THE CONSTRUCTION OF GENERAL TERMS FOR SHAPE PATTERNS: STRATEGIES ADOPTED BY CHILDREN ATTENDING FOURTH CLASS IN TWO IRISH PRIMARY SCHOOLS

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DECLARATION

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of a PhD degree is entirely my own work, and that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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ABSTRACT

The construction of general terms for shape patterns: strategies adopted by children attending fourth class in two Irish primary schools

Aisling Twohill

Generalisation is considered by many to be a highly significant component of algebraic thinking (Carpenter & Levi, 2000; Kaput et al., 2008). In particular, constructing general terms for shape patterns supports children in reasoning algebraically about covariance and rates of change (Rivera & Becker, 2011). Generalisation is not mentioned within the current Irish Primary School Mathematics Curriculum, and shape patterns are not presented as learning activities, beyond simple repeating patterns explored with very young children (Government of Ireland, 1999). This thesis reports on a research study that explored the strategies Irish children adopted in seeking to construct general terms for shape patterns.

Within the context of task-based group interviews, patterning tasks were presented to groups of three or four children (aged nine or ten years old), where the children were asked to describe and extend the patterns, and to consider near and far generalisations. Following the interviews I analysed the strategies that children used to generalise from the patterns, drawing from the work of Lannin (2005) and Rivera and Becker (2011). Taking a hermeneutic phenomenological approach, I included identification and analysis of factors that contributed to children's strategy choices, which included explicit, recursive, numerical and figural approaches.

Findings evidence the potential of children to respond in complex and sophisticated ways to novel tasks, when rich engagement with the mathematics was facilitated through the context of a task-based group interview. The research study demonstrates the necessity for conceptualising children's thinking as multi-faceted whereby observations of relationships within patterns interact with attention to numerical and figural aspects. In addition, evidence is presented to support the contention that observations of structure exist along a continuum from numerical to figural. The study highlights the potential of task-based group interviewing coupled with hermeneutic phenomenological analysis in facilitating in-depth research into children's emerging mathematical thinking.

CHAPTER1: INTRODUCTION

> The end of human science research for educators is a critical pedagogical competence: knowing how to act tactfully in pedagogic situations on the basis of a carefully edified thoughtfulness (Van Manen, 1990, p. 8).

This thesis describes a research study that aims to explore children's strategy use as they seek to construct general terms for shape patterns. In this way, it is hoped that the findings presented in this thesis will contribute to the academic exploration of how children develop understanding in algebra, and also support policy makers and teachers in acting "tactfully in pedagogic situations" by both stimulating and supporting children as they navigate the rich and rewarding field of mathematics which is algebra (ibid., p. 8).

My focus on patterning commenced with my initial observations of the contrast between the early informal algebraic thinking demonstrated by my own children, and the challenge experienced by children in senior primary and secondary school when they encountered algebra within the school curriculum. My concern at this contrast is shared by others who identify a potentially negative role played by some approaches to the teaching of mathematics on children's propensity to reason flexibly (Boaler, 2016; Hiebert et al., 1997). Boaler (2009) describes how a view of mathematics that centres around rote-learned procedures and imitation of the teacher's technique may actively discourage many children from engaging in the flexible, imaginative thinking I witnessed in my own preschool children. In contrast, Askew (2012) describes his personal experience of encountering a mathematics which centred around problem-solving and inquiry when he taught with John Mason on the Open University Mathematics Foundation course. Askew describes how this approach which welcomed flexible thinking and figuring problems out reignited his love for mathematics and he concurs with Boaler (2009) who advocates for facilitating all children in experiencing mathematics that draws on their thinking rather than superimposing the methods and procedures demonstrated by a teacher or within a textbook.

In order to lay the foundations for this study the next section of this chapter draws on the ideas expressed by Boaler and Askew, and outlines my perspective on mathematics as a subject to be taught and learned, along with an explication of the position this research holds regarding the nature of knowledge. In subsequent sections, I present a context and rationale for the research study, and outline preliminary theory on the core concepts of algebraic thinking, patterning, developmental pathways and task-based group interviews.

Mathematics

Mathematics as a field of study has evolved over thousands of years, at least since a baboon fibula was used as a tally stick some 37,000 years ago in the Lebombo mountains (Huylebrouck, 1996). In a discussion on the Mathematical Tradition, Freudenthal (1973) tells us that by the end of the third millennium B.C. well-groomed elementary arithmetic and algebra existed in Babylonia, while Stewart (1996) points to the classification of a well-established field of mathematics into distinct branches by the Pythagorean Academy in the sixth century BC, in Greece. During the evolution of the field of mathematics, there was no single explosion of understanding whereby the individual in the Lebombo mountains millennia ago suddenly developed an insight into prime numbers which allowed him/her to encode vast amounts of information. Very many small steps have been taken as civilisations aimed to identify, comprehend and describe structure within multiple facets of their environment (Kilpatrick, Swafford, & Findell, 2001).

When considering what constitutes mathematics, particularly in the context of teaching, it is interesting to consider this evolutionary nature of mathematics. Conway & Sloane (2005) suggest that “rather than being viewed as a timeless edifice mathematics might more accurately be viewed as a complex growing tree" (p. 8). While children should be facilitated in exploring the rich heritage they inherit in the mathematics that has been explored and discovered heretofore, it may be more apt to present mathematics as a dynamic, evolving field to which they may contribute, as opposed to a dull artefact handed to them by all-knowing institutions. In this way, they will have an opportunity to engage with mathematics as a field of enquiry. When a nine-year-old discovers mathematical entities such as the existence of a constant relationship between diameter and circumference of a circle, this discovery is equally as genuine and original as it was for the first person to discover π.

The potential for all of us, including children, to discover mathematics suggests that it may be more appropriate to view mathematics as a body of knowledge contributed to by a variety of people from all walks of life rather than the lofty pursuit of elite academics. Boaler (2009) laments the categorisation of children into those who can and cannot "do maths" that occurs within English schools and emphasises that in some countries, such as Japan and Finland, children are encouraged to believe that *everyone* can achieve at mathematics. In Ireland there is no state mandated streaming and typically children are not segregated formally in schools according to assessments of mathematics attainment. However, at set points during a child's primary education the results of standardised assessments are shared with parents in order to inform them about their child's progress (Government of Ireland, 2011). While it may be beneficial to parents to be informed about their children's attainment in mathematics, summative results are by their nature short on detail, and many children may be discouraged by evidence of their average or below-average attainment. In research conducted as part of the National Assessment of Mathematics and English Reading, the response of the 4000 Sixth class participants to questions relating to their self-belief in mathematics would indicate that many of the children had a poor self-image in relation to mathematics, with 25.6% of boys and

41.5% of girls deemed to have a ‘low self-concept’ in relation to mathematics (Clerkin & Gilleece, 2010). Throughout this study, I drew on an understanding of mathematics as available to all, and while I gathered information relating to the children’s mathematics attainment I did not do so until after I had completed the interviews. In this way I hoped to avoid implications of unconscious bias, for example any possibility that I would protect children I might expect to struggle by limiting their engagement with the rich patterning tasks that were planned.

