2} \leq \frac{\sigma_D^2(1-y)^2}{\gamma^2}, \quad y \in [0,1].$$

The Mean-Value Theorem gives $\theta(y) \in [y, 1]$ such that

$$\begin{aligned} s(1) - s(y) &= s'\big(\theta(y)\big)(1-y) \\ &= \rho\big(\theta(y)\big)(1-y) \\ &= \Big(\frac{1-a}{1-\theta(y)}\Big)^{1-\gamma} e^{C(\theta(y))}(1-y) \\ &\geq \Big(\frac{1-a}{1-\theta(y)}\Big)^{1-\gamma} e^{c_1}(1-y). \end{aligned}$$

For some constant $c_3 > 0$, by (3.11), (3.12), and $h(1) = 1$, we have

$$\begin{aligned} \frac{s(1)-s(y)}{\rho(y)\sigma_Y(y)^2} &\geq \frac{c_3(1-y)}{\big(1-\theta(y)\big)^{1-\gamma}(1-y)^{1+\gamma}} \\ &\geq \frac{c_3(1-y)}{(1-y)^{1-\gamma}(1-y)^{1+\gamma}} \\ &= \frac{c_3}{1-y}, \end{aligned} \tag{3.14}$$

which is not integrable for y near 1. Therefore, Theorem 8A in [9] gives that the boundary point $y = 1$ is inaccessible.

From (3.11), we $\rho(x) \geq c_4(1-x)^{\gamma-1}$ for some constant $c_4 > 0$. This gives the lower bound

$$\frac{s(y)}{\rho(y)\sigma_Y(y)^2} \geq c_5 \frac{\int_a^y (1-x)^{\gamma-1}dx}{(1-y)^{1+\gamma}} = c_5 \frac{(1-a)^\gamma - (1-y)^\gamma}{\gamma(1-y)^{1+\gamma}}$$

for some constant c_5. Because this lower bound is not integrable for y near 1, Theorem 8C in [9] gives that $y = 1$ is natural (i.e., $y = 1$ is not entrance).

Finally, Theorem 8D in [9] gives that $y = 1$ is attracting because $s(1) < \infty$ and the lower bound

$$\frac{1}{\rho(y)\sigma_Y(y)^2} \geq c_6 \frac{1}{(1-y)^{1+\gamma}}$$

is not integrable for y near 1, where c_6 is another constant. □

3.1.3. Proof of strong SDE existence of Y

The proof of the next result uses a variation of Le Gall's uniqueness argument based on local times (see, e.g., Exercise 3.7.12 in [12]).

Proof of Theorem 3.1. For $Y_0 \in (0,1)$, the Engelbert and Schmidt conditions (i.e., non-degeneracy and local integrability) hold and ensure that a weak solution exists, unique in law, see, e.g., Theorem 5.5 in [12]. For $Y_0 = 0$, there is also a weak solution, unique in law, by Lemma 3.2.

To upgrade these weak solutions to strong solutions it suffices to prove pathwise uniqueness locally. To this end, for $n \in \mathbb{N}$ such that $Y_0 \in [0, 1-\frac{1}{n}]$, we define the stopping times $\tau_n := \inf\{t \geq 0 : Y_t = 1-\frac{1}{n}\}$ and will prove that strong uniqueness holds for $t \in [0, \tau_n]$. Because $h, h' \in \mathcal{C}([0,1))$ with $h \geq \gamma$, the formula

$$\sigma_Y'(y) = -\sigma_D \frac{(1-y)h'(y) + h(y)}{h(y)^2}, \quad y \in [0, 1-\tfrac{1}{n}],$$

shows that $\sigma_Y(y)$ is Lipschitz for $y \in [0, 1-\frac{1}{n}]$. Pathwise uniqueness follows from Proposition IX.3.2 in [20] because the local-time condition $L^0(Y^1 - Y^2) = 0$ holds by σ_Y's Lipschitz property and Corollary IX.3.4 in [20]. $\square$

3.2. Investors' maximization problems

There is a single consumption good that serves as our model's numéraire (i.e., all prices are real). As in [2], the stock pays dividends at rate $D = (D_t)_{t\geq 0}$ given by the geometric Brownian motion

$$dD_t := D_t\big(\mu_D dt + \sigma_D dB_t\big), \quad D_0 > 0, \tag{3.15}$$

where $D_0 > 0$, $\mu_D \in \mathbb{R}$, and $\sigma_D > 0$ are exogenous constants.[4]

There are two assets: a money market account with price process $S^{(0)} = (S^{(0)})_{t\geq 0}$ and a stock with price process $S = (S_t)_{t\geq 0}$ paying dividends at rate D. We conjecture (and verify) that there exist Itô processes $(S^{(0)}, S)$ with dynamics

$$dS_t^{(0)} = r_t S_t^{(0)} dt, \quad S_0^{(0)} := 1, \tag{3.16}$$

$$dS_t = (S_t r_t - D_t)dt + S_t \sigma_{S,t}(\kappa_t dt + dB_t), \quad S_0 \in (0,\infty), \tag{3.17}$$

where S_0 is a constant and (r, κ, σ_S) are stochastic processes. Below, $(S_0, r, \kappa, \sigma_S)$ will be endogenously determined in equilibrium. To ensure that (3.16) is well-defined, we require $r \in \mathcal{L}^1_{\text{loc}}$.[5] To ensure that (3.17) is well-defined, we require $\sigma_S \in \mathcal{L}^2_{\text{loc}}$ and $\kappa \in \mathcal{L}^2_{\text{loc}}$.[6]

As in [2], we consider a model with only two investors (equivalently, two groups of homogeneous investors), and both investors are price takers. Investor 1 can trade the stock whereas investor 2 cannot hold or trade the stock. Investor 1's self-financing wealth process is defined as

4 As in [2], the investors do not receive personal income and so D is the model's aggregate consumption-rate process.

5 A process $Q = (Q_t)_{t\geq 0}$ belongs to $\mathcal{L}^p_{\text{loc}}$, $p \in \{1,2\}$, if Q is progressively measurable and satisfies $\mathbb{P}(\int_0^T |Q_t|^p dt < \infty) = 1$ for all $T > 0$.

