Markowitz's mean-variance asset-liability management with regime switching: A time-consistent approach

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Abstract

In this article, we provide the first study in the time consistent solution of the mean-variance asset-liability management (MVALM). The framework is even considered under a continuous time Markov regime-switching setting. Using the extended Hamilton-Jacobi-Bellman equation (HJB) (see Björk and Murgoci (2010)), we show that the time consistent equilibrium control is state dependent in the sense that it depends on the uncontrollable liability process, which is in substantial contrast with the time consistent solution of the similar problem in Björk and Murgoci (2010), in which it is independent of the state. Finally, we give a numerical comparison between our work with the corrected version (as obtained here) of pre-commitment strategy in Chen et al. (2008).

Keywords: Asset-liability management, Mean-variance, Regime switching, Time consistent feedback control, Extended Hamilton-Jacobi-Bellman.

1. Introduction

In the present work, we aim to solve for the mean-variance asset-liability management problem with regime switching modelling via a time inconsistent formulation as in Björk and Murgoci (2010).

Asset-liability management under the mean-variance criteria has recently been widely studied, in which the underlying surplus is equal to the difference of liability from the asset; indeed, this is an optimization problem of selecting the optimal portfolio that can acquire sufficient return (by maximizing the expectation of the terminal surplus) in compensating the company's liability with minimal risk measured by the terminal surplus variance. Keel and Müller (1995) studied portfolio selection problem with asset liability setting. Leippold et al. (2004) considered the multi-period asset-liability portfolio selection problem. By formulating as a stochastic linear-quadratic control problem, Chiu and Li (2006) studied the similar problem in a continuous-time setting. For modelling the continuous time evolution of liability, Chiu

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and Li (2006) and Chen et al. (2008) considered the mean-variance asset-liability management by using geometric Brownian motions (GBMs), rather than drifted diffusions, to model the liability process which can never be negative.

Since a decade ago, regime switching models have become popular in finance and related fields, which can reflect different market (say "Bullish" versus "Bearish"). Most scholars attempt to use a finite number of regimes to describe different mode of the market state; indeed, various market parameters, such as the bank interest rate, appreciation rates and volatilities of stock and liability, will take different values under different market modes. For the application of regime switching modelling in option pricing, see Boyle and Draviam (2007). While its application in bond valuation, see Elliott and Siu (2009a). For its use in portfolio selection problem, see Zhou and Yin (2003), Chen et al. (2008), and Chen and Yang (2011); in particular, by involving both the asset-liability feature and Markovian regime switching modelling, Chen et al. (2008) and Chen and Yang (2011) revisited the mean-variance asset-liability management problem in continuous-time and in multi-period settings respectively.

Since the introduction by Markowitz (1952), the mean-variance portfolio selection problem has become one of key research topics in finance. The investor aims to determine the optimal portfolio, which minimizes the risk measured by the variance of the terminal wealth subject to a predetermined budget constraint and at an arbitrary level of terminal expected return. Later on, Merton (1971) extended the Markowitz's model to continuous time settings, he investigated both the optimal consumption and portfolio selection problems by formulating them as a stochastic control problem.

Due to the non-linear nature of the mean-variance utility (see the definition in (3) in Section 2), usual Tower Property fails to hold, and the corresponding optimal portfolio selection problem is time-inconsistent in the sense that it does not admit the Bellman optimality principle. In other words, an optimal control that optimizes the mean-variance utility at time zero needs not to remain to be optimal for the mean-variance utility at any latter time.

Time-inconsistency in optimization problems was first studied in Strotz (1955). There are basically three different approaches on providing solution concept for these time-inconsistent problems. Firstly, under the notion of pre-commitment, only the feasible control, which is an adapted L^2 control, that optimizes the initial objective function would be considered, whether it is optimal for the objective function at the latter time or not is not relevant; related literature includes Zhou and Yin (2003), Chen et al. (2008) and Chen and Yang (2011). It is remarked that in these mentioned works, the optimization problems are considered over the class of all adapted L^2 controls, which need not to be Markovian; nevertheless, they eventually established that the pre-commitment solution of such a mean-variance optimization problem depends only on both the current and the initial states (for example, a smooth enough function $u(t, X_t, L_t, I_t; x_0, l_0, i_0)$, in Chen et al. (2008) or in our Section 6, of the current state at time t, (X_t, L_t, I_t), and the initial state, (x_0, l_0, i_0)). Recently, Bensoussan et al. (2013b) obtained the pre-commitment solution by dealing with a couple of Fokker-Planck and HJB equations via the mean field game techniques. Secondly, an agent primarily adopts the strategy that optimizes the objective function on the first day, and then on the next day, he will give up this strategy and uses a new one that optimizes the objective function on the second day, and so on.

The third approach was originated by Strotz (1955) and Pollak (1968), who provided the primitive idea of time-consistent strategies for time-inconsistent problems; Later on, Peleg and Yaari (1973) treated time-inconsistent problems as a non-cooperative game, in which strategies at different time points are planned by different players who aim to optimize their own objective functions; Nash equilibrium of these strategies was then utilized to define as the time-consistent strategy for the agent of the original problem. This game theoretic approach and its extensions could be found in some recent works with an application in solving for the time inconsistent consumption problems with non-classical discounting utility, such as Barro (1999), Ekeland and Pirvu (2008) and Björk and Murgoci (2010). Specifically, Ekeland and Pirvu (2008) provided a precise definition of the (time consistent) equilibrium control in continuous time setting, such that the control will be still an equilibrium one for any subproblems over an arbitrary confined time interval before the planning horizon. Following their idea, Björk and Murgoci (2010) studied the time-inconsistent control problem in a general Markov framework, who derived the extended HJB together with the verification theorem, which gives the necessary and sufficient condition of "equilibrium controls". Furthermore, Kryger and Steffensen (2010) extended the class of problems to general objective function of first two moments and provided the corresponding verification theorem.

In this paper, we study the MVALM problem under the Markovian regime-switching in continuous time. Similar mean-variance portfolio selection problems with neither asset-liability nor regime switching feature had been studied in Basak and Chabakauri (2010) and Björk and Murgoci (2010). It is important to note that their obtained common equilibrium control is completely independent of the current state; in spite of this independence, Björk et al. (2012) and Bensoussan et al. (2013a) reformulated the portfolio selection problem with state-dependent risk aversion, in particular, they illustrated that the equilibrium control is dependent on the current wealth if the risk aversion is inversely proportional to the current wealth. However, recently Bensoussan et al. (2013a) discover a limitation of the discrete counterpart of such a state dependent risk aversion model in Björk et al. (2012), Bensoussan et al. (2013a) have a discussion for such a subtle issue.

Here, we adopt the asset and liability modelling in Chen et al. (2008), in which asset and liability processes are described by geometric Brownian motions with regime switching market parameters[1]. Under the present mean-variance utility setting, the optimization problem is clearly time inconsistent, and we aim to solve for the time consistent optimal control by solving for the extended HJB (see Equations (10)-(14) in Section 3), which is similar to that in Björk and Murgoci (2010). We shall explicitly establish that the equilibrium control is affine in current liability, while the equilibrium value function is affine in current surplus and is quadratic in current liability, that is to say they are both depending on current state; the coefficients in their representations can be shown to satisfy a system of linear first order (backward) ODE with predetermined terminal conditions.

In Section 2, we introduce the formulation of our MVALM problem. We derive the system of extend HJB equations for our MVALM problem in Section 3. In Section 4, we make use of a suitable *Ansatz* in solving for the extend HJB system obtained in Section 3. The closed form expressions of both equilibrium control and value function will be illustrated in Theorem 4.1. In Section 5, a simple example with a single risky asset and time-homogeneous market parameters will be given. Further numerical and graphical illustrations will be provided in

[1] To avoid unnecessary technical details, we assume that the interest rate is independent of the modulating Markov chain.

Section 6. Finally, we conclude in Section 7.

2. Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete probability space, and $W(t) = (W_1(t), \ldots, W_d(t))'$ be a $d-$dimensional standard Brownian motion. The dynamics of the market state is modeled by a stationary continuous-time Markov chain process $I(t)$. We assume that the processes $W(t)$ and $I(t)$ are independent of each other, which may be justifiable since the market could be so huge that individual evolution of any few number of security evolution may not affect much on the whole market, while the market influences any securities through the market parameters. Define $\mathcal{F}_t \triangleq \sigma\{(W(s), I(s)) : 0 \leq s \leq t\}$.

Assume that there are N regimes for the market state, that is to say that the Markov chain I_t takes one of the values in $\mathcal{M} \triangleq \{1, 2, \cdots, N\}$ at a time. Also assume that the Markov chain has a generator $Q = (q_{ij})_{N\times N}$ with stationary transition probabilities given by the following equations:

$$\begin{aligned} p_{ij}(t) &= \mathbb{P}(I(s+t) = j | I(s) = i), \quad \text{for } s, t \geq 0,\ i, j = 1, 2, \ldots, N, \\ q_{ij} &= \frac{d}{dt} p_{ij}(t)\bigg|_{t=0}. \end{aligned}$$

For any matrix M, we denote M' as the transpose of M.