The vision of mathematics that underpins this study is also of a subject that is discoverable through interaction with rich tasks, where a teacher would facilitate children’s learning through active engagement and refrain from leading children to replicate the teacher’s interpretation. This vision draws heavily from the sociocultural perspective, wherein the social and cultural are emphasised as “inseparable contexts in which learning can be understood” (Dunphy, Dooley & Shiel, 2014, p. 43) and where collaboration and the establishment of joint understanding are key. Sociocultural perspectives embrace theories of mathematics education that see children’s sense-making and developing thinking as generated by social activity (Lerman, 2000). Children are active agents in constructing understanding through engaging with their environment, which includes their peers. During the task-based group interviews that the children engaged in there is no hard line between extant knowledge and the understanding they developed as they teased out solutions with their peers in the interview setting. Rather their understanding, their interpretations, and their responses grow and develop during the engagement with the tasks, and require research methods that reflect an understanding of this fluidity.

Knowledge, Reality and Research

The research approach adopted in this study is hermeneutic phenomenology, and in this section I will explicate the philosophical assumptions I have made regarding the nature of enquiry, and of reality in adopting this approach. Van Manen (1990) juxtaposes the human science approach to research and the approach appropriate to the natural sciences. Natural science research methods involve detached and objective perspectives, experiments, and quantitative analysis, and when applied to social science result in instrumental knowledge principles, or “rules-for-acting” (p. 156). In contrast, human science methods, such as hermeneutic phenomenology involve “description, interpretations, and self-reflective or critical analysis” (p. 4), and thereby generate deep understanding and perspectives that are situated in the lived experience of the participants and the phenomenon of interest. Phenomenology is a rigorous and structured approach to explicate the meaning within a phenomenon, to uncover the structures underpinning a lived experience. In this study the experience is the children’s responses to the mathematical tasks presented to them and in adopting a phenomenological approach I attribute value to the multi-layered complexities of the meanings underpinning the children’s responses, and the factors contributing to their thinking, and to their responses. As the phenomenology I adopt requires an unpicking of the children’s utterances, gestures and constructions it is hermeneutic in nature, as it relies on my careful, open and rigorous interpretation of the children’s communications. Hermeneutic phenomenology acknowledges that research is interpretive, and that within the social sciences reality is described through the lens of the author. In Chapter 3 I expand further on these themes of hermeneutics and phenomenology and their contribution to this study.

Algebra: the International Context

> If there is a heaven for school subjects, algebra will never go there. It is the one subject in the curriculum that has kept children from finishing high school, from developing their special interests and from enjoying much of their home study work. It has caused more family rows, more tears, more heartaches, and more sleepless nights than any other school subject. (Anonymous editorial writer, cited by Kilpatrick and Izsák, 2008, p. 3).

In contrast with the view expressed above of mathematics as attainable by all, traditionally in many countries algebra has created barriers for some children from progressing to high levels of educational success (Cai & Knuth, 2011a). Cai and Knuth point to the incomplete understandings and lack of preparation children have received before commencing formal algebra typically after six to eight years of school. While algebra has been taught in primary school in some countries since the 60s, for example China and Russia, it is only in the last decade or two that the early algebra movement has gained pace, with publications such as *Algebra in the Early Grades* (Kaput, Carraher & Blanton, 2008) and *Early Algebraization: A Global Dialogue from Multiple Perspectives* (Cai & Knuth, 2011b; Mason, 2008).

In Chapter 2, I explore and discuss the terms 'algebra', 'algebraic thinking', 'algebraic reasoning' and 'early algebra' but for clarity at this point I would like to suggest a definition of *formal* algebra as "the science of equation solving" (Kieran, 2004, p. 139), including the manipulation of abstract symbols, and the application of symbolic expressions to solve problems. In contrast, Kieran (2004) identifies algebraic thinking as ways of thinking which include "analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modelling, justifying, proving, and predicting" (p. 149). During the last twenty years, many publications advocating 'early algebra' suggest that children's success in engaging with formal algebra is improved by opportunities to develop their algebraic thinking early in their education (Cai & Knuth, 2011a; Kaput, 1998; Carpenter, Franke & Levi, 2003). The research presented in this thesis overlaps with the field of early algebra, but is positioned within the broader field of algebraic thinking, as I explored the emerging thinking of participants attending senior primary school.

While there is a lot of support for a theory which advocates the facilitation of young children in thinking algebraically, Carraher and Schliemann (2007) caution that "the research basis needed for integrating algebra into the early mathematics curriculum is still emerging, little known, and far from consolidated" (p. 671). Progress in improving students' engagement with formal algebra appears to be slow. In 2003 the Programme for International Student Assessment (PISA 2003) conducted tests of mathematical attainment of 15-year-olds from 30 member countries of the Organisation for Economic Co-operation and Development (OECD). A review of the findings by OECD (2009) found that for students from all countries, the items assessing either Algebra or Measure caused generally more difficulty than items assessing Data, Geometry, and Number[1].

Irish Context

The current Primary School Mathematics Curriculum (PSMC, hereafter) was introduced for Irish schools in 1999 (Government of Ireland, 1999). The mathematical content of the PSMC is presented in five strands, namely Number, Algebra, Data, Measure and Space and Shape. While the publication of the curriculum predates many of the publications of research into 'early algebra', the algebra strand is introduced in Junior Infants, and thus fulfils the core recommendation of facilitating children in thinking algebraically from early on in their education. In 2003 the Inspectorate of the DES investigated how the revised primary school curriculum was being implemented by teachers. They found that the implementation of the algebra strand was 'good' in most classrooms (Government of Ireland, 2005). However, the findings of national and international assessments of attainment paint a different picture.

In 2014, the National Assessment of Mathematics and English Reading (NAMER 2014) assessed the attainment of almost 4000 Sixth class pupils. Most of the children in the Sixth class cohort (74%) could solve multi-step word problems and "demonstrate understanding of a letter as a placeholder in algebraic expressions, and complete two-step number sentences involving addition and subtraction" (Shiel, Kavanagh & Millar,

[1] The PISA 2003 assessment items were classified under five general mathematics curriculum topics, namely Number, Algebra, Measurement, Geometry and Data (OECD, 2009).

2014, p. 38). 42% succeeded in translating word problems into number sentences and vice versa and 15% achieved a mastery of algebra which enabled them to evaluate “linear expressions and one-step equations” (ibid., p. 38). It is a cause for concern that 85% of the children who are within months of entering secondary school failed to evaluate a one-step algebraic equation, and that 58% could not translate between word and number sentences. As a result it would seem that problem-solving strategies which involve deriving a number sentence from a word problem are not available to them (Polya, 1990; Carpenter & Levi, 2000). This situation may have contributed to the poor success of children in this study in the process skill of ‘Apply & Problem-Solve’. Of questions posed to children in Sixth class requiring an application of problem-solving skills, only 49% were answered correctly, which is the lowest percent correct score of the five process skills (the other four being Understand & Recall, Implement, Integrate & Connect, and Reason) (Shiel et al., 2014).