6 Unlike the literature on endogenous market completeness (see, e.g., [1]), we do not require $\sigma_S \neq 0$.

$$\begin{aligned} dX_{1,t} &:= r_t X_{1,t} dt + \theta_{1,t} S_t \sigma_{S,t}(\kappa_t dt + dB_t) - c_{1,t} dt, \\ X_{1,0} &:= \theta^{(0)}_{1,0-} + S_0 \in \mathbb{R}, \end{aligned} \tag{3.18}$$

where the nonnegative consumption-rate process $c_1 \in \mathcal{L}^1_{\text{loc}}$ and the number of shares of stock held θ_1 are investor 1's control processes. Both c_1 and θ_1 are endogenously determined in equilibrium and must be such that the dynamics (3.18) are well-defined. In (3.18), the exogenous constant $\theta^{(0)}_{1,0-}$ denotes investor 1's endowed holdings in the money market account. Furthermore, for simplicity, we normalize the stock supply to one share outstanding so that $X_{1,0}$ in (3.18) reflects investor 1's endowed stock holdings being $\theta_{1,0-} = 1$.

The dynamics of investor 2's self-financing wealth process are simpler because she cannot access the stock market at any time. Investor 2's self-financing wealth process is defined by

$$\begin{aligned} dX_{2,t} &:= r_t X_{2,t} dt - c_{2,t} dt, \\ X_{2,0} &:= \theta^{(0)}_{2,0-} \in (0, \infty). \end{aligned} \tag{3.19}$$

In (3.19), the nonnegative process $c_2 \in \mathcal{L}^1_{\text{loc}}$ is investor 2's consumption-rate process and is endogenous, and $\theta^{(0)}_{2,0-}$ is investor 2's endowed holdings in the money market account (exogenous). The process c_2 is endogenously determined in equilibrium and must be such that the dynamics (3.19) are well-defined. Because the money market account is in zero-net supply, the investors' endowed money market holdings satisfy $\theta^{(0)}_{1,0-} + \theta^{(0)}_{2,0-} = 0$. The restriction on the initial money market holdings $\theta^{(0)}_{2,0-} = -\theta^{(0)}_{1,0-} > 0$ in (3.19) is taken from [2].

As in [2], investor 1 and investor 2 are assumed to have identical preferences. However, unlike the log-log utilities in [2], our model uses a common time preference parameter $\beta > 0$ and a common constant relative risk aversion coefficient $\gamma \in (0, 1)$.[7] The investors' maximization problems are

$$\sup_{\theta_1, c_1 \in \mathcal{A}_1} \tfrac{1}{1-\gamma} \mathbb{E}\left[\int_0^\infty e^{-\beta t} c_{1,t}^{1-\gamma} dt \right], \quad \sup_{c_2 \in \mathcal{A}_2} \tfrac{1}{1-\gamma} \mathbb{E}\left[\int_0^\infty e^{-\beta t} c_{2,t}^{1-\gamma} dt \right], \tag{3.20}$$

where the admissible sets $\mathcal{A}_1$ and $\mathcal{A}_2$ in (3.20) are defined as:

Definition 3.4 *(Admissibility).* We deem progressively measurable processes (θ_1, c_1) admissible for investor 1 and we write $(\theta_1, c_1) \in \mathcal{A}_1$ if $c_{1,t} \geq 0$, $t \geq 0$, and the solution of (3.18) is nonnegative, i.e., $X_{1,t} \geq 0$ for all $t \geq 0$. Likewise, we deem a progressively measurable process c_2 admissible for investor 2 and we write $c_2 \in \mathcal{A}_2$ if $c_{2,t} \geq 0$, $t \geq 0$, and the solution of (3.19) is nonnegative, i.e., $X_{2,t} \geq 0$ for all $t \geq 0$. ◇

3.3. Radner equilibrium

Because both investors are price takers, the equilibrium is referred to as a Radner equilibrium. This equilibrium concept is standard and can be found in, e.g., Chapter 4 in the textbook [13].

[7] There are two natural extensions of our setting: (i) extend the model to different powers for the two investors, and (ii) extend the model to $\gamma \in (1, \infty)$. However, both such extensions produce more complicated governing ODEs than (1.10) and so we leave these extensions to future research.

We recall that the stock supply is in a net supply of one, the money market account is in zero net supply, and the aggregate consumption rate is D in (3.15).

Definition 3.5 *(Radner equilibrium).* A constant $S_0 > 0$ and processes (r, κ, σ_S) and $(\hat{\theta}_1, \hat{c}_1, \hat{c}_2)$ constitute a Radner equilibrium if:

(i) The processes $(\hat{\theta}_1, \hat{c}_1) \in \mathcal{A}_1$ maximize (3.20) for $i = 1$; that is,

$$(\hat{\theta}_1, \hat{c}_1) \in \operatorname{argmax}_{(\theta,c)\in\mathcal{A}_1} \tfrac{1}{1-\gamma}\mathbb{E}\Big[\int_0^\infty e^{-\beta t} c_t^{1-\gamma}\, dt\Big] < \infty. \tag{3.21}$$

(ii) The process $\hat{c}_2 \in \mathcal{A}_2$ maximizes (3.20) for $i = 2$; that is,

$$\hat{c}_2 \in \operatorname{argmax}_{c\in\mathcal{A}_2} \tfrac{1}{1-\gamma}\mathbb{E}\Big[\int_0^\infty e^{-\beta t} c_t^{1-\gamma}\, dt\Big] < \infty. \tag{3.22}$$

(iii) The stock and consumption markets clear in the sense

$$\hat{\theta}_{1,t} = 1 \quad \text{and} \quad \hat{c}_{1,t} + \hat{c}_{2,t} = D_t, \quad \text{for all } t \geq 0. \tag{3.23}$$

◇

Walras' law ensures that clearing in the stock market and clearing in the goods market imply clearing in the money market account. In other words, the two clearing conditions in (3.23) are sufficient for clearing the money market account too.

The next theorem gives existence of a Radner equilibrium under the additional restriction $\theta^{(0)}_{2,0-} < \frac{g(0)}{D_0}$ where g solves the below ODE (3.25). This restriction is a generalization of the restriction in Eq. (6) in [2]. In the following, we specialize Theorem 1.1 to the constant A defined as

$$A := \frac{2\beta + \sigma_D^2 - (1-\gamma)(2\mu_D - \gamma\sigma_D^2)}{\sigma_D^2}. \tag{3.24}$$

This particular expression for A comes from equating the below ODE (3.25) with the ODE (1.10). The restriction $A > 1$ comes from (4.1) below. Instead of h_{ξ_0}, we use h to denote the function produced by Theorem 1.1 in the following.