Suppose that the market has $m + 1$ assets only with m risky assets with price process $S(t) = (S_1(t), \ldots, S_m(t))'$ and one riskless money account with price process $B(t)$ such that $(B(t), S(t))$ satisfy the following SDE:

$$\begin{aligned} dB(t) &= r(t)B(t)dt \\ dS_k(t) &= \mu_k(t, I(t))S_k(t)\, dt + S_k(t) \sum_{j=1}^{d} \sigma_{kj}(t, I(t))dW_j(t) \end{aligned}$$

where $r(t)$ is the riskless interest-rate, $\mu_k(t, i)$ and $\sigma_k(t, i) \triangleq (\sigma_{k1}(t, i), \ldots, \sigma_{kd}(t, i))$ are the appreciation-rate and volatility processes of the $k-$th risky asset according to different market state i. In the present paper, we assume that for every $i \in \mathcal{M}$, $r(t), \mu_k(t, i)$ and $\sigma_k(t, i)$ are deterministic on $[0, T]$ for any $k \in \{1, \ldots, m\}$. Also define a volatility matrix of assets as $\sigma(t, i) = (\sigma_{kj}(t, i))_{m\times d}$. We also assume that $r(t) \leq \mu_k(t, i)$ for all $(t, i) \in [0, T] \times \mathcal{M}$ and $k \in \{1, \ldots, m\}$.

Denote $u(t) = (u_1(t), \ldots, u_m(t))'$ as the asset portfolio of the company, where $u_k(t)$ is the dollar amount invested in the $k-$th risky assets at time t, while the remaining asset will be invested in riskless money account. We assume that there is no transaction cost, and hence the asset value process A_t is given by:

$$dA(t) = [r(t)A(t) + \theta(t, I(t))'u(t)]\, dt + u(t)'\sigma(t, I(t))dW(t), \tag{1}$$

where $\theta(t, I_t) \triangleq (\theta_1(t, I_t), \ldots, \theta_m(t, I_t))'$ and $\theta_k(t, I_t) \triangleq \mu_k(t, I_t) - r(t)$ for any $k \in \{1, \ldots, m\}$.

The liability process L_t is not controllable and is modeled by GBM:

$$dL(t) = \alpha(t, I(t))L(t)dt + L(t)\beta(t, I(t))'dW(t), \quad (2)$$

where $\alpha(t,i)$ and $\beta(t,i) = (\beta_1(t,i), \ldots, \beta_d(t,i))'$ are the appreciation-rate and volatility of liability according to different market state i. Again, we assume that for every $i \in \mathcal{M}$, $\alpha(t,i)$ and $\beta_j(t,i)$ are deterministic on $[0,T]$ for any $j \in \{1,\ldots,d\}$.

In this context, we restrict ourselves to feedback control law, i.e., the controls are in the form $u_t = u(t, X_t, L_t, I_t)$, where the control law $u : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathcal{M} \to \mathbb{R}^m$ is a deterministic function of its arguments (variables). Denote the surplus process of the company by $X^u(t) \triangleq A(t) - L(t)$, which is the difference of liability from the asset value using control law $u(t) = u(t,x,l,i)$ when the current state $(X(t), L(t), I(t)) = (x,l,i)$. Hence, the dynamics of $X(t)$ and $L(t)$ are given by:

$$\begin{aligned} dX(t) &= [r(t)X(t) + \eta(t, I(t))L(t) + \theta(t, I(t))'u(t)]\, dt \\ &\quad + [u(t)'\sigma(t, I(t)) - L(t)\beta(t, I(t))']\, dW(t), \\ dL(t) &= \alpha(t, I(t))L(t)dt + L(t)\beta(t, I(t))'dW(t), \end{aligned}$$

where $\eta(t, I(t)) \triangleq r(t) - \alpha(t, I(t))$.

The reward (objective) functional we adopted here is the mean-variance utility of the agent's terminal surplus, which is given by:

$$\begin{aligned} J(t,x,l,i,u(\cdot)) &\triangleq \mathbb{E}_{t,x,l,i}[X^u(T)] - \frac{\gamma(t,i)}{2} Var_{t,x,l,i}[X^u(T)] \\ &= \mathbb{E}_{t,x,l,i}[X^u(T)] - \frac{\gamma(t,i)}{2}\left\{\mathbb{E}_{t,x,l,i}\left[(X^u(T))^2\right] - \left[\mathbb{E}_{t,x,l,i}(X^u(T))\right]^2\right\}, \quad (3) \end{aligned}$$

where $\mathbb{E}_{t,x,l,i}[\cdot]$ and $Var_{t,x,l,i}[\cdot]$ are the expectation and variance conditioned on the event $\{X_t = x, L_t = l, I_t = i\}$ respectively, $\gamma(t,i)$ are the risk aversion process under the market state i.

To ensure the existence of both the equilibrium value function and the equilibrium control, one suffices to assume the following uniform boundedness and non-degeneracy conditions on coefficients:

Condition 2.1. *(i) $r(t)$, (ii) $\alpha(t,i)$, (iii) $\gamma(t,i)$, (iv) $\mu_k(t,i)$, (v) $\beta_j(t,i)$, (vi) all entries in $\sigma(t,i)$, and (vii) all entries in $(\sigma(t,i)\sigma(t,i)')^{-1}$ are uniformly bounded on $[0,T]$, for each $i \in \mathcal{M}$, $k \in \{1,\ldots,m\}$, and $j \in \{1,\ldots,d\}$. $\gamma(t,i)$ is also non-degenerate, i.e. there exists a $\delta > 0$ such that $\gamma(t,i) > \delta$ for any $t \in [0,T]$ and $i \in \mathcal{M}$.*

The agent aims to look for an admissible control law that optimizes the above reward (objective) functional. Due to the fact that this objective functional is non-linear in the expectation of the terminal surplus, the corresponding optimization problem is clearly a time-inconsistent one in the sense that it fails to satisfy the Bellman optimality principle (as pointed out by Björk and Murgoci (2010)); in other words, even though the obtained optimal control can optimize the reward functional at time zero, since this functional changes over time with the state, it is not possible to expect that the same control remains to be optimal at latter times.

In order to deal with the time-inconsistent nature of various optimization problems in finance, it becomes popular in the literature to formulate the problem as a non-cooperative game (with each time point as a player), and then look for some equilibrium controls that will also be equilibrium for any (time) subproblems. Ekeland and Pirvu (2008) provided the precise definition of equilibrium control in continuous time setting for a class of time-inconsistent investment-consumption problems. Björk and Murgoci (2010) generalized their notion of equilibrium control to the one that caters for more general time-inconsistent control problems. In the present work, we adopt the definition of equilibrium control given by Björk and Murgoci (2010), namely:

Definition 2.2. *A control law $\hat{u}$ is said to be an equilibrium control if for every admissible control law u, and $h > 0$, one can define a new control law u_h by*

$$u_h(s,y,z,i) = \begin{cases} u(s,y,z,i), & \text{for} \quad t \le s < t+h, \\ \hat{u}(s,y,z,i), & \text{for} \quad t+h \le s \le T, \end{cases}$$

such that

$$\liminf_{h\to 0^+} \frac{J(t,x,l,i,\hat{u}(\cdot)) - J(t,x,l,i,u_h(\cdot))}{h} \ge 0 \text{ for any } (t,x,l,i) \in [0,T]\times\mathbb{R}^2\times\mathcal{M}. \tag{4}$$

In the language of game theory, we consider our problem as a game problem in which every $t \in [0,T]$ is regarded as a player who chooses a strategy $u(t,x,l,i)$ (only at t) and has the objective function $J(t,x,l,i,u(\cdot))$ for every $(x,l,i) \in \mathbb{R}^2 \times \mathcal{M}$. Note that $u(\cdot)$ in $J(t,x,l,i,u(\cdot))$ is a control law composed by: (1) $u(t,y,z,j)$, the strategy chosen by player t at $(X(t),L(t),I(t)) = (y,z,j)$; and (2) all $u(s,y,z,j)$ with $s > t$, which is the strategy chosen by player s at $(X(s),L(s),I(s)) = (y,z,j)$). According to Definition 2.2, the notion equilibrium control implies that if for all players $s > t$ have already chosen $\hat{u}$, it is optimal for the player t to choose $\hat{u}$; based on this definition, one actually uses the concept of a "subgame perfect Nash equilibrium point", and hence the corresponding equilibrium control is time-consistent by definition. Further details of Definition 2.2 and the game theoretic approach on tackling some other time inconsistent problems, one can consult the work by Peleg and Yaari (1973), Ekeland and Pirvu (2008), and Björk and Murgoci (2010).

In our paper, we aim to establish explicitly the admissible equilibrium control in accordance with Definition 2.2.

3. Extended HJB equation

We now apply the result by Björk and Murgoci (2010) to derive the extended HJB equation for our present MVALM problem. Firstly, we rewrite the objective functional as:

$$J(t,x,l,i,u(\cdot)) = \mathbb{E}_{t,x,l,i}\left[F\left(t,i,X^u(T)\right)\right] + G\left(t,i,\mathbb{E}_{t,x,l,i}\left[X^u(T)\right]\right), \tag{5}$$

where

$$
\begin{aligned}
F(t,i,y) &= y - \frac{\gamma(t,i)}{2}y^2, \\
G(t,i,y) &= \frac{\gamma(t,i)}{2}y^2.
\end{aligned}
$$

Therefore, the equilibrium value function is given by

$$
V(t,x,l,i) = J(t,x,l,i,\hat{u}(\cdot)). \tag{6}
$$

For the simplicity of notation, in the rest of the paper, we shall denote $r, \eta_i, \sigma_i, \theta_i, \alpha_i, \beta_i, \gamma_i$ by $r(t), \eta(t,i)$, $\sigma(t,i), \theta(t,i), \alpha(t,i), \beta(t,i), \gamma(t,i)$ respectively unless other specifications.