In 2011, the Trends in International Mathematics and Science Study (hereafter TIMSS 2011) measured the mathematics attainment of 4560 children in Fourth class in Ireland. In the design of the mathematical assessment presented to participating children, the researchers targeted the cognitive processes of Knowing, Applying and Reasoning (Eivers and Clerkin, 2012). Mean scores were published as a subscale under each cognitive process. Reasoning was identified as including “intuitive and inductive reasoning based on patterns and regularities that can be used to arrive at solutions to non-routine problems” (Mullis, Martin, Ruddock, O'Sullivan and Preuschoff, 2009, p. 45). The Irish cohort achieved a mean score of 510 on this subscale which was significantly lower than the corresponding overall mean of 527 achieved for mathematics (Eivers and Clerkin, 2012).

There seems to be a mismatch between the perception of how algebra is taught and the findings from research in terms of how children are learning. Corcoran (2005) discusses the disparity between what is perceived to be taught and what is in actuality learned in Irish primary classrooms. She suggests that a variety of contributing factors may be resulting in a situation where the curriculum as designed is not implemented in classrooms. Similarly, Remillard (2005) outlines the many processes involved in implementing a published curriculum which has been designed for use by teachers. She states that to investigate the 'enacted curriculum' requires consideration of how teachers' pedagogical content knowledge, subject matter knowledge and personal beliefs impact on their construction of the curriculum within lessons. In using curriculum documents teachers not only draw from the contents but also bring to them a personal interpretation.

The Role of Teaching in the Development of Algebraic Thinking

Mason (2008) contends that before commencing school children already demonstrate the ability to imagine and express, to focus and de-focus, to specialize and generalize, to conjecture and convince, to classify and characterize and these skills are fundamental to algebraic thinking. Radford (2011) suggests education should facilitate children in developing algebraic thinking and Schifter, Bastable, Russell, Seyferth and Riddle (2008) provide examples of how this may occur in classrooms with very young children. Central to the 'early algebra' publications of Kaput and Blanton (1998, 2008) was a theory of algebraization of the curriculum, where teachers aim at all times to seek out opportunities for children to consider the general case.

Unfortunately, traditionally what happened in classrooms in Ireland did not bear much resemblance to this ideal of engaging children in developing higher order thinking regarding the generalisability of the observed properties they encounter. Delaney (2010) and Eivers et al. (2010) indicated concerns about the application of mathematical knowledge. Their findings suggested a situation where much mathematics was taught by rote and as a 'given', which children were not encouraged to question. In particular, Eivers et al. (2010) found that for the Sixth class cohort of children involved in the NAMER 2009 study, "whole class textbook-based teaching still predominates" (p. 64). Additionally, only 69% of children in this cohort indicated that they often or always talked about a mathematics problem before doing it and only 8% reported that they often or always worked in pairs or small groups. The findings suggest a model of mathematics teaching which was overly reliant on textbooks and lacking in opportunities for mathematical discussion. We know from research such as Conway and Sloane (2005) and Eivers et al. (2010) that excessive textbook use may be counterproductive to the development of children's reasoning skills and that discussion about how to solve problems supports such development.

A comparison of the more recent findings from NAMER 2014 to those from NAMER 2009 indicates that some progress may have been made in relation to children's mathematics thinking and classroom practices which support children's reasoning skills. In the Context Report published in connection with NAMER 2014, Kavanagh, Shiel, Gilleece and Kiniry (2015) emphasise that while progress has been identified in children's mathematics attainment between NAMER 2009 and NAMER 2014, there are persistently disappointing findings in relation to problem-solving and higher order thinking. NAMER 2014 presents findings from teacher interviews, wherein problem-solving and reasoning were identified by teachers as the aspects of mathematics for which they felt most in need of support. It is alarming, therefore, to think that for many children a flexibility of thinking which supports their understanding and application of algebraic reasoning is being underdeveloped, rather than nurtured and enhanced through classroom practices that focus on reasoning.

The Need for Algebra in Primary Schools

In considering the learning outcomes of children in Irish primary schools, it is pertinent to consider the children's needs and the purposes of the algebra curriculum. In educational terms, opportunities to engage in algebraic thinking support children's developing understanding in mathematics (Schifter et al., 2008). Internationally, there is concern that formal algebra when encountered, often at 12 or 13 years of age, hinders the progress of some students in mathematics (Kaput, 1998; Mason, 2008; Cai and Knuth, 2011a). There is no research into the impact on children in Ireland of strong or fragile skills of algebraic reasoning on entering secondary school. In order to consider whether algebra may be important for the study of mathematics in secondary school, I will now consider the content of the Junior Certificate[2] syllabus and also sample Junior Certificate examination papers.

The Junior Certificate mathematics syllabus consists of five strands (State Examinations Commission, 2013). Algebra comprises one strand and a second strand entitled 'Functions' is identified as progressing from the algebra strand of the primary school curriculum. Within the remaining three strands (Number, Statistics and Probability, and Geometry and Trigonometry) there are many components for which algebra is required. Examining the Junior Certificate examination of 2012, it is evident that algebra plays a crucial role in children's success in the Junior Certificate mathematics examination. In 2012 questions dedicated to algebra and functions comprised 35% of the marks in the examination at higher level (State Examinations Commission, 2012). If we extend the investigation to consider questions for which algebra was required but for which algebra was not the focus, for example the generation and manipulation of expressions in trigonometry questions, 49% of the marks in the examination depended on students' successful engagement with algebraic concepts. Therefore, when children leave primary school, they are at a distinct disadvantage if their ability in algebraic reasoning is

[2] In Ireland students sit formal examinations after three years of attending second-level school. This examination is referred to as the Junior Certificate, and mathematics is a compulsory subject for all students. Students have the option to follow a 'Higher Level' or 'Ordinary Level' program in mathematics, and to sit the corresponding examination.

underdeveloped, and there is a possibility that algebra may fulfil the role of ‘gatekeeper’ in students’ choices of engagement in mathematics at ordinary or higher levels.

Furthermore, in all area of mathematics, the ability to reason algebraically allows children to identify values as entities within an expression rather than feeling pressure to arrive at a single numerical answer. Without the pressure to seek closure in a mathematical expression, students can employ the expression to work for them as a tool or device in the solution of a problem (Gray and Tall, 1994). Such control may give students confidence in their ability and render mathematics a less daunting and thus more enjoyable pursuit. This, I believe, should be the aim of the primary mathematics curriculum, to nurture skills, present but underdeveloped, of the 4-year-old child and to facilitate the development of sophisticated and confident skills in algebraic reasoning.