Theorem 3.6. *Let $\gamma \in (0, 1)$, $\sigma_D^2 > 0$, and $\beta > \frac{1}{2}(1-\gamma)(2\mu_D - \gamma\sigma_D^2)$.*

1. *There exists a nonincreasing and nonnegative solution $g \in \mathcal{C}^1([0, 1]) \cap \mathcal{C}^2([0, 1))$ of the ODE*

$$\begin{cases} \beta g(y) = (1-\gamma)\mu_D g(y) - \frac{1}{2}(1-\gamma)\gamma\sigma_D^2 g(y) + \mu_Y(y) g'(y) \\ \qquad\qquad + \frac{1}{2}\sigma_Y(y)^2 g''(y) + (1-\gamma)\sigma_D\sigma_Y(y) g'(y) \\ \qquad\qquad + (1-y)^{1-\gamma}, \quad y \in (0, 1), \\ g'(0) = 0, \quad g(1) = 0. \end{cases} \tag{3.25}$$

2. *For $\theta^{(0)}_{2,0-} \in \big(0, \frac{g(0)}{D_0}\big)$, let $Y_0 \in (0,1)$ solve $g(Y_0)D_0(1-Y_0)^\gamma = \theta^{(0)}_{2,0-}$. Then, there exists a Radner equilibrium in which the equilibrium interest rate process is $r_t = r(Y_t) \in \mathcal{L}^1_{loc}$ and the equilibrium market price of risk process is $\kappa_t = \kappa(Y_t) \in \mathcal{L}^2_{loc}$ for the deterministic functions*

$$\begin{aligned} r(y) &:= \beta + \gamma\mu_D - \frac{1}{2}\gamma(\gamma+1)\sigma_D^2 - \frac{\gamma(\gamma+1)\sigma_D^2(1-y)}{2yh(y)^2}, \quad y \in (0,1), \\ \kappa(y) &:= \gamma\sigma_D\left(\frac{1-y}{yh(y)}+1\right), \quad y \in (0,1), \end{aligned} \tag{3.26}$$

and the equilibrium consumption-rate processes in (3.21) and (3.22) are

$$\hat{c}_{1,t} := D_t Y_t, \quad \hat{c}_{2,t} := D_t(1-Y_t), \quad t \geq 0. \tag{3.27}$$

Proof of Theorem 3.6(1). Let h be the solution produced by Theorem 1.1 for A defined in (3.24) and insert

$$g(y) := \frac{2}{\xi_0}\exp\Big\{-\int_0^y \frac{h(q)}{1-q}dq\Big\}(1-y)^{-\gamma}, \quad y \in [0,1), \tag{3.28}$$

and insert σ_Y and μ_Y from (3.1) into the ODE (3.25) to recover the ODE (1.10).

Next, we verify the boundary conditions in (3.25). The representation (3.28) means that the boundary condition $g(1) = 0$ in (3.25) requires

$$\lim_{y\uparrow 1} \xi_0 \exp\Big\{\int_0^y \frac{h(q)}{1-q}dq\Big\}(1-y)^\gamma = \infty. \tag{3.29}$$

Because $h(1) = 1 > \gamma$, Eq. (3.29) holds. To see this, we pick $\epsilon > 0$ and $y_\epsilon \in (0,1)$ such that

$$\forall y \in (y_\epsilon, 1) : h(y) > \gamma + \epsilon.$$

For $y \in (y_\epsilon, 1)$, we have

$$\begin{aligned} \exp\Big\{\int_0^y \frac{h(q)}{1-q}dq\Big\}(1-y)^\gamma &\geq \exp\Big\{\int_{y_\epsilon}^y \frac{h(q)}{1-q}dq\Big\}(1-y)^\gamma \\ &\geq \exp\Big\{\int_{y_\epsilon}^y \frac{\gamma+\epsilon}{1-q}dq\Big\}(1-y)^\gamma \\ &= \frac{(1-y_\epsilon)^{\gamma+\epsilon}}{(1-y)^\epsilon}. \end{aligned}$$

Passing $y \uparrow 1$ gives the boundary condition $g(1) = 0$. Because g defined in (3.28) has derivative

$$g'(y) = \frac{2}{\xi_0} e^{-\int_0^y \frac{h(q)}{1-q} dq} \frac{1}{(1-y)^{1+\gamma}} \big(\gamma - h(y)\big), \quad y \in (0,1), \tag{3.30}$$

we see that the boundary condition $g'(0) = 0$ in (3.25) holds because $h(0) = \gamma$.

Finally, because $h(y) \geq \gamma$ for $y \in [0,1]$, we see from (3.30) that $g'(y) \leq 0$ for $y \in (0,1)$.

The proof of Theorem 3.6(2) is given in the next section. $\square$

4. Proof of Theorem 3.6(2)

The duality techniques we use in the following proofs are standard and have been used in various contexts, see, e.g., the textbook [13]. Because the optimization problems (3.20) are on an infinite time horizon $t \in [0,\infty)$, there is a nontrivial transversality condition as $t \to \infty$; see, e.g., Section 9D in [7]. As we shall see, the transversality condition is ensured by the parameter restriction $A > 1$ for A in (3.24), which is equivalent to $\beta > \frac{1}{2}(1-\gamma)\left(2\mu_D - \gamma\sigma_D^2\right)$. Among other things, the restriction $A > 1$ ensures that

$$\mathbb{E}\Big[\int_0^\infty e^{-\beta t} D_t^{1-\gamma} dt\Big] = \frac{D_0^{1-\gamma}}{\beta - \frac{1}{2}(1-\gamma)\left(2\mu_D - \gamma\sigma_D^2\right)} < \infty, \tag{4.1}$$

where the geometric Brownian motion D is from (3.15).

4.1. State-price densities

For an arbitrary process $\eta \in \mathcal{L}^2_{\text{loc}}$, we define the process

$$dZ_{\eta,t} := -Z_{\eta,t}\big(r(Y_t)dt + \eta_t dB_t\big), \quad Z_{\eta,0} \in (0,\infty). \tag{4.2}$$

State-price density processes underly the duality theory we use to prove Theorem 3.6(2) below. A state-price density process Z_η has the defining property that the process $Z_{\eta,t}X_{i,t} + \int_0^t Z_{\eta,u}c_{i,u}du$, $t \geq 0$, is a supermartingale for any admissible consumption process c_i where $i = 1$ or $i = 2$ and X_i is the corresponding wealth process in (3.18) or (3.19).