In spite of Lemma 3.1 in Chen et al. (2008), we deduce that, for any test function $v(t,x,l,i)$ and any admissible control u, the controlled infinitesimal generator is given by:

$$
\begin{aligned}
& \mathcal{A}^u v(t,x,l,i) \\
\triangleq\ & v_t(t,x,l,i) + (r(t)x + \eta(t,i)l + \theta(t,i)'u)\, v_x(t,x,l,i) + \alpha(t,i) l v_l(t,x,l,i) \\
& + \frac{1}{2}\left(u'\sigma(t,i) - l\beta(t,i)'\right) v_{xx}(t,x,l,i) \left(\sigma(t,i)'u - l\beta(t,i)\right) \\
& + \left(u'\sigma(t,i) - l\beta(t,i)'\right) v_{xl}(t,x,l,i) l\beta(t,i) \\
& + \frac{1}{2}\beta(t,i)' l^2 v_{ll}(t,x,l,i)\beta(t,i) + \sum_{j=1}^{N} q_{ij} v(t,x,l,j).
\end{aligned} \tag{7}
$$

By making use of the definition of equilibrium control as given in (4) in Definition 2.2 and the infinitesimal generator $\mathcal{A}^u$ in (7), following the application in Björk and Murgoci (2010), we can derive the extended HJB and its verification theorem as follows:

Theorem 3.1 (Verification theorem). *Suppose that there are functions $V, g : [0,T]\times\mathbb{R}^2\times\mathcal{M} \to \mathbb{R}$, $f : [0,T]\times\mathbb{R}^2\times\mathcal{M}\times[0,T]\times\mathcal{M} \to \mathbb{R}$, $\hat{u} : [0,T]\times\mathbb{R}^2\times\mathcal{M} \to \mathbb{R}^m$ such that they satisfy the following system of equations:*

$$
\begin{aligned}
\sup_{u\in\mathbb{R}^m} \big\{ \mathcal{A}^u V(t,x,l,i) - \mathcal{A}^u f(t,x,l,i,t,i) + \mathcal{A}^u f^{t,i}(t,x,l,i) & \\
- \mathcal{A}^u (G\circ g)(t,x,l,i) + \mathcal{H}^u g(t,x,l,i) \big\} &= 0, \\
\mathcal{A}^{\hat{u}} f^{s,k}(t,x,l,i) &= 0, \\
\mathcal{A}^{\hat{u}} g(t,x,l,i) &= 0, \\
V(T,x,l,i) &= x, \\
f^{s,j}(T,x,l,i) &= x - \frac{\gamma(s,j)}{2}x^2, \\
g(T,x,l,i) &= x,
\end{aligned}
$$

where the supremum of the first equation is attained at $\hat{u}(t,x,l,i)$ for all $(x,l,i)\in\mathbb{R}^2\times\mathcal{M}$, $f^{s,j}(t,x,l,i) \triangleq f(t,x,l,i,s,j)$, $(G\circ g)(t,x,l,i) \triangleq G(t,i,g(t,x,l,i)) = \frac{\gamma(t,i)}{2}g(t,x,l,i)^2$, and $\mathcal{H}^u g(t,x,l,i) \triangleq G_y(t,i,g(t,x,l,i))\cdot\mathcal{A}^u g(t,x,l,i)$.

Then $\hat{u}$ is an equilibrium control law, and V is the corresponding equilibrium value function.

Moreover, f and g have the following probabilistic representations:

$$\begin{aligned} f(t,x,l,i,s,j) &= \mathbb{E}_{t,x,l,i}\left[F\left(s,j,X^{\hat{u}}(T)\right)\right], \\ g(t,x,l,i) &= \mathbb{E}_{t,x,l,i}\left[X^{\hat{u}}(T)\right]. \end{aligned} \tag{8}$$

In accordance with the probabilistic representations in Theorem 3.1, we clearly have:

$$V(t,x,l,i) = f(t,x,l,i,t,i) + G(t,i,g(t,x,l,i)). \tag{9}$$

Moreover, the infinitesimal generator (7) is linear, and so

$$\mathcal{A}^u V(t,x,l,i) = \mathcal{A}^u f(t,x,l,i,t,i) + \mathcal{A}^u (G \circ g)(t,x,l,i).$$

Therefore, the first equation in Theorem 3.1 can be simplified further as follows,

$$\begin{aligned} & \mathcal{A}^u V(t,x,l,i) - \mathcal{A}^u f(t,x,l,i,t,i) + \mathcal{A}^u f^{t,i}(t,x,l,i) - \mathcal{A}^u (G \circ g)(t,x,l,i) \\ & + \mathcal{H}^u g(t,x,l,i) \\ = \; & \mathcal{A}^u f^{t,i}(t,x,l,i) + \mathcal{H}^u g(t,x,l,i). \end{aligned}$$

By using the infinitesimal generator (7), we can rewrite the extended HJB system in Theorem 3.1 as:

$$\begin{aligned} \sup_u \Big\{ f_t^{t,i}(t,x,l,i) + \gamma_i g_t(t,x,l,i) g(t,x,l,i) \\ + (rx + \eta_i l + \theta_i' u)\left[f_x^{t,i}(t,x,l,i) + \gamma_i g_x(t,x,l,i) g(t,x,l,i)\right] \\ + \alpha_i l\left[f_l^{t,i}(t,x,l,i) + \gamma_i g_l(t,x,l,i) g(t,x,l,i)\right] \\ + \frac{1}{2}(u'\sigma_i - l\beta_i')(\sigma_i' u - l\beta_i)\left[f_{xx}^{t,i}(t,x,l,i) + \gamma_i g_{xx}(t,x,l,i) g(t,x,l,i)\right] \\ + (u'\sigma_i - l\beta_i') l\beta_i\left[f_{lx}^{t,i}(t,x,l,i) + \gamma_i g_{lx}(t,x,l,i) g(t,x,l,i)\right] \\ + \frac{1}{2}\beta_i' l^2 \beta_i\left[f_{ll}^{t,i}(t,x,l,i) + \gamma_i g_{ll}(t,x,l,i) g(t,x,l,i)\right] \\ + \sum_{j=1}^N q_{ij}[f^{t,i}(t,x,l,j) + \gamma_i g(t,x,l,i) g(t,x,l,j)] \Big\} &= 0 \end{aligned} \tag{10}$$

$$\begin{aligned} f_t^{s,k}(t,x,l,i) + (rx + \eta_i l + \theta_i'\hat{u}) f_x^{s,k}(t,x,l,i) + \alpha_i l f_l^{s,k}(t,x,l,i) \\ + \frac{1}{2}(\hat{u}'\sigma(t,i) - l\beta(t,i)')(\sigma(t,i)'\hat{u} - l\beta(t,i)) f_{xx}^{s,k}(t,x,l,i) \\ + (\hat{u}'\sigma(t,i) - l\beta(t,i)') l\beta_i f_{xl}^{s,k}(t,x,l,i) \\ + \frac{1}{2}\beta_i' l^2 \beta_i f_{ll}^{s,k}(t,x,l,i) + \sum_{j=1}^N q_{ij} f^{s,k}(t,x,l,j) &= 0 \end{aligned} \tag{11}$$

$$
\begin{aligned}
g_t(t,x,l,i) + (rx + \eta_i l + \theta_i' \hat{u})\, g_x(t,x,l,i) + \alpha_i l g_l(t,x,l,i) & \\
+\frac{1}{2}\left(\hat{u}'\sigma(t,i) - l\beta(t,i)'\right)\left(\sigma(t,i)'\hat{u} - l\beta(t,i)\right) g_{xx}(t,x,l,i) & \\
+ \left(\hat{u}'\sigma(t,i) - l\beta(t,i)'\right) l\beta_i g_{xl}(t,x,l,i) & \\
+\frac{1}{2}\beta_i' l^2 \beta_i g_{ll}(t,x,l,i) + \sum_{j=1}^{N} q_{ij} g(t,x,l,j) &= 0 \qquad (12) \\
f(T,x,l,i,s,j) &= x - \frac{\gamma_j(s)}{2} x^2 \qquad (13) \\
g(T,x,l,i) &= x \qquad (14)
\end{aligned}
$$

Since the equilibrium control $\hat{u}$ maximizes the LHS of (10) which is a concave quadratic function of u, thus the equilibrium control $\hat{u}$ is found to be:

$$
\begin{aligned}
&\hat{u}(t,x,l,i) \\
= \; &(\sigma_i\sigma_i')^{-1}\left[\left(\frac{f_{xx}^{t,i} + \gamma_i g_{xx} g - f_{xl}^{t,i} - \gamma_i g_{xl} g}{f_{xx}^{t,i} + \gamma_i g_{xx} g}\right)\sigma_i\beta_i l - \left(\frac{f_x^{t,i} + \gamma_i g_x g}{f_{xx}^{t,i} + \gamma_i g_{xx} g}\right)\theta_i\right] \qquad (15)
\end{aligned}
$$

given

$$
f_{xx}^{t,i}(t,x,l,i) + \gamma(t,i) g_{xx}(t,x,l,i) g(t,x,l,i) \;<\; 0 \qquad (16)
$$

for all (t,x,l,i).

Obviously, if one can solve for f and g via solving Equations (10)-(14) with the equilibrium control $\hat{u}$ as given in (15) and f and g satisfy (16), then V given by (9) is the equilibrium value function, and f and g have the probabilistic representations as in (8), and finally, we can also conclude that $\hat{u}$ is the equilibrium control.

4. Solution of MVALM problem

In this section, we attempt to resolve the extended HJB (10)-(14) by making use of the following *Ansatz*:

$$
\begin{aligned}
g(t,x,l,i) &= a(t,i)x + b(t,i)l + k(t,i) \\
f(t,x,l,i,s,j) &= a(t,i)x + b(t,i)l + k(t,i) - \frac{\gamma(s,j)}{2}\left[A(t,i)x^2 + B(t,i)l^2\right. \\
&\quad \left. + 2C(t,i)xl + 2D(t,i)x + 2E(t,i)l + K(t,i)\right] \qquad (17)
\end{aligned}
$$

We assume that $A(t,i) > 0$ for all (t,i) so that (16) is satisfied.