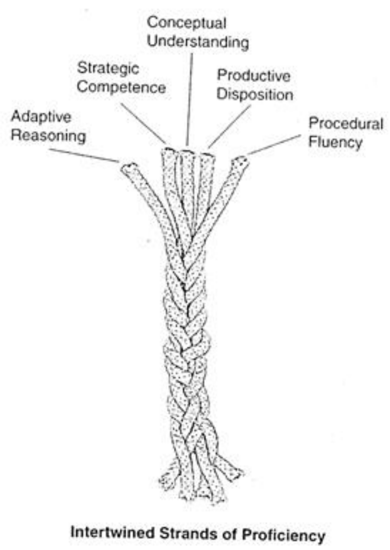
There is uncertainty therefore regarding the extent to which children attending Irish primary schools are developing algebraic reasoning skills and the results of international assessments of the attainment of Irish students do not paint a very positive picture. Algebra plays a significant role in supporting students’ engagement in mathematics education. Further research is warranted therefore to explore how children’s algebraic reasoning skills are developing in primary school in Ireland.

Developmental Pathways

When discussing curricula for algebraic thinking through primary and secondary school, it may be pertinent to consider how children’s thinking develops over time. One approach to this consideration is to plot a developmental pathway for children’s algebraic thinking (Dunphy, Dooley & Shiel, 2014). In this study I endeavoured to plot such a pathway for children’s developing thinking in patterning and in this section I introduce some of the theory relating to such a mechanism for tracking children’s learning.

There are many and varied theories regarding the consequences of plotting a developmental pathway. In order to consider how a developmental pathway may be envisaged, it is useful to firstly discuss what it is to learn and in particular to learn mathematics. Schoenfeld (1992) presents a range of conceptualizations of mathematics, highlighting two extremes. Mathematics can be seen as a body of facts and procedures to be mastered or the polar opposite, a discipline emphasising pattern-seeking on the basis of empirical evidence. These two terms, ‘mastered’ and ‘pattern-seeking’ are particularly relevant in a discussion of what mathematics is in order to investigate children’s development. If mathematics is to be mastered as a body of facts and procedures, then it may be appropriate to outline a deterministic linear pathway wherein growth points are advocated for specific age or class levels. This would not correlate however with my experience as a teacher or with my view on learning. Rather I would concur with the dynamic view of mathematical proficiency consisting of the five interrelated and interdependent strands as suggested by Kilpatrick, Swafford and Findell (2001), presented in Figure 1.1.

Figure 1.1. The five strands comprising mathematical proficiency (Kilpatrick et al., 2001, p. 117)



In Chapter 2, I return to this multi-strand understanding of mathematical proficiency as I seek to outline the myriad facets of children's thinking which must be attended to in the presentation of a developmental pathway in algebraic thinking.

Task-based Group Interview

When I commenced this research study, I intended to interview children individually, adopting a clinical interview method, as I endeavoured to get beyond "the child's initial fear or defensiveness" and unpick children's thinking without the barriers of cultural differences (Ginsburg, 1997, p. ix). In many recent studies clinical interviews have been utilised to investigate children's thinking in mathematics, including research by Rivera and Becker of the algebraic thinking of children attending sixth grade in California (mean age of 11) and the Early Numeracy Research Project (ENRP) in Australia (Rivera & Becker, 2011; Gervasoni, 2005). Bobis et al. (2005) researched the impact that conducting clinical interviews as part of the ENRP had on the teacher's knowledge for teaching. While they found no significant increase in the subject matter knowledge, they found that the teachers "increased their knowledge of how children learn mathematics in general, they had a much clearer picture of their own children's understanding" (p. 42). Bobis et al. also found in research emanating from the New Zealand Numeracy Development Project (NDP) that the framework of growth points and the interviews used helped to move the emphasis of professional development from "the notion of children carefully reproducing taught procedures to an emphasis on children's thinking, with teachers as researchers" (p. 50).

As I will explore in detail in Chapter 3, while clinical interviews offer great potential for rich data collection, I was concerned that individual interviews in this context may not fulfil the requirements of this research. Building on advice from Dolores Corcoran who examined my Progress Viva, along with feedback from conferences of the British Society for Research in the Learning of Mathematics (BSRLM) and the British

Congress of Mathematics Education (BCME), I researched alternatives to individual clinical interviews. From the advice I received, my analysis of this first pilot, and my personal reflection on the pilot, I felt that richer mediation was necessary to truly support children in operating within the highest cognitive ranges of their Zones of Proximal Development (ZPD) (Vygotsky, 1978). To fulfil this need, I turned to task-based group interviewing and drew heavily from the writings of Goldin (2000) and Mueller, Yankelewitz and Maher (2012). A detailed explication of this change in approach is outlined in Chapter 3 along with presentation of the rich affordances of the task-based group interview as a research instrument.

Mediation

A central theme of task-based interviewing is the mediated interaction between each child and the mathematics. John-Steiner and Mahn (1996) present an overview of mediation as it was characterised by Vygotsky (1978) wherein semiotic means such as language, tools and tasks connect external representations of a concept with internal understanding. Such mediators are central to the child's appropriation of the concepts represented, and must be acknowledged and foregrounded in researching how children understand mathematical concepts. In Chapter 2 I expand upon the role of mediation and the various means by which the children's thinking was mediated.

Role of My Research

Mathematics is both increasingly present and rapidly evolving in international society, for example, in the algorithms that underpin economic decisions and the formulae which power computer systems. Facilitating all children in developing mathematical skills which will allow them to engage with new areas and ideas in mathematics is now an essential role of education (Kaput, 1998). "Algebra lies at the heart of taking control of as-yet-unknown situations" (Mason, 2011, p. 564) and has a crucial part to play in the mathematics education which is required if we wish to give all children access to powerful and evolving mathematics. Additionally, thinking algebraically supports children's mathematical thinking in many ways. Carraher and Schliemann (2007) state that algebraic generalisations are necessary for one to understand arithmetic deeply. Arcavi (2008) extends this view to suggest that knowledge of algebra, in particular, is vital for the "development of a critical appraisal of the large amount of information and arguments with which we are confronted at all times" (p. 37). The aim of my research is to explore the algebraic thinking of children attending primary school in Ireland, and to include findings from this research with international research in the area of algebraic thinking in order to make recommendations for future review of the PSMC.

Overview of Thesis

In this chapter I have introduced the area of mathematics within which this study is situated. I have presented a rationale for this research, drawing from national and international research. I have explicated many of the terms and concepts relating to algebraic thinking. The focus of this study is children's response to patterning tasks and in Chapter 2 I fully outline the role patterning plays within the broader area of algebraic thinking. Also in Chapter 2 I attend to the existing research literature in the field of algebraic thinking, generalisation and developmental pathways including their potential role in the assessment of children's thinking. In Chapter 3 and 4 I outline the methodological decisions I made in carrying out this research, and describe the relevant theory underpinning my decisions. Chapters 5, 6 and 7 contain my analysis of the children's responses to the patterns and the thesis concludes with a synthesis of this analysis in Chapter 8. Also in Chapter 8, I identify limitations to the findings presented, and I point to further study that may build upon the research presented here.