In the below proofs, we establish the following first-order condition for investor 1

$$\hat{Z}_{1,t} := e^{-\beta t}(D_t Y_t)^{-\gamma} = e^{-\beta t}\hat{c}_{1,t}^{-\gamma}, \quad t \geq 0, \tag{4.3}$$

$$d\hat{Z}_{1,t} = -\hat{Z}_{1,t}\big(r(Y_t)dt + \kappa(Y_t)dB_t\big), \tag{4.4}$$

where we inserted $\hat{c}_1 := DY$ from (3.27). The dynamics in (4.4) follow from Itô's lemma and

$$\begin{aligned} d\hat{c}_{1,t} &= dD_t Y_t \\ &= D_t\big(\mu_Y(Y_t) + \mu_D Y_t + \sigma_D\sigma_Y(Y_t)\big)dt + D_t\big(\sigma_Y(Y_t) + \sigma_D Y_t\big)dB_t. \end{aligned} \tag{4.5}$$

Similarly, we establish the following first-order condition for investor 2

$$\hat{Z}_{2,t} := e^{-\beta t}\big(D_t(1-Y_t)\big)^{-\gamma} = e^{-\beta t}\hat{c}_{2,t}^{-\gamma}, \quad t \geq 0, \tag{4.6}$$

$$d\hat{Z}_{2,t} = -\,\hat{Z}_{2,t}\left(r(Y_t)dt - \gamma\sigma_D \frac{1-h(Y_t)}{h(Y_t)} dB_t\right), \tag{4.7}$$

where we inserted $\hat{c}_2 := D(1-Y)$ from (3.27). The dynamics in (4.7) follow from Itô's lemma and

$$\begin{aligned} d\hat{c}_{2,t} &= dD_t(1-Y_t) \\ &= D_t\big(\mu_D - \mu_Y(Y_t) - \mu_D Y_t - \sigma_D\sigma_Y(Y_t)\big)dt \\ &\quad + D_t\big(\sigma_D - \sigma_Y(Y_t) - \sigma_D Y_t\big)dB_t. \end{aligned} \tag{4.8}$$

While the dt terms of both $\frac{d\hat{Z}_1}{\hat{Z}_1}$ and $\frac{d\hat{Z}_2}{\hat{Z}_2}$ are identically equal to $-r(Y)$, the dB terms differ. This is because investor 2 cannot access the stock market, which renders the model incomplete.

4.2. Investor 2's optimization problem

Let g denote the function from Theorem 3.6(1) and define $r_t := r(Y_t)$ and $\kappa_t := \kappa(Y_t)$ using the functions in (3.26).

Lemma 4.1. *Let $\gamma \in (0,1)$, $\sigma_D^2 > 0$, and $\beta > \frac{1}{2}(1-\gamma)(2\mu_D - \gamma\sigma_D^2)$.*

1. *For $Y_0 \in (0,1)$, we have*

$$\mathbb{E}_t\left[\int_t^\infty e^{-\beta u}\big(D_u(1-Y_u)\big)^{1-\gamma} du\right] = e^{-\beta t} g(Y_t) D_t^{1-\gamma}, \quad t \geq 0. \tag{4.9}$$

2. *When $Y_0 \in (0,1)$ satisfies $g(Y_0)D_0(1-Y_0)^\gamma = \theta^{(0)}_{2,0-}$, investor 2's value function is*

$$\begin{aligned} \sup_{c_2 \in \mathcal{A}_2} \tfrac{1}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\beta u} c_{2,u}^{1-\gamma} du\right] &= \tfrac{1}{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\beta u}\big(D_u(1-Y_u)\big)^{1-\gamma} du\right] \\ &= \tfrac{1}{1-\gamma} g(Y_0) D_0^{1-\gamma}. \end{aligned} \tag{4.10}$$

Proof. 1: For $Y_0 \in (0,1)$ and $n \in \mathbb{N}$ such that $Y_0 \in (\frac{1}{n}, 1-\frac{1}{n})$, we define the nondecreasing sequence of stopping times $\tau_n(\omega) := \inf\big\{s \geq t : Y_s(\omega) \notin (\frac{1}{n}, 1-\frac{1}{n})\big\}$, $t \geq 0$ and $\omega \in \Omega$, where $\inf \emptyset := \infty$. Lemma 4.1 and Lemma 4.2 ensure that both $y=0$ and $y=1$ are inaccessible and so $\mathbb{P}\big(Y_t \in (0,1)\ \forall t \geq 0\big) = 1$. We replace τ_n with $\tau_n \wedge n$, to get that τ_n is bounded by n uniformly in $\omega \in \Omega$, while $\lim_{n\to\infty} \tau_n(\omega) = \infty$ continues to hold.

The ODE (3.25) and Itô's lemma produce the following dynamics

$$
\begin{aligned}
& d\Big(e^{-\beta s} g(Y_s) D_s^{1-\gamma} + \int_0^s e^{-\beta u}\big(D_u(1-Y_u)\big)^{1-\gamma} du\Big) \\
&= e^{-\beta s} D_s^{1-\gamma}\Big\{\Big((1-\gamma)\sigma_D g(Y_s) + \sigma_Y(Y_s) g'(Y_s)\Big) dB_s \\
&+ \Big(-\beta g(Y_s) + (1-\gamma)\mu_D g(Y_s) - \tfrac{1}{2}(1-\gamma)\gamma\sigma_D^2 g(Y_s) + \mu_Y(Y_s) g'(Y_s) \\
&+ \tfrac{1}{2}\sigma_Y(Y_s)^2 g''(Y_s) + (1-\gamma)\sigma_D\sigma_Y(Y_s) g'(Y_s) + (1-Y_s)^{1-\gamma}\Big) ds\Big\}, \\
&= e^{-\beta s} D_s^{1-\gamma}\Big((1-\gamma)\sigma_D g(Y_s) + \sigma_Y(Y_s) g'(Y_s)\Big) dB_s,
\end{aligned} \tag{4.11}
$$

for $s \in [0, \tau_n]$. We integrate (4.11) over $u \in [t, \tau_n]$ to get the representation

$$
\begin{aligned}
& e^{-\beta\tau_n} g(Y_{\tau_n}) D_{\tau_n}^{1-\gamma} + \int_t^{\tau_n} e^{-\beta u}\big(D_u(1-Y_u)\big)^{1-\gamma} du \\
&= e^{-\beta t} g(Y_t) D_t^{1-\gamma} + \int_t^{\tau_n} e^{-\beta u} D_u^{1-\gamma}\Big((1-\gamma)\sigma_D g(Y_u) + \sigma_Y(Y_u) g'(Y_u)\Big) dB_u.
\end{aligned} \tag{4.12}
$$

Taking conditional expectation and using the martingale property of the stopped dB integral give

$$
\mathbb{E}_t[e^{-\beta\tau_n} g(Y_{\tau_n}) D_{\tau_n}^{1-\gamma}] + \mathbb{E}_t\left[\int_t^{\tau_n} e^{-\beta u}\big(D_u(1-Y_u)\big)^{1-\gamma} du\right] = e^{-\beta t} g(Y_t) D_t^{1-\gamma}.
$$