For simplicity, we again adopt a simpler notation scheme by denoting $a(t,i), b(t,i), k(t,i)$, $A(t,i), B(t,i), C(t,i), D(t,i), E(t,i)$ and $K(t,i)$ by $a_i, b_i, k_i, A_i, B_i, C_i, D_i, E_i$ and K_i respectively. By substituting $\hat{u}$, as given in (15), into (10), it is obvious that Equation (10) is redundant, which can be expressed as an algebraic sum of Equations (11) and (12); indeed, $\{(10) \text{ with } u = \hat{u}\} = \{(11) \text{ with } s = t \text{ and } k = i\} + \gamma(t,i)g(t,x,l,i) \times (12)$. It now suffices to determine $a(t,i), b(t,i), k(t,i), A(t,i), B(t,i), C(t,i), D(t,i)$, $E(t,i)$ and $K(t,i)$ so that f

and g in the form in (17) can solve for Equations (11) and (12) and the terminal conditions: $A(T,i) = a(T,i) = 1$ and $B(T,i) = C(T,i) = D(T,i) = E(T,i) = K(T,i) = b(T,i) = k(T,i) = 0$.

With the previously mentioned *Ansatz* in mind, using the expression in (15), the equilibrium control can be rewritten as:

$$\begin{aligned}
&\hat{u}(t,x,l,i) \\
= \;& (\sigma_i\sigma_i')^{-1}\left(\frac{A_i - C_i}{A_i}\right)\sigma_i\beta_i l + (\sigma_i\sigma_i')^{-1}\left(\frac{\frac{a_i}{\gamma_i} - (A_i x + C_i l + D_i) + a_i\left(a_i x + b_i l + k_i\right)}{A_i}\right)\theta_i \\
= \;& \left((\sigma_i\sigma_i')^{-1}\frac{(a_i^2 - A_i)}{A_i}\theta_i\right)x + \left((\sigma_i\sigma_i')^{-1}\left[\left(\frac{A_i - C_i}{A_i}\right)\sigma_i\beta_i + \left(\frac{a_i b_i - C_i}{A_i}\right)\theta_i\right]\right)l \\
& + (\sigma_i\sigma_i')^{-1}\frac{1}{A_i}\left(\frac{a_i}{\gamma_i} - D_i + a_i k_i\right)\theta_i.
\end{aligned}$$

Based on this expression, we can now rewrite $rx + \eta_i l + \theta_i'\hat{u}$ and $\sigma_i'\hat{u} - l\beta_i$ as:

$$\begin{aligned}
rx + \eta_i l + \theta_i'\hat{u} \;&=\; \left(r + \frac{(a_i^2 - A_i)}{A_i}\theta_i'\left(\sigma_i\sigma_i'\right)^{-1}\theta_i\right)x \\
&\quad + \left(\eta_i + \frac{A_i - C_i}{A_i}\theta_i'\left(\sigma_i\sigma_i'\right)^{-1}\sigma_i\beta_i + \frac{a_i b_i - C_i}{A_i}\theta_i'\left(\sigma_i\sigma_i'\right)^{-1}\theta_i\right)l \\
&\quad + \frac{1}{A_i}\left(\frac{a_i}{\gamma_i} - D_i + a_i k_i\right)\theta_i'\left(\sigma_i\sigma_i'\right)^{-1}\theta_i \\
&\triangleq\; \overline{r}_i^x x + \overline{r}_i^l l + \overline{r}_i^k \\
\sigma_i'\hat{u} - l\beta_i \;&=\; \left(\frac{(a_i^2 - A_i)}{A_i}\sigma_i'\left(\sigma_i\sigma_i'\right)^{-1}\theta_i\right)x \\
&\quad + \left(\frac{A_i - C_i}{A_i}\sigma_i'\left(\sigma_i\sigma_i'\right)^{-1}\sigma_i\beta_i + \frac{a_i b_i - C_i}{A_i}\sigma_i'\left(\sigma_i\sigma_i'\right)^{-1}\theta_i - \beta_i\right)l \\
&\quad + \frac{1}{A_i}\left(\frac{a_i}{\gamma_i} - D_i + a_i k_i\right)\sigma_i'\left(\sigma_i\sigma_i'\right)^{-1}\theta_i \\
&\triangleq\; \overline{\sigma}_i^x x + \overline{\sigma}_i^l l + \overline{\sigma}_i^k
\end{aligned}$$

Further, we also have alternative expressions for $\mathcal{A}^{\hat{u}} f^{s,k}$ and $\mathcal{A}^{\hat{u}} g$:

$$
\begin{aligned}
& \mathcal{A}^{\hat{u}} f^{s,k} \\
= \; & -\frac{\gamma_k(s)}{2}\left(\dot{A}_i + 2\overline{r}_i^x A_i + (\overline{\sigma}_i^x)'(\overline{\sigma}_i^x)A_i + \sum_{j=1}^N q_{ij}A_j\right)x^2 \\
& -\frac{\gamma_k(s)}{2}\left(\dot{B}_i + 2\overline{r}_i^l C_i + 2\alpha_i B_i + (\overline{\sigma}_i^l)'(\overline{\sigma}_i^l)A_i + 2(\overline{\sigma}_i^l)'\beta_i C_i + \beta_i'\beta_i B_i + \sum_{j=1}^N q_{ij}B_j\right)l^2 \\
& -\gamma_k(s)\left(\dot{C}_i + \overline{r}_i^x C_i + \overline{r}_i^l A_i + \alpha_i C_i + (\overline{\sigma}_i^x)'\,\overline{\sigma}_i^l A_i + (\overline{\sigma}_i^x)'\beta_i C_i + \sum_{j=1}^N q_{ij}C_j\right)xl \\
& +\Big\{\dot{a}_i - \gamma_k(s)\dot{D}_i + \overline{r}_i^x\left(a_i - \gamma_k(s)D_i\right) - \gamma_k(s)A_i\overline{r}_i^k - \gamma_k(s)\left(\overline{\sigma}_i^x\right)'\overline{\sigma}_i^k A_i \\
& +\sum_{j=1}^N q_{ij}\left(a_j - \gamma_k(s)D_j\right)\Big\}x \\
& +\Big\{\dot{b}_i - \gamma_k(s)\dot{E}_i + \overline{r}_i^l\left(a_i - \gamma_k(s)D_i\right) - \gamma_k(s)C_i\overline{r}_i^k + \alpha_i(b_i - \gamma_k(s)E_i) - \gamma_k(s)\left(\overline{\sigma}_i^l\right)'\overline{\sigma}_i^k A_i \\
& -\gamma_k(s)\left(\overline{\sigma}_i^k\right)'\beta_i C_i + \sum_{j=1}^N q_{ij}\left(b_j - \gamma_k(s)E_j\right)\Big\}l \\
& +\dot{k}_i - \frac{\gamma_k(s)}{2}\dot{K}_i + \overline{r}_i^k\left(a_i - \gamma_k(s)D_i\right) - \frac{\gamma_k(s)}{2}\left(\overline{\sigma}_i^k\right)'\left(\overline{\sigma}_i^k\right)A_i + \sum_{j=1}^N q_{ij}\left(k_j - \frac{\gamma_k(s)}{2}K_j\right), \\
& \mathcal{A}^{\hat{u}} g \\
= \; & \left(\dot{a}_i + a_i\overline{r}_i^x + \sum_{j=1}^N q_{ij}a_j\right)x + \left(\dot{b}_i + a_i\overline{r}_i^l + \alpha_i b_i + \sum_{j=1}^N q_{ij}b_j\right)l + \left(\dot{k}_i + a_i\overline{r}_i^k + \sum_{j=1}^N q_{ij}k_j\right).
\end{aligned}
$$

By equating $\mathcal{A}^{\hat{u}} f^{s,k}$ and $\mathcal{A}^{\hat{u}} g$ with zero, due to the arbitrariness of variables x, l, s and k, we deduce the following system of ODEs with the terminal conditions: $A(T,i) = a(T,i) = 1$ and $B(T,i) = C(T,i) = D(T,i) = E(T,i) = K(T,i) = b(T,i) = k(T,i) = 0$. In particular, we have further reduction in the number of equations, because the coefficients in $\mathcal{A}^{\hat{u}} g$ can be completely covered by those in $\mathcal{A}^{\hat{u}} f^{s,k}$:

$$
\dot{A}_i + \left(2r - \theta_i'\left(\sigma_i\sigma_i'\right)^{-1}\theta_i\right)A_i + \theta_i'\left(\sigma_i\sigma_i'\right)^{-1}\theta_i\frac{a_i^4}{A_i} + \sum_{j=1}^N q_{ij}A_j = 0, \quad (18)
$$

$$
\begin{aligned}
& \dot{B}_i + \left(2\alpha_i + \beta_i'\beta_i\right)B_i + \left(-\beta_i'\sigma_i'\left(\sigma_i\sigma_i'\right)^{-1}\sigma_i\beta_i + \beta_i'\beta_i\right)A_i \\
& +2\left(\eta_i + \theta_i'\left(\sigma_i\sigma_i'\right)^{-1}\sigma_i\beta_i + \beta_i'\sigma_i'\left(\sigma_i\sigma_i'\right)^{-1}\sigma_i\beta_i - \beta_i'\beta_i\right)C_i + \theta_i'\left(\sigma_i\sigma_i'\right)^{-1}\theta_i\frac{a_i^2 b_i^2}{A_i} \\
& -\left(\theta_i'\left(\sigma_i\sigma_i'\right)^{-1}\theta_i + 2\theta_i'\left(\sigma_i\sigma_i'\right)^{-1}\sigma_i\beta_i + \beta_i'\sigma_i'\left(\sigma_i\sigma_i'\right)^{-1}\sigma_i\beta_i\right)\frac{C_i^2}{A_i} + \sum_{j=1}^N q_{ij}B_j = 0, \quad (19)
\end{aligned}
$$