CHAPTER 2: LITERATURE REVIEW

Introduction

In 1676, Issac Newton paraphrased the twelfth-century philosopher, Bernard of Chartres, when he said "if I have seen further it is by standing on the sholders of Giants". In undertaking research in the field of mathematics education, it was imperative that I drew from and built upon existing work in the field. In this chapter, I present a review of the literature most relevant to my research, and I aim to show how this literature has informed and supported the direction I took, and decisions I made.

As the focus of this research study is the algebraic thinking of Irish children, in this chapter I will discuss my interpretation of the terminology surrounding algebra and also some constituent elements of algebraic thinking of relevance to primary school. The specific aspect of algebraic thinking examined within this thesis is the construction of general terms from shape patterns, and a comprehensive discussion of the interplay between patterning and algebraic thinking is included in this chapter. I will also explore the content of the algebra strand of the Irish Primary School Mathematics Curriculum (PSMC) and critique the representation of algebra therein in light of the preceding discussion. Building upon my review of the literature relating to algebraic thinking, patterning and the construction of a general term, I discuss theories of developmental pathways and present a preliminary developmental framework of growth points in patterning. Also in this chapter I discuss the role a task-based group interview may play in investigating children's algebraic thinking and to conclude I present my research questions.

Algebra

To commence a discussion regarding children's algebraic thinking, it is pertinent to consider the terminology in use regarding algebra and to clarify the meanings of terms used within this research. To many the term 'algebra' is synonymous with symbol manipulation and abstracted arithmetic, an embodiment of algebra which I will refer to as 'formal algebra'. There are however many other perspectives on algebra and in this section I would like to present my conceptualisation of the terminology to clarify its use within my research. I will commence with a discussion of the role that algebra traditionally played within education and proceed to consider 'algebraic thinking', and the 'Early Algebra' movement which arose from research indicating the benefits of facilitating children in reasoning and thinking algebraically from a young age. The ideas encompassed by Early Algebra have great relevance to this research as they draw attention to children's capacity for algebraic thinking, and highlight when children have demonstrated such capacity before encountering instruction aimed at developing their algebraic thinking. Nonetheless, this research study involved children in the senior end of primary school, and it is thus appropriate to position discussion and findings within the broader field of algebraic thinking.

Traditional teaching of algebra

In the introduction of this report, I referred to traditional or formal approaches towards the teaching of algebra as including the manipulation of abstract symbols, and the application of symbolic expressions to solve problems, that is "the science of equation solving" (Kieran, 2004, p. 139). Some mathematical tasks involve the manipulation of abstract symbols without a context while others ask students to apply symbolic expressions to solve a problem. Figures 2.1 and 2.2 present examples of these two contrasting tasks, both focusing on algebraic thinking, but employing contrasting approaches.

Figure 2.1. An example of a task involving the manipulation of abstract symbols without a context

Factorise $4xy - 6x^2y^2$

Figure 2.2. A task where students are asked to apply symbolic expressions to solve a problem[3]

A company employs two drivers, John and David. Each has use of a company car and small van. The company buys €30 worth of Toll Tags for each driver. Each time that a vehicle goes through the M50 Toll, a charge will be deducted from the Toll Tags. John goes through the M50 toll five times in his car and four times in his small van. He then has €7.90 remaining on his Toll Tags. David goes through the M50 Toll twice in his car and six times in his small van. He then has €8.40 left on his Toll Tags. Calculate how much it costs for a car and for a small van to go through the M50 Toll (State Examinations Commission, 2013, p. 10).

In the example given in Figure 2.1, the task is to find factors of the expression such that possible solutions may be found for 'x' by manipulating the abstract symbols in appropriate ways. For example, knowledge of the distributive property might indicate to students that 2xy may be isolated as it is a factor of both terms, 4xy, and $6x^2y^2$. The skills involved in such context-free manipulation of abstract symbols have long occupied a central role in formal algebra, and in some cases educational establishments continue to value these specific skills. The University of Limerick (UL), for example, requires incoming undergraduate students on science-based and technology-based degree courses to complete a test purporting to diagnose their preparedness for the service mathematics modules of their courses (Treacy, Faulkner & Prendergast, 2016). Eight of the 40 questions relate to algebra, and all algebra questions involve decontextualised manipulation of abstract symbols. In the UL diagnostic test there are no questions to assess students' propensity to mathematise real life situations as required in the example given in Figure 2.2 above.

Kieran (1996) presents a model for deconstructing 'algebra' into three constituent elements: generational activities, transformational activities and global meta-level activities. Generational activities involve the production of algebraic objects, for example expressions of generality in arithmetic, expressions of generalities in patterns,

[3] This problem is taken from a Junior Certificate paper, used in 2013 in Irish state examinations for students aged 15 or 16. In solving this problem, a student sitting the examination was expected to use symbolic expressions, and a student was required to present simultaneous equations to achieve partial credit (State Examinations Commission, 2013a, p. 22).

or expressions containing unknowns that represent problems to be solved. Transformational activities involve manipulation of abstract symbols in order to simplify, expand and/or find solutions. Global meta-level activities involve the use of algebra as a tool within other areas of mathematics, and beyond mathematics, such as "problem solving, modelling, finding structure, justifying, proving and predicting" (p. 272). Of these three constituent elements, the manipulation of abstract symbols required in order to solve the example given in Figure 2.1 above, and also the algebra questions of the UL diagnostic test, fall within the transformational activities alone.

However, the example given in Figure 2.2 above requires generational and transformational activities, as a student must generate expressions for the toll charges incurred by both drivers. S/he must then manipulate the expressions in order to figure out how much each vehicle costs as it passes through the M50 toll booth. It the student were then asked to identify an optimal amount which the company could budget for a collection of drivers with varying driving patterns, s/he would engage in the global meta-level activity of exploring the underlying mathematical structure of a situation in order to answer conjectural questions (Kieran, 1996).

Watson, Jones and Pratt (2013) discuss the important role played by algebraic manipulation for students. Manipulation of symbolic expressions facilitates students in expressing the relationships between quantities in ways which allow them to know more about the relationships. The difficulty occurs when the skills inherent in expressing the relationships between quantities are taught without meaning, as with the example given in Figure 2.1 above. Watson et al. contend that the teaching of algebra in schools needs to build upon understandings of number and to allow students to utilise algebra as a tool in reasoning within mathematics. They suggest that activities focused on facilitating algebraic reasoning skills should be rooted in meaningful situations such as the example given in Figure 2.2, and that activities involving context-free abstract symbol manipulation should be introduced when students understand that "expressions stand for relations between numbers" (p. 37). If the teaching of algebra focuses largely on manipulation of expressions without meaning, Watson et al. (2013) suggest that many students are turned off mathematics by the challenge such abstraction presents.