The Monotone Convergence Theorem gives

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{E}_t[e^{-\beta\tau_n} g(Y_{\tau_n}) D_{\tau_n}^{1-\gamma}] + \mathbb{E}_t\left[\int_t^{\infty} e^{-\beta u}\big(D_u(1-Y_u)\big)^{1-\gamma} du\right] \\
= e^{-\beta t} g(Y_t) D_t^{1-\gamma}.
\end{aligned} \tag{4.13}
$$

Because D is a geometric Brownian motion, we have

$$
\begin{aligned}
e^{-\beta t} D_t^{1-\gamma} &= D_0^{1-\gamma} e^{-\beta t + (1-\gamma)(\mu_D - \frac{1}{2}\sigma_D^2)t + (1-\gamma)\sigma_D B_t} \\
&= D_0^{1-\gamma} e^{(-\beta + (1-\gamma)(\mu_D - \frac{1}{2}\sigma_D^2 + \sigma_D \frac{B_t}{t}))t}.
\end{aligned}
$$

The law of the iterated logarithm gives $\lim_{t\to\infty} \frac{B_t}{t} = 0$. Furthermore, because $\beta > (1-\gamma)(\mu_D - \frac{1}{2}\sigma_D^2)$, we have $\mathbb{P}$-a.s. the limit $\lim_{t\to\infty} e^{-\beta t} D_t^{1-\gamma} = 0$. Therefore, because g is uniformly bounded, it suffices to show that the family of random variables $(e^{-\beta\tau_n} D_{\tau_n}^{1-\gamma})_{n\in\mathbb{N}}$ is uniformly integrable so that we can interchange limit and expectation in (4.13) to see $\mathbb{P}$-a.s. the following transversality condition

$$\lim_{n\to\infty}\mathbb{E}_t[e^{-\beta\tau_n}g(Y_{\tau_n})D_{\tau_n}^{1-\gamma}]=\mathbb{E}_t[\lim_{n\to\infty}e^{-\beta\tau_n}g(Y_{\tau_n})D_{\tau_n}^{1-\gamma}]=0.$$

To establish uniform integrability, we use the inequality $\beta>(1-\gamma)\left(\mu_D-\frac{1}{2}\gamma\sigma_D^2\right)$ to find $\epsilon>0$ but so small such that

$$\beta>(1-\gamma)\Big(\mu_D-\frac{1}{2}\sigma_D^2+\frac{1}{2}(1+\epsilon)(1-\gamma)\sigma_D^2\Big). \tag{4.14}$$

We can write

$$\begin{aligned}&(e^{-\beta t}D_t^{1-\gamma})^{1+\epsilon}\\&=D_0^{(1+\epsilon)(1-\gamma)}e^{-(1+\epsilon)\beta t+(1+\epsilon)(1-\gamma)(\mu_D-\frac{1}{2}\sigma_D^2+\frac{1}{2}(1+\epsilon)(1-\gamma)\sigma_D^2)t+(1+\epsilon)(1-\gamma)\sigma_D B_t-\frac{1}{2}(1+\epsilon)^2(1-\gamma)^2\sigma_D^2 t}.\end{aligned}$$

The bound in (4.14) gives the inequality

$$\begin{aligned}&\mathbb{E}[(e^{-\beta\tau_n}D_{\tau_n}^{1-\gamma})^{1+\epsilon}]\\&=D_0^{(1+\epsilon)(1-\gamma)}\\&\qquad\times\mathbb{E}[e^{-(1+\epsilon)(\beta-(1-\gamma)(\mu_D-\frac{1}{2}\sigma_D^2+\frac{1}{2}(1+\epsilon)(1-\gamma)\sigma_D^2))\tau_n+(1+\epsilon)(1-\gamma)\sigma_D B_{\tau_n}-\frac{1}{2}(1+\epsilon)^2(1-\gamma)^2\sigma_D^2\tau_n}]\\&\le D_0^{(1+\epsilon)(1-\gamma)}\mathbb{E}[e^{(1+\epsilon)(1-\gamma)\sigma_D B_{\tau_n}-\frac{1}{2}(1+\epsilon)^2(1-\gamma)^2\sigma_D^2\tau_n}]\\&=D_0^{(1+\epsilon)(1-\gamma)},\end{aligned}$$

where the last equality uses Doob's optional sampling theorem (recall that τ_n is a bounded stopping time and that $e^{pB_t-\frac{1}{2}p^2t}$ is a martingale for any $p\in\mathbb{R}$). De la Valleé Poussin criterion's gives uniform integrability.

2: As in (3.27), we define $\hat{c}_2:=D(1-Y)$ and we claim that the corresponding wealth process is given by

$$\hat{X}_{2,t}:=\frac{1}{\hat{Z}_{2,t}}\mathbb{E}_t\Big[\int_t^\infty e^{-\beta u}\big(D_u(1-Y_u)\big)^{1-\gamma}du\Big]=D_t g(Y_t)(1-Y_t)^\gamma,\quad t\ge 0, \tag{4.15}$$

where $\hat{Z}_2$ is defined in (4.6). We need to show

$$d\hat{X}_{2,t}=\Big(\hat{X}_{2,t}r(Y_t)-\hat{c}_{2,t}\Big)dt,\quad \hat{X}_{2,0}=\theta_{2,0-}^{(0)}. \tag{4.16}$$

The initial condition $\hat{X}_{2,0}=\theta_{2,0-}^{(0)}$ holds by the assumption that Y_0 satisfies $g(Y_0)D_0(1-Y_0)^\gamma=\theta_{2,0-}^{(0)}$. By inserting $\hat{X}_2=Dg(Y)(1-Y)^\gamma$ from (4.15) and $\hat{c}_2=D(1-Y)$ from (3.27) into the dynamics (4.16), the dynamics (4.16) follow as soon as we show

$$d\Big(\underbrace{D_t g(Y_t)(1-Y_t)^\gamma}_{\hat{X}_{2,t}}\Big) = \Big(\underbrace{D_t g(Y_t)(1-Y_t)^\gamma}_{\hat{X}_{2,t}} r(Y_t) - \underbrace{D_t(1-Y_t)}_{\hat{c}_{2,t}}\Big)dt. \tag{4.17}$$

Itô's lemma and the functions in (3.26) produce the dynamics (4.17).