$$\dot{C}_i + \left(r + \alpha_i - \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \sigma_i\beta_i - \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i\right) C_i + \left(\eta_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \sigma_i\beta_i\right) A_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i \frac{a_i^3 b_i}{A_i} + \sum_{j=1}^N q_{ij} C_j = 0, \quad (20)$$

$$\dot{D}_i + \left(r - \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i\right) D_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i \left(\frac{a_i^3}{\gamma_i A_i} + \frac{a_i^3 k_i}{A_i}\right) + \sum_{j=1}^N q_{ij} D_j = 0, \quad (21)$$

$$\dot{E}_i + \alpha_i E_i + \left(\eta_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \sigma_i\beta_i\right) D_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i \left(\frac{a_i^2 b_i}{\gamma_i A_i} + \frac{a_i^2 b_i k_i}{A_i}\right) - \left(\theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \sigma_i\beta_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i\right) \frac{C_i D_i}{A_i} + \sum_{j=1}^N q_{ij} E_j = 0, \quad (22)$$

$$\dot{K}_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i \frac{1}{A_i} \left(\left(\frac{a_i}{\gamma_i} + a_i k_i\right)^2 - D_i^2\right) + \sum_{j=1}^N q_{ij} K_j = 0, \quad (23)$$

$$\dot{a}_i + \left(r - \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i\right) a_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i \frac{a_i^3}{A_i} + \sum_{j=1}^N q_{ij} a_j = 0, \quad (24)$$

$$\dot{b}_i + \alpha_i b_i + \left(\eta_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \sigma_i\beta_i\right) a_i - \left(\theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \sigma_i\beta_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i\right) \frac{C_i a_i}{A_i} + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i \frac{a_i^2 b_i}{A_i} + \sum_{j=1}^N q_{ij} b_j = 0, \quad (25)$$

$$\dot{k}_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \theta_i \frac{a_i}{A_i} \left(\frac{a_i}{\gamma_i} - D_i + a_i k_i\right) + \sum_{j=1}^N q_{ij} k_j = 0. \quad (26)$$

Although the system of ODEs looks tedious, we can still resolve all of them one by one:

1. Solve for a_i and A_i from Equations (18) and (24).
2. Solve for b_i and C_i from Equations (20) and (25).
3. Solve for k_i and D_i from Equations (21) and (26).
4. Solve for all the remaining unknowns from Equations (19), (22) and (23).

Firstly, from Equations (18) and (24), we can directly establish the solution of this latter system, namely $A(t, i) = \exp\left(2 \int_t^T r(s)ds\right)$ and $a(t, i) = \exp\left(\int_t^T r(s)ds\right)$ for all $i \in \mathcal{M}$ because $q_{ii} = -\sum_{j \neq i} q_{ij}$.

Secondly, from Equations (20) and (25) with solved expression of A_i and a_i, we can deduce that $C(t, i) = \exp\left(\int_t^T r(s)ds\right) b(t, i)$. Then, further algebra leads us to a linear first order ODE satisfied by b_i:

$$\dot{b}_i + \left(\alpha_i - \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \sigma_i\beta_i\right) b_i + \sum_{j=1}^N q_{ij} b_j + \left(\eta_i + \theta_i' \left(\sigma_i\sigma_i'\right)^{-1} \sigma_i\beta_i\right) \exp\left(\int_t^T r(s)ds\right) = 0.$$

The above system of first order ODE is well studied, thus b_i and C_i can be solved explicitly. Similarly, we can again deduce that $D(t, i) = \exp\left(\int_t^T r(s)ds\right) k(t, i)$ for all $i \in \mathcal{M}$ from

Equations (21) and (26) with known coefficients, then the system satisfied by k_i can be reduced to the system:

$$\dot{k}_i = -\sum_{j=1}^{N} q_{ij}k_j - \theta_i' \left(\sigma_i\sigma_i'\right)^{-1}\theta_i \frac{1}{\gamma_i},$$

Again, this system can be solved explicitly.

Finally, since b_i, k_i can be obtained explicitly, we can solve B_i, E_i and K_i from Equations (19), (22) and (23) by just tackling the system of first order linear ODEs. To conclude this section with our main theorem, we first define the following system of first order linear ODEs satisfied by $b(t,i), k(t,i), B(t,i), E(t,i), K(t,i)$:

$$\begin{aligned}
\dot{b}(t,i) + \Big(\alpha(t,i) - \theta(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\sigma(t,i)\beta(t,i)\Big)\, b(t,i) + \sum_{j=1}^{N} q_{ij}b(t,j), & \\
+ \Big(\eta(t,i) + \theta(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\sigma(t,i)\beta(t,i)\Big)\, e^{\int_t^T r(s)ds} &= 0, \quad (27)\\
b(T,i) &= 0, \quad (28)\\
\dot{k}(t,i) + \sum_{j=1}^{N} q_{ij}k(t,j) + \frac{1}{\gamma(t,i)}\theta(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\theta(t,i), &= 0, \quad (29)\\
k(T,i) &= 0, \quad (30)\\
\dot{B}(t,i) + \Big(2\alpha(t,i) + \beta(t,i)'\beta(t,i)\Big)B(t,i) + \sum_{j=1}^{N} q_{ij}B(t,j) & \\
+ \Big(-\beta(t,i)'\sigma(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\sigma(t,i)\beta(t,i) + \beta(t,i)'\beta(t,i)\Big)\, e^{\int_t^T 2r(s)ds} & \\
+2\Big(\eta(t,i) + \theta(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\sigma(t,i)\beta(t,i) - \beta(t,i)'\beta(t,i) & \\
+\beta(t,i)'\sigma(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\sigma(t,i)\beta(t,i)\Big)\, e^{\int_t^T r(s)ds}b(t,i) & \\
- \Big(2\theta(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\sigma(t,i)\beta(t,i) & \\
+\beta(t,i)'\sigma(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\sigma(t,i)\beta(t,i)\Big)\, b(t,i)^2 &= 0, \quad (31)\\
B(T,i) &= 0, \quad (32)\\
\dot{E}(t,i) + \alpha(t,i)E(t,i) + \sum_{j=1}^{N} q_{ij}E(t,j) & \\
+ \Big(\eta(t,i) + \theta(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\sigma(t,i)\beta(t,i)\Big)\, e^{\int_t^T r(s)ds}k(t,i) & \\
- \Big(\theta(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\sigma(t,i)\beta(t,i)\Big)\, b(t,i)k(t,i) & \\
+ \left(\frac{1}{\gamma(t,i)}\theta(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\theta(t,i)\right) b(t,i) &= 0, \quad (33)\\
E(T,i) &= 0, \quad (34)
\end{aligned}$$

$$\dot{K}(t,i) + \sum_{j=1}^{N} q_{ij} K(t,j) + \left(\frac{2}{\gamma(t,i)} \theta(t,i)' \left(\sigma(t,i)\sigma(t,i)'\right)^{-1} \theta(t,i) \right) k(t,i)$$

$$+ \frac{1}{\gamma(t,i)^2} \theta(t,i)' \left(\sigma(t,i)\sigma(t,i)'\right)^{-1} \theta(t,i) = 0, \quad (35)$$

$$K(T,i) = 0. \quad (36)$$

Theorem 4.1. *The equilibrium control for the MVAL problem is given by*

$$\hat{u}(t,x,l,i) = \left(1 - e^{-\int_t^T r(s)ds} b(t,i)\right) \left(\sigma(t,i)\sigma(t,i)'\right)^{-1} \sigma(t,i)\beta(t,i) l + \frac{1}{\gamma(t,i)} e^{-\int_t^T r(s)ds} \left(\sigma(t,i)\sigma(t,i)'\right)^{-1} \theta(t,i), \quad (37)$$

and the corresponding equilibrium value function is given by

$$V(t,x,l,i) = e^{\int_t^T r(s)ds} x + b(t,i) l + k(t,i) - \frac{\gamma(t,i)}{2} \left\{ \left[B(t,i) - b(t,i)^2\right] l^2 + 2\left[E(t,i) - b(t,i)k(t,i)\right] l + K(t,i) - k(t,i)^2 \right\}, \quad (38)$$

where $b(t,i), k(t,i), B(t,i), E(t,i)$ and $K(t,i)$ satisfy the system of first order linear ODEs in (27)-(36)

Remark 4.2. *In principle, the solution of our MVALM problem can be solved explicitly; With the assumption of condition 2.1, the existence of finite $b(t,i), k(t,i), B(t,i), E(t,i), K(t,i)$ can essentially be guaranteed by uniform boundedness condition on the coefficients (over $[0,T]$) of the first order linear ODE system in Theorem 4.1.*

Remark 4.3. *The equilibrium control (37) stated in Theorem 4.1 does depend on the current state namely, the current liability though it does not depend on the current surplus; indeed, the expression $e^{\int_t^T r(s)ds}$ does not satisfy the terminal conditions of $b(T,i) = 0$, and therefore the coefficient of l in $\hat{u}$ can never be zero, at least, at the time T.*