Kaput (1998) discusses a similar experience among students in the United States of America and he presents a dichotomy between two dimensions of algebra in the education system. Kaput distinguishes between 'Algebra the Institution' and 'Algebra the Web of Knowledge and Skill' and he suggests that it is necessary to be cognisant of the presence of both in order to reform the mathematics education of many students. 'Algebra the Institution' includes the course materials relating to algebra, the cultural expectations of success in algebra and also the perceived relationship between success in algebra and overall academic ability. This institutionalised algebra creates an academic capital which acts as a gatekeeper to exclude many from the highest reaches of academic success (ibid.). Similar references are found in literature from other jurisdictions. Cai and Knuth (2011a) discuss the international concern regarding the gatekeeper role played by algebra and Mason (2008) describes algebra as a 'watershed' because it is on encountering algebra that many students decide that mathematics is not a subject in which they will succeed, and thus turn away from pursuing further study.

In the Irish context, a perception that algebra is only accessible to very high-functioning students may confer it with an undue high degree of status. In a study involving mathematics students at NUI Galway, Pfeiffer (2009) found that third-level students of mathematics were more convinced of validity in arguments which involved algebra or evidence of algebraic thinking, even when the proofs were in fact incorrect. The power of algebra, in the inherent ability to transform a seemingly inaccessible puzzle into a solvable equation, seemed to be a source of intimidation. There may, however, be some grounds for students to view algebra as less accessible than other areas of the mathematics curriculum. As mentioned in my introduction, in PISA 2003 the items assessing Algebra and Measure generally caused more difficulty for participants than items assessing Data, Geometry, and Number (OECD, 2009). Kieran (1996) and Watson, Jones and Pratt (2013) suggest that the relative level of challenge experienced by students and the sense of awe students demonstrated with regard to algebraic proofs may be typical products of traditional teaching methods. In the next section I explore an alternative to the traditional approach of symbol manipulation, whereby the focus of teachers and children is on ‘thinking algebraically’.

Algebraic thinking

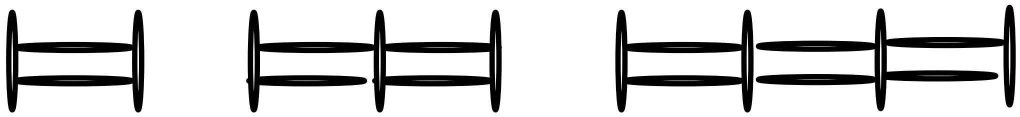
In her seminal paper presented at the 8th International Conference on Mathematics Education, Kieran (1996) defines algebraic thinking as “the use of any of a variety of representations in order to handle quantitative situations in a relational way” (p. 275). Kieran states that much of the abstract symbol manipulation traditionally emphasised in algebra may be at times distinctly procedural and not require ‘algebraic thinking’ whereas the mathematisation of real-life situations, or the use of algebra in other fields of mathematics and science require a way of thinking that involves change and relationships, and is thus distinctly algebraic. In a similar vein, Blanton and Kaput (2004) present their conceptualisation of algebraic reasoning as “a habit of mind that permeates all of mathematics and that involves students' capacity to build, justify, and express conjectures about mathematical structure and relationship” (p. 142). Combining the perspectives of Kieran, and Blanton and Kaput, I posit that algebraic thinking is a broad and multi-faceted way of thinking (or habit of mind), that facilitates the application of observations and understandings about change and relationships in order to explore structure. In subsequent sections of this chapter I discuss generalisation as a core aspect of algebraic thinking, within the domain of functional thinking, and specifically patterning.

Attempts to define algebraic thinking often falter over the boundary between arithmetic and algebra. Radford (2012) asserts that "there is something inherently arithmetic in algebra and something inherently algebraic in arithmetic" (p. 676), but they are far from identical and it is useful to highlight what differentiates the two. For an activity to be considered algebraic, the child must make a leap from the exploration of the pattern to the identification of a generalisation (Radford, 2011). Additionally, Radford (2012) highlights the algebraic action of operating upon an unknown. When simplifying an expression of the form Ax+D=B, it is possible to subtract D from B and divide the answer by A without engaging in thinking which is algebraic. With an expression such as Ax+D=Bx+C, however, an arithmetic approach is not sufficient and the solution of this expression requires an algebraic approach where one must operate upon an unknown quantity (ibid.). Similarly, for thinking to be algebraic, it must involve "*indeterminate* quantities conceived of in *analytic* ways" (Radford, 2011, p. 310, my emphasis).

Blanton, Brizuela and Stephens (2016) identified four key practices of algebraic thinking as: a) generalising, b) representing, c) justifying, and d) reasoning with generalisations, emphasising that these practices must focus upon structures and relationships. Blanton, Brizuela, Murphy Gardiner, Sawrey, & Newman-Owens, (2015) presented the following three content domains within which children may apply the key practices of algebraic thinking: a) generalised arithmetic, b) equations, and c) functional thinking. Functional thinking embodies an approach that sees functions as descriptions of relationships about how the values of some quantities depend in some way upon the values of other quantities (Chazan 1996). Blanton et al. (2015) highlight the relevance of functional thinking to the algebraic thinking of primary school children, stressing the connection between functional thinking and the four key practices of algebraic thinking highlighted above.

In shape patterns, children are asked to discover or explore a function that relates the number of elements of some component of a term, to the position of that term in the pattern. For example, in the pattern of fences depicted in Figure 2.3, the number of posts in each fence is a function of the position number (x), where the specific function is $f(x)=3x+1$.

Figure 2.3. A pattern of fences, wherein the number of posts in each fence is a function of the position number of the fence



Blanton et al. (2015) highlight the role of functional thinking in children's algebraic thinking by stating that functional thinking includes generalisations of co-varying quantities and their relationship; representations of these relationships, and reasoning with the relationships in order to predict functional behaviour. Concurrent with the definition of functional thinking above, many of these constituent skills of functional thinking focus on quantities, and the relationship between quantities. For children to explore these relationships, in order to define and apply functions, their thinking is supported by broad and multi-faceted observations of structure within patterns. As I will outline later in this chapter, functional thinking, and specifically shape patterning, is absent from the PSMC, and it is therefore highly improbable that most children attending Irish primary schools will have engaged with activities designed to develop functional thinking.