To see that $\hat{c}_2$ is optimal, we use the following standard duality argument. Set $U(x) := \frac{1}{1-\gamma}x^{1-\gamma}$ and let $V(t,y)$ be the Fenchel conjugate

$$V(t,b) := \sup_{x>0}\big(e^{-\beta t}U(x) - xb\big) = \frac{\gamma}{1-\gamma}e^{-\frac{\beta}{\gamma}t}b^{\frac{\gamma-1}{\gamma}} > 0, \quad b \geq 0.$$

In the below inequalities, we use $\mathbb{P}(Y_t \leq 1) = 1$ for $t \geq 0$, so that

$$V(t,\hat{Z}_{2,t}) = \frac{\gamma}{1-\gamma}e^{-\beta t}\big(D_t(1-Y_t)\big)^{1-\gamma} \leq \frac{\gamma}{1-\gamma}e^{-\beta t}D_t^{1-\gamma}, \quad \mathbb{P}\text{-a.s.},$$

which is integrable over $(t,\omega) \in [0,\infty)\times\Omega$ by (4.1). Let $T \in (0,\infty)$ and let $c \in \mathcal{A}_2$ be arbitrary with corresponding wealth process $X_2 \geq 0$ given in (3.19). Then,

$$\begin{aligned}
\mathbb{E}\left[\int_0^T e^{-\beta t}U(c_t)dt\right] &\leq \mathbb{E}\left[\int_0^T \Big(V(t,\hat{Z}_{2,t}) + \hat{Z}_{2,t}c_t\Big)dt\right] \\
&\leq \mathbb{E}\left[\int_0^T \Big(V(t,\hat{Z}_{2,t}) + \hat{Z}_{2,t}c_t\Big)dt + \hat{Z}_{2,T}X_{2,T}\right] \\
&\leq \mathbb{E}\left[\int_0^T V(t,\hat{Z}_{2,t})dt\right] + \hat{Z}_{2,0}X_{2,0},
\end{aligned}$$

where the first inequality uses Fenchel's inequality, the second inequality follows from $X_{2,T} \geq 0$, and the last inequality is produced by the supermartingale property of the state-price density $\hat{Z}_2$. We pass $T \uparrow \infty$ using the Monotone Convergence Theorem to get

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty e^{-\beta t}U(c_t)dt\right] &\leq \mathbb{E}\left[\int_0^\infty V(t,\hat{Z}_{2,t})dt\right] + \hat{Z}_{2,0}\hat{X}_{2,0} \\
&= \mathbb{E}\left[\int_0^\infty \Big(V(t,\hat{Z}_{2,t}) + \hat{Z}_{2,t}\hat{c}_{2,t}\Big)dt\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-\beta t}U(\hat{c}_{2,t})dt\right],
\end{aligned}$$

where the first equality uses (4.15) and the second equality uses (4.6). □

4.3. Investor 1's optimization problem

Let g denote the function from Theorem 3.6(1) and define $r_t := r(Y_t)$ and $\kappa_t := \kappa(Y_t)$ using the functions in (3.26).

Lemma 4.2. *Let $\gamma \in (0,1)$, $\sigma_D^2 > 0$, $\beta > \frac{1}{2}(1-\gamma)(2\mu_D - \gamma\sigma_D^2)$, and assume $Y_0 \in (0,1)$ satisfies $g(Y_0)D_0(1-Y_0)^\gamma = \theta_{2,0-}^{(0)}$.*

1. *There exists a volatility process $\sigma_S \in \mathcal{L}^2_{loc}$ such that the nonnegative processes*

$$
\begin{aligned}
S_t &:= \hat{X}_{1,t} + \hat{X}_{2,t}, \quad t \geq 0,\\
\hat{X}_{1,t} &:= \frac{1}{\hat{Z}_{1,t}} \mathbb{E}_t\Big[\int_t^\infty e^{-\beta u} (D_u Y_u)^{1-\gamma} du \Big], \quad t \geq 0,
\end{aligned}
\tag{4.18}
$$

satisfy (3.17) *and*

$$
\begin{aligned}
d\hat{X}_{1,t} &= r(Y_t)\hat{X}_{1,t}dt + S_t \sigma_{S,t}\big(\kappa(Y_t)dt + dB_t\big) - D_t Y_t dt,\\
\hat{X}_{1,0} &= \theta_{1,0-}^{(0)} + S_0 \in \mathbb{R}.
\end{aligned}
\tag{4.19}
$$

2. *Investor 1's optimal consumption rate process is $\hat{c}_{1,t} := D_t Y_t$, $t \geq 0$.*

Proof. 1. The Martingale Representation Theorem produces an integrand $\Delta \in \mathcal{L}^2_{\text{loc}}$ such that the second equation in (4.18) can be rewritten as

$$
\begin{aligned}
\hat{Z}_{1,t}\hat{X}_{1,t} + \int_0^t \hat{Z}_{1,u} D_u Y_u du &= \mathbb{E}_t\Big[\int_0^\infty e^{-\beta u}(D_u Y_u)^{1-\gamma} du \Big]\\
&= \mathbb{E}\Big[\int_0^\infty e^{-\beta u}(D_u Y_u)^{1-\gamma} du \Big] + \int_0^t \Delta_s dB_s\\
&= \hat{Z}_{1,0}\hat{X}_{1,0} + \int_0^t \Delta_s dB_s.
\end{aligned}
\tag{4.20}
$$

Itô's lemma produces the following dynamics of $\hat{X}_1$ in (4.18)

$$
\begin{aligned}
d\hat{X}_{1,t} &= d\left(\frac{\hat{Z}_{1,t}\hat{X}_{1,t}}{\hat{Z}_{1,t}} \right)\\
&= \Big(\hat{X}_{1,t}\big(r(Y_t) + \kappa(Y_t)^2\big) - D_t Y_t + \frac{\kappa(Y_t)\Delta_t}{\hat{Z}_{1,t}} \Big) dt\\
&\quad + \Big(\hat{X}_{1,t}\kappa(Y_t) + \frac{\Delta_t}{\hat{Z}_{1,t}} \Big) dB_t.
\end{aligned}
\tag{4.21}
$$

By defining the volatility process

$$\sigma_{S,t} := \frac{1}{S_t}\Big(\hat{X}_{1,t}\kappa(Y_t) + \frac{\Delta_t}{\hat{Z}_{1,t}}\Big), \quad t \geq 0, \tag{4.22}$$

we see that the dynamics (4.21) equal those in (3.18). In other words, $d\hat{X}_1$ are wealth dynamics for investor 1 when using $\theta_{1,t} := 1$ and $c_{1,t} := D_t Y_t$ for $t \geq 0$.