Remark 4.4. *In spite of the probabilistic representation (8) of the solution of the extended HJB, both the conditional expectation and conditional variance of the terminal surplus, given the filtration of information up to time t, are given by:*

$$\mathbb{E}_{t,x,l,i}\left[X^{\hat{u}}(T)\right] = e^{\int_t^T r(s)ds} x + b(t,i) l + k(t,i), \quad (39)$$

$$Var_{t,s,l,i}\left[X^{\hat{u}}(T)\right] = \left[B(t,i) - b(t,i)^2\right] l^2 + 2\left[E(t,i) - b(t,i)k(t,i)\right] l + K(t,i) - k(t,i)^2. \quad (40)$$

5. Example: one bond and one risky asset with time homogeneous coefficients

We now consider an example in which there is only one risky asset and a riskless bank account. We assume that $W(t) \triangleq (W^1(t), W^2(t))$ is the 2-dimensional standard Brownian motion, and all market parameters are time-homogeneous, i.e. all market parameters only depend on the regime switching. The dynamics of the surplus and liability can now be rewritten as:

$$
\begin{aligned}
dX(t) &= [rX(t) + \eta(I(t))L(t) + \theta(I(t))u(t)]\,dt \\
&\quad + [u(t)\sigma(I(t)) - L(t)\beta(I(t))\rho(I(t))]\,dW^1(t) - L(t)\beta(I(t))\sqrt{1-\rho(I(t))^2}dW^2(t), \\
dL(t) &= \alpha(I(t))L(t)dt + L(t)\beta(I(t))\rho(I(t))dW^1(t) + L(t)\beta(I(t))\sqrt{1-\rho(I(t))^2}dW^2(t),
\end{aligned}
$$

where $\theta(I(t)) \triangleq \mu(I(t)) - r$ and $\eta(I(t)) \triangleq r - \alpha(I(t))$. Here $\rho(I(t)) \in [-1,1]$ is the correlation coefficient between the increments of risky asset and liability over the unit time. We also assume the risk aversion to be time-homogeneous, and therefore the reward functional is given by:

$$
J(t,x,l,i,u(\cdot)) = \mathbb{E}_{t,x,l,i}\left[X^u(T)\right] - \frac{\gamma(i)}{2} Var_{t,s,l,i}\left[X^u(T)\right]. \tag{41}
$$

For the notational simplicity, we again denote $\eta(i), \sigma(i), \theta(i), \alpha(i), \beta(i), \gamma(i)$ and $\rho(i)$ by $\eta_i, \sigma_i, \theta_i, \alpha_i$, β_i, γ_i and ρ_i respectively. Now, $\sigma(t,i) = (\sigma_i, 0)$ and $\beta(t,i) = \left(\beta_i\rho_i, \beta_i\sqrt{1-\rho_i^2}\right)'$, we also define the system of first order linear ODEs satisfied by $b(t,i), k(t,i), B(t,i), E(t,i)$ and $K(t,i)$:

$$
\begin{aligned}
\dot{b}(t,i) + \left(\alpha_i - \frac{\theta_i\beta_i\rho_i}{\sigma_i}\right) b(t,i) + \sum_{j=1}^{N} q_{ij} b(t,j) + \left(\eta_i + \frac{\theta_i\beta_i\rho_i}{\sigma_i}\right) e^{r(T-t)} &= 0, && (42)\\
b(T,i) &= 0, && (43)\\
\dot{k}(t,i) + \sum_{j=1}^{N} q_{ij} k(t,j) + \frac{\theta_i^2}{\gamma_i\sigma_i^2} &= 0, && (44)\\
k(T,i) &= 0, && (45)\\
\dot{B}(t,i) + \left(2\alpha_i + \beta_i^2\right) B(t,i) + \sum_{j=1}^{N} q_{ij} B(t,j) - \left(2\frac{\theta_i\beta_i\rho_i}{\sigma_i} + \beta_i^2\rho_i^2\right) b(t,i)^2 & \\
+2\left(\eta_i + \frac{\theta_i\beta_i\rho_i}{\sigma_i} - \beta_i^2\left(1-\rho_i^2\right)\right) e^{r(T-t)} b(t,i) + \beta_i^2\left(1-\rho_i^2\right) e^{2r(T-t)} &= 0, && (46)\\
B(T,i) &= 0, && (47)\\
\dot{E}(t,i) + \alpha_i E(t,i) + \sum_{j=1}^{N} q_{ij} E(t,j) + \left(\eta_i + \frac{\theta_i\beta_i\rho_i}{\sigma_i}\right) e^{r(T-t)} k(t,i) & \\
-\frac{\theta_i\beta_i\rho_i}{\sigma_i} b(t,i)k(t,i) + \frac{\theta_i^2}{\gamma_i\sigma_i^2} b(t,i) &= 0, && (48)\\
E(T,i) &= 0, && (49)\\
\dot{K}(t,i) + \sum_{j=1}^{N} q_{ij} K(t,j) + \frac{2\theta_i^2}{\gamma_i\sigma_i^2} k(t,i) + \frac{\theta_i^2}{\gamma_i^2\sigma_i^2} &= 0, && (50)\\
K(T,i) &= 0. && (51)
\end{aligned}
$$

Theorem 5.1. *For our present example, the equilibrium control of the MVAL problem is* *given by*

$$\hat{u}(t,x,l,i) = \frac{\beta_i \rho_i}{\sigma_i}\left(1 - e^{-r(T-t)} b(t,i)\right) l + \frac{\theta_i}{\gamma_i \sigma_i^2} e^{-r(T-t)},$$

while the corresponding equilibrium value function is given by

$$\begin{aligned} V(t,x,l,i) &= e^{r(T-t)}x + b(t,i)l + k(t,i) \\ &\quad -\frac{\gamma_i}{2}\left(\left[B(t,i) - b(t,i)^2\right] l^2 + 2\left[E(t,i) - b(t,i)k(t,i)\right] l + K(t,i) - k(t,i)^2\right), \end{aligned}$$

where $b(t,i), k(t,i), B(t,i), E(t,i)$ and $K(t,i)$ satisfy the system of first order linear ODEs as defined above in (42)-(51).

Remark 5.2. *For if the initial liability were zero ($l = 0$) (and so the $L_t \equiv 0$ for all $t \in [0,T]$) with all market parameters remaining constant for every market state, the present example reduces to the simplest mean-variance control problem as considered in Björk and Murgoci (2010); indeed, the solution of our present example becomes:*

$$\begin{aligned} \hat{u}(t,x) &= \frac{\mu - r}{\gamma \sigma^2} e^{-r(T-t)}, \\ V(t,x) &= e^{r(T-t)}x + k(t) - \frac{\gamma}{2}\left(K(t) - k(t)^2\right), \end{aligned}$$

where $k(t)$ and $K(t)$ satisfy the ODEs:

$$\begin{aligned} \dot{k}(t) + \frac{(\mu - r)^2}{\gamma \sigma^2} &= 0, \\ k(T) &= 0; \\ \dot{K}(t) + \frac{2(\mu - r)^2}{\gamma \sigma^2} k(t) + \frac{(\mu - r)^2}{\gamma^2 \sigma^2} &= 0, \\ K(T) &= 0; \end{aligned}$$

which lead to:

$$\begin{aligned} k(t) &= \frac{(\mu - r)^2}{\gamma \sigma^2}(T-t), \\ K(t) &= \left(\frac{(\mu - r)^2}{\gamma \sigma^2}(T-t)\right)^2 + \frac{(\mu - r)^2}{\gamma^2 \sigma^2}(T-t), \end{aligned}$$

which are equivalent to those result as obtained in Björk and Murgoci (2010).

6. Numerical Illustration

As in Chen et al. (2008), we assume that $N = 2$, i.e. the market state can be either "bullish" or "bearish". Regime 1 and Regime 2 are correspond to "bullish" and "bearish" of the market respectively. We shall illustrate through the results obtained in Section 5 in which the market consists of one bond and one risky asset with the coefficients being constant. In this section, we shall make the comparison of our result with that in Chen et al. (2008), we have two subsections of two categories of comparisons. In the first subsection, we illustrate the changes of the equilibrium strategy and the equilibrium value function by varying initial time and liability respectively. At the same time, we compare them to the mean-variance strategy (Mean-variance investor only aims to optimize his current utility (41) and will always revise his strategy.) and its corresponding utility function respectively. In the second subsection, we illustrate the comparison of the mean-variance distribution of time-consistent investors with the efficient frontier of the mean-variance investors.

All parameters specified for our illustrative purpose are shown in Table 1. Note that $q_{11} = -q_1, q_{12} = q_1, q_{21} = q_2, q_{22} = -q_2$.