Early algebra

In order to lessen the 'gatekeeper' effect of algebra, considerable research has been conducted internationally into commencing engagement in algebraic thinking early in children's education (Mason, 2008). The 'Early Algebra' movement advocates that when children experience mathematics teaching which facilitates them in reasoning and thinking algebraically, the children benefit both in terms of the transition to formal algebra and in their computation skills in primary school (Kaput, 2008; Mason, 2008; Radford, 2011; Kieran, 2007; Cai & Knuth, 2011a). Kaput (1998) discusses the similarities between language acquisition and early development of algebraic reasoning skills. Language learning is most successful when commenced early, and particularly "language learned before puberty is learned without an accent and is deeply integrated with one's patterns of thinking" (p. 4). Kaput suggests that for children to develop algebraic reasoning skills with which they can unlock many sophisticated areas of mathematics, it is important that they engage with algebraic processes over a length of time and that this engagement is purposeful (ibid.).

Carpenter, Levi, Franke and Zeringue (2005) make the case for the early introduction of algebra as a necessary contributor to the development of relational thinking. The suggestion is that when children learn to see mathematics as a study of patterns and relationships, rather than the pursuit of answers, skills are developed which enhance children's learning in both algebra and arithmetic. When children understand and can apply the underlying properties of mathematical operations, arithmetic is made easier for them and their understanding is broadened beyond the computational (ibid.; Gray & Tall, 1994). Gray and Tall (1994) also warned that learners who fail to perceive algebraic expressions as other than requests for an answer, encounter difficulty when unknowns are introduced and an answer is not achievable.

In advocating such a radical change from the traditional view of algebra as appropriate for high achieving secondary school students, to a view of algebra for all and from the beginnings of education, two fundamental questions must be addressed.

- Do young children possess the necessary skills to succeed when encountering algebra?

- What is the ‘algebra’ that is appropriate for children in this age-group of 4 to 12 years of age?

In consideration of the first question, Mason (2008) outlines powers which are present in young children and which are relevant to algebraic thinking including specializing, generalising, conjecturing, convincing, and expressing. Similarly, Hewitt (2009) discusses how very young children learn rules in their acquisition of language, how they generalise these rules and apply them in new contexts, sometimes incorrectly. Mason (2009) asserts that “young children are able to generalise, because without it they could not function in the world and certainly could not grasp language” (p. 159). It is possible however that not all children are adept at applying the ‘spontaneous concept’ of generalisation which underpins their language learning to the ‘scientific’ arena of school mathematics (Vygotsky, 1962). In considering the role of visual representations, Rivera and Becker (2011) discuss the inconsistency between children’s observed ability to generalise in the learning of language and their application of generalisation skills in thinking algebraically. They cite Papert (2002) who argued that as language developed in tandem with ‘genetic tools’ and without the intervention of linguists, it is not directly comparable to algebraic thinking which has been adapted and developed to suit the needs of mathematicians.

While it may not be appropriate to expect all young children to transfer their spontaneous concept of generalisation in language to school based mathematics, it is possible to observe young children generalising. Schifter et al. (2008) discuss their observations of children in a Kindergarten classroom, as they spontaneously applied algebraic thinking to a game they were playing. It is appropriate therefore to anticipate that at least some very young children may spontaneously demonstrate algebraic thinking in any setting, and to prepare curricula to support and nurture algebraic thinking in all young children.

Generalisation, and the identification of a generality for a pattern or phenomenon, emerged from the research of the Early Algebra movement as a key proficiency of algebraic thinking (Kieran, Pang, Schifter & Fong Ng, 2016). Generalisation of a phenomenon involves the analysis of visible instances of the phenomenon, and the application of conclusions to cases that are not observable. The capacity to generalise supports us in utilizing the structure of the world around us so that we can conjecture and predict. In mathematics, young children are often encouraged to generalise properties of computation, such as the addition or multiplication of zero, so that they may build on what they have observed with some numbers, in order to perform mental computations with other numbers. Indeed, Mason (2017) asserts that in order to make sense of much of mathematics children must generalise. Mason highlights evidence pointing to children's early ability to generalise, and describes typical instances when young children work from the abstract to the particular, whereby they apply generalisations to specific instances. As I referred to above, in exploring children's construction and application of generalisation, many research studies have focused on shape patterns, for example Radford (2011), Rivera and Becker (2011) and Warren and Cooper (2008). Lannin, Barker and Townsend (2006) suggests that generalising through patterning activities may create a bridge between children's knowledge of arithmetic and their understanding of symbolic representations.

Patterning and Generalisation

Much discussion regarding extending algebra to primary school involves patterns and patterning. The ability to generalise from and work with generalisations is fundamental to children's developing thinking in mathematics (Kaput, 1998; Mason, 1996) and Lannin (2005) suggests that generalising through patterning activities may create a bridge between children's knowledge of arithmetic and their understanding of symbolic representations. Equally, patterns may serve as "dynamic representation of variables" when children are introduced to algebra, and asked to think algebraically (Lannin, 2005, p. 233). In this section I will discuss theories from research regarding the role of patterning in supporting the development of children's algebraic thinking.

Patterns: An overview

> Mathematics is the science of patterns. The mathematician seeks patterns in number, in space, in science, in computers, and in imagination. Mathematical theories explain the relations among patterns; functions and maps, operators and morphisms bind one type of pattern to another to yield lasting mathematical structures. Applications of mathematics use these patterns to "explain" and predict natural phenomena that fit the pattern (Steen, 1988, p. 616).

Patterns play many roles in how we interact with our world. Patterns offer us both the simple reassurance that spring will follow summer, and also the sophisticated predictions that underpin stock-market products. Throughout time people have incorporated patterns in the aesthetics of design, drawing on the inherent attraction of the structure of patterns. Particularly now, in the era of big data, as technology allows us to analyse enormous numbers of records, pattern-spotting is supporting discoveries in health, in crime prevention, and in many aspects of human behaviour.

While we encounter, and find purpose for, patterns in everyday life, it is pertinent to be specific about what we mean by the term 'pattern', and how that meaning is relevant to the tasks underpinning the research discussed in this thesis. Stromskag (2015) defines a shape pattern as a sequence of terms, composed of 'constituent parts', where some or all elements of such parts may be increasing, or decreasing, in quantity in systematic ways. While a limited number of terms of a shape pattern may be presented for consideration, the pattern is perceivable as extending until infinity. In order to construct a general term for a shape pattern, children must "grasp a regularity" in the structure of the terms presented, and generalize this regularity to terms beyond their perceptual field (Radford 2010, p. 6). Similarly, Mulligan and Mitchelmore (2009) present "structure" as the definition of a pattern, which is most often expressed as a generalization, that is a "numerical, spatial or logical relationship which is always true in a certain domain" (p.

34). Patterns may be categorised into two distinct groups: a) patterns which contain a sequence of identical units are regularly referred to as ‘repeating patterns’; and b) patterns consisting of terms which increase or decrease in the number of components of some element may be referred to as ‘growing patterns’. In the following sections, I will explore these two types of patterns.

Repeating patterns

Within repeating patterns, the recurring item may be simple, as in the repeating pattern presented in Figure 2.4, or complex as in Figure 2.5. The unit of repeat is identified in both figures.