The initial value in (3.18) follows from the first formula in (4.18) and initial clearing in the money market (i.e., $\theta^{(0)}_{1,0-} + \theta^{(0)}_{2,0-} = 0$) because

$$\begin{aligned}\theta^{(0)}_{1,0-} + S_0 &= \theta^{(0)}_{1,0-} + \hat{X}_{1,0} + \hat{X}_{2,0}\\ &= \theta^{(0)}_{1,0-} + \hat{X}_{1,0} + \theta^{(0)}_{2,0-}\\ &= \hat{X}_{1,0}.\end{aligned} \tag{4.23}$$

To see that (4.22) ensures that the dynamics dS of the first equation in (4.18) agree with (3.17), we use (3.18) and $\hat{c}_2 = D(1-Y)$ to see

$$\begin{aligned}dS_t &= d\hat{X}_{1,t} + d\hat{X}_{2,t}\\ &= r(Y_t)\hat{X}_{1,t}dt + S_t\sigma_{S,t}\big(\kappa(Y_t)dt + dB_t\big) - D_tY_tdt + \big(r(Y_t)\hat{X}_{2,t} - \hat{c}_{2,t}\big)dt\\ &= r_tS_tdt + S_t\sigma_{S,t}\big(\kappa(Y_t)dt + dB_t\big) - D_tdt.\end{aligned} \tag{4.24}$$

2. To see $\hat{c}_1 := DY$ and $\hat{\theta}_1 := 1$ are optimal for investor 1, we again use the following standard duality argument as we did in the proof of Lemma 4.1. Let $T > 0$ and let $(c,\theta) \in \mathcal{A}_1$ be arbitrary with corresponding wealth process X_1 given in (3.18). Then,

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{-\beta t}U(c_t)dt\right] &\leq \mathbb{E}\left[\int_0^T \Big(V(t,\hat{Z}_{1,t}) + \hat{Z}_{1,t}c_t\Big)dt\right]\\ &\leq \mathbb{E}\left[\int_0^T \Big(V(t,\hat{Z}_{1,t}) + \hat{Z}_{1,t}c_t\Big)dt + \hat{Z}_{1,T}X_{1,T}\right]\\ &\leq \mathbb{E}\left[\int_0^T V(t,\hat{Z}_{1,t})dt\right] + \hat{Z}_{1,0}X_{1,0},\end{aligned}$$

where the first inequality uses Fenchel's inequality, the second inequality follows from $X_{1,T} \geq 0$, and the last inequality is produced by the supermartingale property of the state-price density $\hat{Z}_1$. We pass $T \uparrow \infty$ using the Monotone Convergence Theorem to get

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}U(c_t)dt\right] \leq \mathbb{E}\left[\int_0^\infty V(t,\hat{Z}_{1,t})dt\right] + \hat{Z}_{1,0}\hat{X}_{1,0}$$

$$= \mathbb{E}\left[\int_0^\infty \Big(V(t,\hat{Z}_{1,t}) + \hat{Z}_{1,t}\hat{c}_{1,t}\Big)dt\right]$$
$$= \mathbb{E}\left[\int_0^\infty e^{-\beta t}U(\hat{c}_{1,t})dt\right],$$

where the first equality uses (4.18) and the second equality uses (4.3). □

4.4. Remaining proof

Proof of Theorem 3.6(2). The regularity conditions $r(Y_t) \in \mathcal{L}^1_{\text{loc}}$ and $\kappa(Y_t) \in \mathcal{L}^2_{\text{loc}}$ follow from the functions in (3.26) and the fact that $y = 0$ and $y = 1$ are inaccessible by Lemma 3.2 and Lemma 3.3. Optimality of (3.27) and $\hat{\theta}_1 := 1$ follows from Lemma 4.1 and Lemma 4.2. Obviously, these control processes satisfy the clearing conditions in (3.23). □

Declaration of generative AI and AI-assisted technologies in the writing process

Declaration of generative AI in scientific writing: No AI nor AI-assisted tools have been used.

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Appendix A. The failure of the comparison principle at $y = 1$

This appendix reports on the failure of the comparison principle at $y = 1$. Theorem 2.5 uses Schauder's fixed-point theorem to construct a local solution of (1.10) starting from $y = 0$. Alternatively, by redefining the set $\mathcal{X} = \mathcal{X}(y_0) \subset \mathcal{C}\big([y_0, 1]\big)$ as those continuous functions $h : [y_0, 1] \to \mathbb{R}$ for which

$$\sup_{y\in[y_0,1)} \frac{|1-h(y)|}{1-y} \le \frac{1-\gamma^2}{A-1}, \tag{A.1}$$

for a constant $y_0 \in [0, 1)$, it is possible to use Schauder's fixed-point theorem to construct a local solution of (1.10) starting from $y = 1$. However, as the next lemma shows, the comparison principle, i.e., the conclusion of Theorem 2.3, can fail when we compare ODEs starting at $y = 1$. This is the reason why we construct local solutions starting for $y = 0$ in Theorem 2.5.

In the next proposition, the function $f \in \mathcal{C}^1([0, 1)) \cap \mathcal{C}([0, 1])$ denotes the unique solution produced by setting $a_3 := -\gamma$ in Theorem 2.4 of the ODE

$$\begin{cases} f'(y) = \frac{1+\gamma}{y}\Big(\gamma - f(y)\Big) + \gamma f(y)\frac{1-f(y)}{1-y}, & y \in (0,1), \\ f(0) = \gamma, \quad f(1) = 1. \end{cases} \tag{A.2}$$

Proposition A.1. *Let $\gamma \in (0,1)$, $A > 1$, let f solve (A.2), and let $h_{\xi_0} \in \mathcal{C}^1([0,1])$ be as in Lemma 2.10. Then, we have*

1. $f \notin \mathcal{C}^1([0,1])$.
2. $\lim_{y\uparrow 1} f'(y) = \infty$, $\int_0^1 |f'(q)|dq < \infty$, *and* $\int_0^1 \frac{1-f(q)}{1-q}dq < \infty$.
3. $f(y)$ *fails the comparison principle with* $h_{\xi_0}(y)$ *for* y *close to* 1.

Proof. 1. To see $f \notin \mathcal{C}^1([0,1])$, we argue by contradiction. By using L'Hospital's rule and $f(1) = 1$ in (A.2), we get

$$f'(1) = (1+\gamma)(\gamma-1) + \gamma \lim_{y\uparrow 1} f(y)\frac{1-f(y)}{1-y} = \gamma^2 - 1 + \gamma f'(1).$$

This equation has the unique negative solution $f'(1) = -(1+\gamma)$. Theorem 2.4 ensures that $f \le 1$. However, because $f(1) = 1$, $f'(1) < 0$ gives the contradiction $f(y) > 1$ for y close to 1.
2. **Step 1/3:** We start by proving $\liminf_{y\uparrow 1} \frac{1-f(y)}{1-y} = \infty$. Because $f(1) = 1$, the first term on the right-hand-side of (A.2) has limit

$$\lim_{y\uparrow 1} \frac{1+\gamma}{y}\Big(\gamma - f(y)\Big) = -(1-\gamma)(1+\gamma) < 0. \tag{A.3}$$