Table 1: The parameter-set

Regime	T	r_i	μ_i	σ_i	α_i	β_i	ρ_i	q_i	γ_i
$i = 1$ (bullish)	10	0.04	0.2	0.3	0.08	0.3	0.6	0.3	0.5
$i = 2$ (bearish)	10	0.04	0.05	0.07	0.04	0.1	0.4	0.7	0.9

Before we make the comparison between time consistent strategy and pre-commitment strategy, we first stress that the correct expressions of efficient portfolio, $\tilde{u}(t, x, l, i; z, x_0, l_0, i_0, T)$ (representing the optimal strategy when the current state is (t, x, l, i), the initial state is $(0, x_0, l_0, i_0)$, the expected terminal surplus benchmark is z, and the expiry time is T), and corresponding optimal variance, $f(z; x, l, i; T) \triangleq Var_{0,x,l,i}[X^{\tilde{u}}(T)]$, should be the following (Those obtained in Chen et al. (2008) are actually in-accurate, which leads to an incorrect expression for the corresponding optimal control):

$$\begin{aligned}
\tilde{u}(t, x, l, i; z, x_0, l_0, i_0, T) &= -\left(\sigma(t, i)P(t, i)\sigma(t, i)'\right)^{-1} \{P(t, i)\theta(t, i)x \\
&\quad + \left[-P(t, i)\sigma(t, i)\beta(t, i) + \frac{1}{2}R(t, i)\left(\theta(t, i) + \sigma(t, i)\beta(t, i)\right)\right] l \\
&\quad + P(t, i)H(t, i)\theta(t, i)\left[\frac{P(0, i_0)H(0, i_0)x_0 + \frac{1}{2}S(0, i_0)l_0 - z}{1 - \rho - P(0, i_0)H(0, i_0)^2}\right]\Bigg\} \\
f(z; x, l, i; T) &= \frac{P(0, i)H(0, i)^2 + \rho}{1 - \rho - P(0, i)H(0, i)^2} \times \left[z - \frac{P(0, i)H(0, i)x + \frac{1}{2}S(0, i)l}{P(0, i)H(0, i)^2 + \rho}\right]^2 \\
&\quad + P(0, i)x^2 + Q(0, i)l^2 + R(0, i)xl \\
&\quad - \frac{\left[2P(0, i)H(0, i)x + S(0, i)l\right]^2}{4\left[P(0, i)H(0, i)^2 + \rho\right]}
\end{aligned}$$

where $\rho := \sum_{j=1}^{d}\sum_{k=1}^{d}\int_0^T p_{i_0j}(t)q_{jk}P(t, k)[H(t, k) - H(t, j)]^2 dt$ and $P(t, i), H(t, i), R(t, i), Q(t, i), S(t, i)$ form the solution of the following ODE system:

$$\begin{aligned}
\frac{\partial}{\partial t}P(t, i) + \left[2r(t) - \theta(t, i)'\left(\sigma(t, i)\sigma(t, i)'\right)^{-1}\theta(t, i)\right]P(t, i) + \sum_{k=1}^{d} q_{ik}P(t, k) &= 0 \\
P(T, i) &= 1 \\
\frac{\partial}{\partial t}H(t, i) - r(t)H(t, i) + \frac{1}{P(t, i)}\sum_{k=1}^{d} q_{ik}P(t, k)\left[H(t, k) - H(t, i)\right] &= 0 \\
H(T, i) &= 1
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial t}R(t,i)+[r(t)+\alpha(t,i)]\,R(t,i)+\sum_{k=1}^{d}q_{ik}R(t,k)+2\eta(t,i)P(t,i)& \\
+\theta(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\left\{2\sigma(t,i)\beta(t,i)P(t,i)-R(t,i)\left[\theta(t,i)+\sigma(t,i)\beta(t,i)\right]\right\} &= 0 \\
R(T,i) &= 0 \\
\frac{\partial}{\partial t}Q(t,i)+[2\alpha(t,i)+\beta(t,i)'\beta(t,i)]\,Q(t,i)+\sum_{k=1}^{d}q_{ik}Q(t,k)& \\
+R(t,i)\left[\eta(t,i)-\beta(t,i)'\beta(t,i)\right]+\beta(t,i)'P(t,i)\beta(t,i)& \\
-\left\{\sigma(t,i)'\beta(t,i)'P(t,i)-\frac{1}{2}R(t,i)\left[\theta(t,i)'+\sigma(t,i)'\beta(t,i)'\right]\right\}& \\
\times\left(\sigma(t,i)P(t,i)\sigma(t,i)'\right)^{-1}\left\{\sigma(t,i)\beta(t,i)P(t,i)-\frac{1}{2}R(t,i)\left[\theta(t,i)+\sigma(t,i)\beta(t,i)\right]\right\} &= 0 \\
Q(T,i) &= 0 \\
\frac{\partial}{\partial t}S(t,i)+\alpha(t,i)S(t,i)+\sum_{k=1}^{d}q_{ik}S(t,k)+2\eta(t,i)P(t,i)H(t,i)& \\
+H(t,i)\theta(t,i)'\left(\sigma(t,i)\sigma(t,i)'\right)^{-1}\left\{2\sigma(t,i)\beta(t,i)P(t,i)-R(t,i)\left[\theta(t,i)+\sigma(t,i)\beta(t,i)\right]\right\} &= 0 \\
S(T,i) &= 0
\end{aligned}
$$

6.1. Comparison between equilibrium and mean-variance strategies

Here, in accordance with our Theorem 5.1, we adopt the equilibrium strategy $\hat{u}(t,x,l,i)$ and the equilibrium value function $V(t,x,l,i)$. For the mean-variance investor always revises his strategy continuously in order to maximize his current utility in (41) for some z, an optimal value $z^*(t)$ will be chosen so that $\tilde{u}(0,x,l,i;z,x,l,i,T-t)|_{z=z^*(t)}$ maximizes his current utility (41). And we define the mean-variance strategy $u^*(t,x,l,i)\triangleq\tilde{u}(0,x,l,i;z^*(t),x,l,i,T-t)$, and the corresponding optimal current utility $V^*(t,x,l,i)\triangleq z^*(t)-\frac{\gamma_i}{2}f(z^*(t);x,l,i;T-t)$.

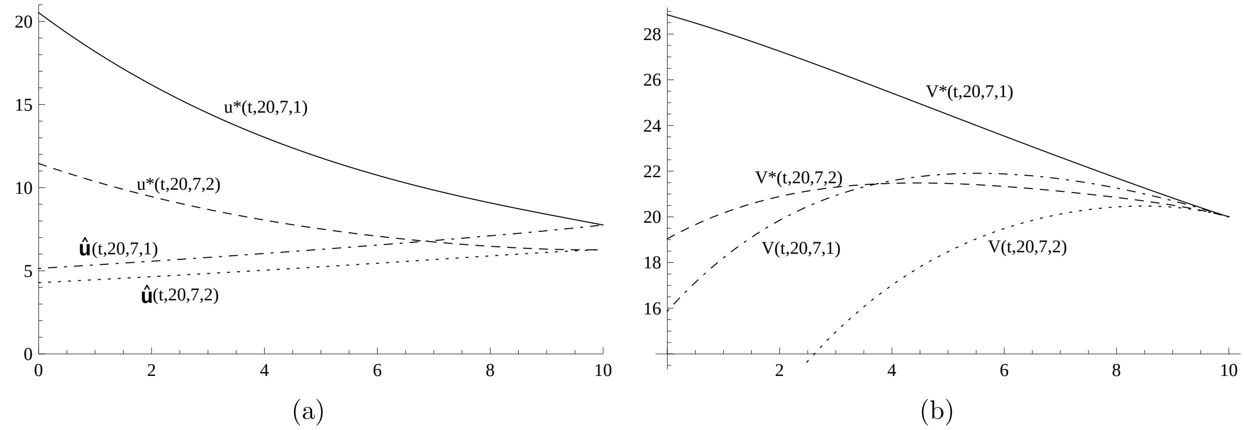

Figure 1: (a) The comparison between the equilibrium strategies $\hat{u}(t,20,7,i)$ and the mean-variance strategy $u^*(t,20,7,i)$ against the initial time for $i=1,2$. (b) The comparison between the equilibrium value functions $V(t,20,7,i)$ and the optimal current utility $V^*(t,20,7,i)$ against the initial time for $i=1,2$.

Figure 1 shows that the mean-variance strategy decreases with time; while the equilibrium strategy increases with time. As the time goes on, the mean-variance investor will have less investment time to maximize his current utility, and he ought to invest lesser and lesser in risky asset, so that he could be quite certain on acquiring a satisfactory terminal surplus. In contrast, the time consistent investor will pay attention on all the utility functions over the whole planning horizon, and will apply a strategy that can optimize his utility once and for all time. Therefore, this time consistent investor will sacrifice his current happiness by holding part of his wealth in riskless bond in exchange to ensure sufficient budget, with a smaller chance of running deep in deficit, for the later investment in the future.

We can observe that the mean-variance control and consistent control converge as the time goes to expiry, since in single period model, the definition of equilibrium control will be the same as the definition of optimal control. Also, the value function of both strategies converges toward expiry because the investment time is lesser so the difference of the performance between both strategies becomes insignificant.

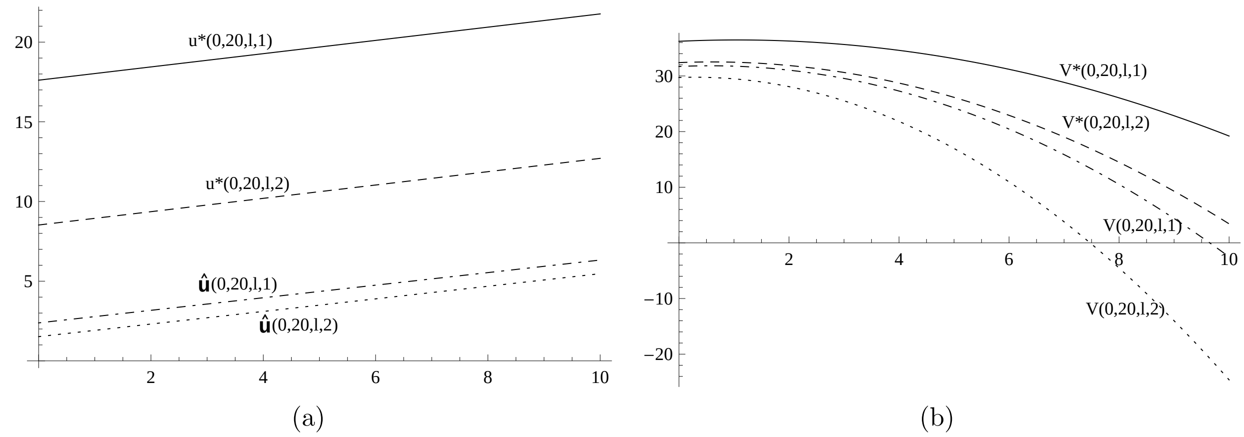

Figure 2: (a) The comparison between the equilibrium strategies $\hat{u}(0, 20, l, i)$ and the mean-variance strategy $u^*(0, 20, l, i)$ against the current liability for $i = 1, 2$. (b) The comparison between the equilibrium value functions $V(0, 20, l, i)$ and the optimal current utility $V^*(0, 20, l, i)$ against the current liability for $i = 1, 2$.