Set $\kappa := (1-\gamma)(1+\gamma) > 0$ and consider any ϵ such that

$$0 < \epsilon < \kappa.$$

The limit in (A.3) gives $y_\epsilon \in (0,1)$ such that the solution of the ODE (A.2) satisfies

$$\begin{aligned} f'(y) &\le -\kappa + \epsilon + \gamma f(y)\frac{1-f(y)}{1-y}, \\ &\le -\kappa + \epsilon + \gamma \frac{1-f(y)}{1-y}, \quad y \in (y_\epsilon, 1), \end{aligned} \tag{A.4}$$

where the second inequality uses $f \le 1$. Because

$$\frac{d}{dy}\left(\frac{1-f(y)}{(1-y)^\gamma}\right) = \gamma \frac{1-f(y)}{(1-y)^{1+\gamma}} - \frac{f'(y)}{(1-y)^\gamma} \ge \frac{\kappa-\epsilon}{(1-y)^\gamma}, \tag{A.5}$$

we have for $y \in (y_\epsilon, 1)$

$$\begin{aligned} \frac{1-f(y)}{(1-y)^\gamma} - \frac{1-f(y_\epsilon)}{(1-y_\epsilon)^\gamma} &= \int_{y_\epsilon}^{y} \frac{d}{dq}\left(\frac{1-f(q)}{(1-q)^\gamma}\right)dq \\ &\ge (\kappa-\epsilon)\int_{y_\epsilon}^{y} \frac{1}{(1-q)^\gamma}dq \\ &= -\frac{\kappa-\epsilon}{1-\gamma}\Big((1-y)^{1-\gamma} - (1-y_\epsilon)^{1-\gamma}\Big). \end{aligned} \tag{A.6}$$

To argue by contradiction, we assume there is an increasing sequence $(y_n)_{n\in\mathbb{N}} \subset (y_\epsilon, 1)$ converging to 1 such that

$$\lim_{n\to\infty} \frac{1-f(y_n)}{1-y_n} = \liminf_{y\uparrow 1} \frac{1-f(y)}{1-y} \in [0,\infty).$$

In that case, we have

$$\lim_{n\to\infty} \frac{1-f(y_n)}{(1-y_n)^\gamma} = \lim_{n\to\infty} \frac{1-f(y_n)}{1-y_n}(1-y_n)^{1-\gamma} = 0.$$

In (A.6), we replace y by y_n and pass $n\uparrow\infty$ to see

$$-\frac{1-f(y_\epsilon)}{(1-y_\epsilon)^\gamma} \geq \frac{\kappa-\epsilon}{1-\gamma}(1-y_\epsilon)^{1-\gamma}.$$

By multiplying by $-(1-y_\epsilon)^\gamma$, we get the contradiction

$$0 \leq 1-f(y_\epsilon) \leq \frac{\epsilon-\kappa}{1-\gamma}(1-y_\epsilon) < 0,$$

where the first inequality holds by Theorem 2.4.

Step 2/3: To prove $\lim_{y\uparrow 1} f'(y) = \infty$, it suffices to show $\liminf_{y\uparrow 1} f'(y) = \infty$. By taking liminf in the ODE (A.2) and using $f(1) = 1$, we get

$$\liminf_{y\uparrow 1} f'(y) = -(1-\gamma)(1+\gamma) + \gamma \liminf_{y\uparrow 1} \frac{1-f(y)}{1-y} = \infty.$$

Step 3/3: Consider $y_\epsilon \in (0,1)$ such that $f'(y) > 0$ for $y \in (y_\epsilon, 1)$. Then, $f(1) = 1$ and the Monotone Convergence Theorem give

$$\int_{y_\epsilon}^{1} f'(q)dq = \lim_{y\uparrow 1} \int_{y_\epsilon}^{y} f'(q)dq = \lim_{y\uparrow 1} f(y) - f(y_\epsilon) = 1 - f(y_\epsilon) < \infty.$$

Because $f(1) = 1$, by increasing y_ϵ if necessary, we can assume $\gamma f(y) \geq \frac{\gamma}{2}$ for $y \in (y_\epsilon, 1)$. The ODE (A.2) gives

$$\begin{aligned}\frac{1+\gamma}{y} f(y) + f'(y) &= \frac{1+\gamma}{y}\gamma + \gamma f(y)\frac{1-f(y)}{1-y} \\ &\geq \frac{\gamma}{2}\frac{1-f(y)}{1-y}, \quad y \in (y_\epsilon, 1).\end{aligned}$$

Because the left-hand-side is integrable over $y \in (y_\epsilon, 1)$, the nonnegative function $\frac{1-f(y)}{1-y}$ is also integrable over $y \in (y_\epsilon, 1)$. For $y \in (0, y_\epsilon)$, the bounds $\gamma \leq f(y) \leq 1$ give integrability of $\frac{1-f(y)}{1-y}$.

3. The ODE for h_{ξ_0} in (2.33) and $h_{\xi_0}(q) \le 1$ for all $q \in [0,1]$ give the lower bound

$$\begin{aligned} &h'_{\xi_0}(y) \\ &= \frac{1+\gamma}{y}\big(\gamma - h_{\xi_0}(y)\big) + h_{\xi_0}(y)\frac{\gamma + \Big((A-\gamma)\exp\Big\{\int_y^1 \frac{1-h_{\xi_0}(q)}{1-q}dq\Big\} - A\Big)h_{\xi_0}(y)}{1-y} \\ &\ge \frac{1+\gamma}{y}\big(\gamma - h_{\xi_0}(y)\big) + \gamma h_{\xi_0}(y)\frac{1-h_{\xi_0}(y)}{1-y}, \quad y \in (0,1). \end{aligned} \tag{A.7}$$

To argue by contradiction, we assume that the comparison principle holds for y near 1. Based on (A.2) and (A.7), the property $f(1) = h_{\xi_0}(1) = 1$ gives $h_{\xi_0}(y) \le f(y)$ for $y \in [0,1]$ and so

$$\frac{1-h_{\xi_0}(y)}{1-y} \ge \frac{1-f(y)}{1-y} \ge 0, \quad y \in (0,1).$$

However, Lemma 2.10(7) ensures that $h'_{\xi_0}(y)$ is continuous at $y=1$ with limit $\frac{1-\gamma^2}{A-1}$ and so we get the contradiction

$$\infty > \frac{1-\gamma^2}{A-1} = h'_{\xi_0}(1) = \lim_{y\uparrow 1}\frac{1-h_{\xi_0}(y)}{1-y} \ge \lim_{y\uparrow 1}\frac{1-f(y)}{1-y} = \infty. \quad \square$$

Data availability

No data was used for the research described in the article.

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