On the other hand, Figure 2 shows that the mean-variance strategy and the equilibrium strategy both increase with the liability. As the liability increases, the investor has more assets in hand to maximize his current utility since the initial surplus level is assumed to be constant; thus, more asset value encourages him from investing in risky assets to attain greater expected optimal utility and overcome the increasing liability.

In both Figures 1 and 2, we observed that the time-consistent investor make a more conservative investment than the mean-variance investor, because time-consistent investor sacrifice his current happiness to ensure a consistent return for the whole time horizon, but he have to give up the chance to invest more to attain greater current utility.

In both Figures 1 and 2, we observed that the equilibrium control and its equilibrium value function in bullish market are greater than those in bearish. It is reasonable because investor should invest more in bullish market and they normally feel more optimistic in bullish market, and thus greater current utility will be resulted.

6.2. *Comparison between the mean-variance distribution for equilibrium strategy and the efficient frontier for mean-variance strategy*

Every investor has different risk aversions at different market status; with no doubt, different risk aversions will lead to different mean-variance and equilibrium strategies. In Figure 3, for the investors using the mean-variance strategy, the expectation and variance of their own terminal surplus are still lying on the same efficient frontier no matter what their market-oriented risk aversions are. In contrast, for the time consistent investors, the expectation and variance of the terminal surplus are dependent on the risk aversions at different market states (see (39) and (40) and the system of ODEs in (27)-(36) satisfied by b_i, k_i, B_i, E_i and K_i), and therefore a shadow two-dimensional region will result, which represents the mean-variance distribution for time consistent investors (due to the limitation of computing infinite value, we only show the mean-variance distribution for both strategy with risk aversion between 0.01 and 10)

In Figure 3, the mean-variance distribution for mean-variance strategy is always above the mean-variance distribution for equilibrium strategy, which is due to the fact that the mean-variance investor aims to maximize the utility function, and hence he also maximizes the current mean-to-variance ratio of terminal surplus among all plausible strategies. The time-consistent investor chooses to give up the possible better current utility (and so higher value of mean-to-variance ratio) in return to keep a consistent satisfaction over the whole planning horizon. Therefore, the current mean-to-variance ratio of the mean-variance investor is obviously greater than that of the time consistent investor. As time goes on, the mean-variance distribution of both strategies moves down and left; moreover, they also approach to each other. Indeed, the mean-variance investor will have lesser and lesser time to increase the gap between his current mean-to-variance ratio and that of time consistent investors.

7. Conclusion

We here studied the mean-variance asset-liability management problem via the approach based on the time consistent equilibrium controls. We adopted the model first proposed by Chen et al. (2008), in which the coefficients in the dynamics of asset price and liability processes also depend on a Markovian modulated regime switching process. We had solved for the equilibrium control that, with no doubt, guarantees an extension of its equilibrium nature over any (time) subproblems. By applying the verification theorem of extended HJB in Björk and Murgoci (2010), we derived the extend HJB for our MVALM problem in Section 3. By utilizing a suitable *Ansatz*, we also established explicitly the solution of the corresponding extended HJB, and hence both the equilibrium control and equilibrium value function could be obtained as stated in Theorem 4.1. We had shown that the equilibrium value function is quadratic in the current liability and affine in the current surplus, while the equilibrium control is affine in the current liability, and the coefficients involved can be obtained by solving a system of first-order linear ODEs with some predetermined terminal conditions.

8. Acknowledgement

The third author, Phillip Yam, acknowledges the financial support from The Hong Kong RGC GRF 404012 with the project title, “Advanced Topics In Multivariate Risk Management In

Finance And Insurance", The Chinese University of Hong Kong Direct Grant 2010/2011 Project ID: 2060422, and The Chinese University of Hong Kong Direct Grant 2010/2011 Project ID: 2060444. Phillip Yam also expresses his sincere gratitude to the hospitality of both Hausdorff Center for Mathematics of the University of Bonn and Mathematicsches Forschungsinstitut Oberwolfach (MFO) in the German Black Forest during the preparation of the present work. The forth author, S. P. Yung, acknowledges the financial support from an HKU internal grants of code 200807176228 and 200907176207.

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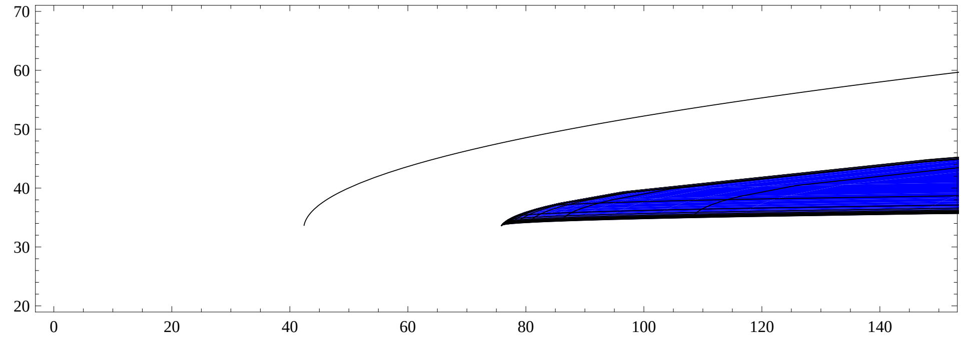
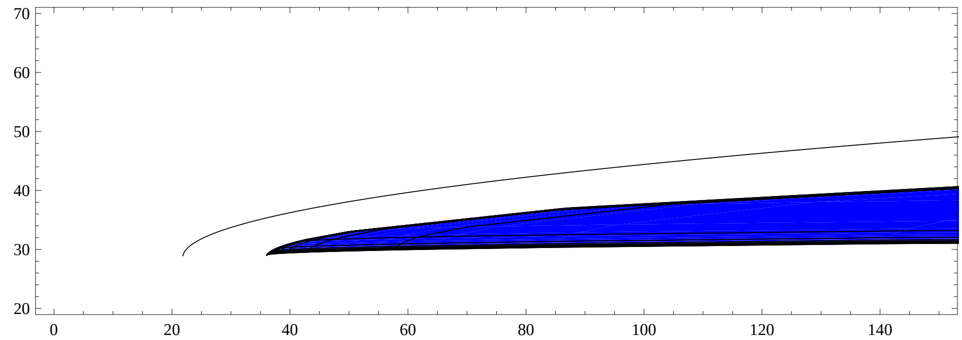


(a)



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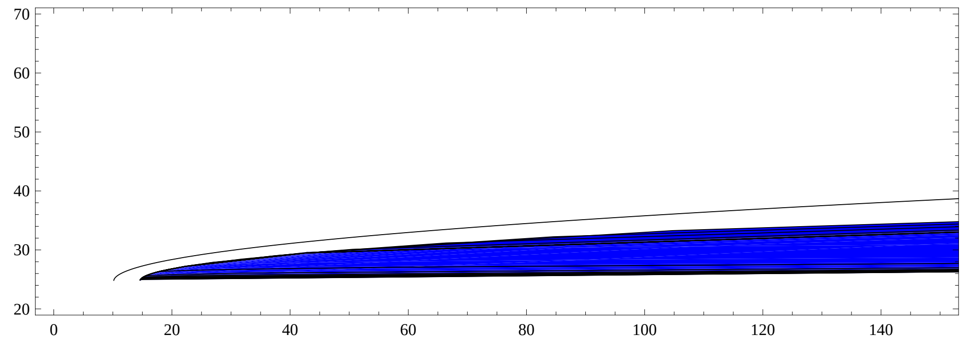

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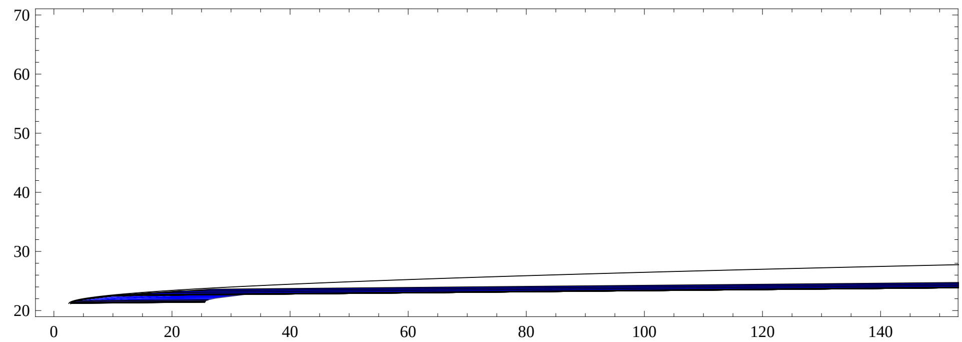

(d)

Figure 3: The comparison between the mean-variance distribution for equilibrium strategy and the efficient frontier for mean-variance strategy when the current market state is bullish with different initial time: (a) $t = 0$, (b) $t = 3$, (c) $t = 6$ and (d) $t = 9$, given the terminal time $T = 10$. The y-axis represents the expectation of terminal surplus, and the x-axis represents the variance of terminal surplus. The black line represents the efficient frontier for mean-variance strategy while the blue region represents the mean-variance distribution for equilibrium strategy