Figure 2.4. A simple repeating pattern

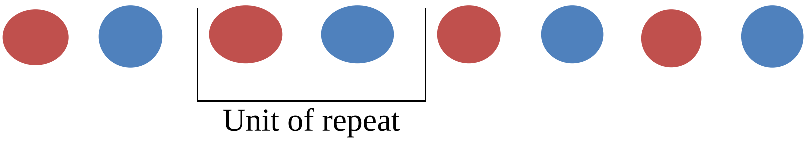
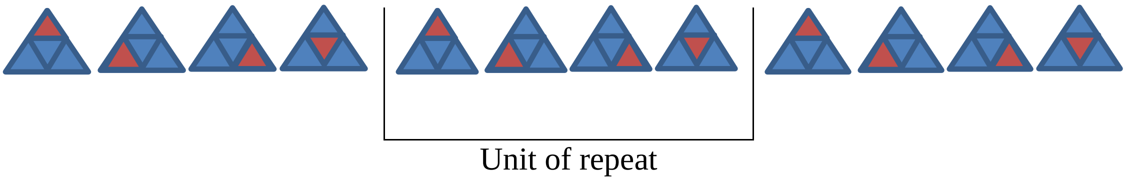


Figure 2.5. A complex repeating pattern



The PSMC algebra strand for Junior and Senior Infants[4] includes the objective that “the children should be enabled to identify, copy and extend patterns in colour, shape and size” where all suggested activities involve repeating patterns of the form A, B, A, B, etc. (Government of Ireland, 1999, p. 26). Fox (2006) explores the patterning activities of children in Australian preschool settings, where the children are of similar age to children in Junior and Senior Infants in Irish primary schools. She concludes that “experiences with identifying, creating, extending and generalising patterns, recognising relationships, making predictions, and abstracting rules provide foundations for future

[4] In Ireland children attend primary school for eight years, typically from the age of four or five. The classes are referred to as Junior Infants, Senior Infants, and 1st through to 6th class.

algebraic development" (p. 6). In her research Fox (2006) found that such patterning activities which are supportive of development in algebraic reasoning were not evident in the preschools involved. The children neither discussed repetition within the patterns they explored, nor did they identify a unit of repeat. Clements and Sarama (2009a) caution that teachers need to be aware of the role of repeating sequential patterns and of where they fit into "the large role of patterning and structure" (p. 190). When teaching patterning to young children, it is necessary that teachers remain cognisant of focusing children's attention on the commonality, or unit of repeat (Papic, 2007).

In considering the role played by repeating patterns in children's early algebraic thinking, there exist two schools of thought. Radford (2011) describes the identification of a commonality within a pattern as central to a child's development of algebraic reasoning but does not consider the activity to be algebraic in itself. Rather, he feels that for an activity to be considered algebraic, the child must make a leap from the exploration of the pattern to the identification of a generality for the pattern. Radford also aims to specify where algebra is involved in activities which involve extending sequences. He researched the actions and underlying thought processes undertaken by children attending Grade 2 who were involved in extending non-numeric patterns. The findings of the study demonstrated that in order to extend a sequence, the children were compelled to "grasp a commonality", and in order to do so, they coordinated spatial and numerical structures. Radford (2011) did not consider the process involved in this sequence extension to be algebraic but suggested that algebraic thinking is introduced when the child is expected to consider "remote figures [from] beyond the perceptual field" (p. 318).

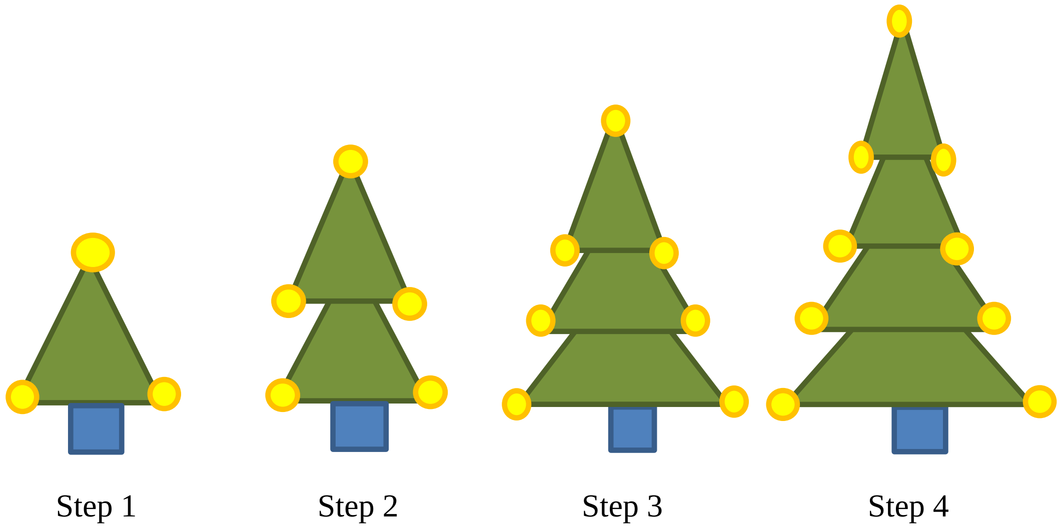
Threlfall (1999) also asserts that patterning has been established as a vital preliminary component for children to progress onto the study of algebra. Diverging somewhat with the positing held by Radford, Threlfall contends that involvement in patterning supports the children in their developing abilities in mathematics “including recognition and prediction, and the generalisation and communication of rules” (p. 21). Owen (1995) concurs with Threlfall’s thinking and highlights areas of mathematical thinking which cannot be developed in children at a pre-number developmental stage through activities other than patterning, namely “recognition of an event, prediction of a future event, generalisation of a rule and communication of that rule” (Owen, 1995, p. 126). Before children can proceed to the algebraic process of identifying or evaluating a remote figure, they must have established an understanding of what a pattern is and how a commonality may be identified (Threlfall, 1999; Owen, 1995). There is a necessity therefore for early patterning activities to facilitate young children in developing these key understandings. Dooley, Dunphy and Shiel (2015) concur with this position and state that activities may involve children progressing from an exploration of the patterns in the environment around them to working with repeating sequences of objects, where such work includes both “both pattern making and pattern perception” (p. 84).

To summarise, there is much research which supports the view of repeating patterns as supporting children’s prerequisite skills in the area of algebraic thinking. An argument may be made that some of the activity carried out is not strictly algebraic in itself and there is a risk that children may be engaging with patterning in a routine, non-explorative way which may not support their developing thinking. There are indications from research literature, however, that children in Infant classes could progress to activities which are algebraic in nature, and that the curriculum need not be limited to low order patterning activities (Britt and Irwin, 2011; Schifter et al., 2008; Papic, 2007). Teachers need to be aware of the role that patterning plays and how it contributes to children’s developing thinking.

Growing patterns

The patterns presented to children within this research project do not consist of sequences of identical items, but rather contain sequences of items which may be seen to 'grow' in some way, as shown in Figure 2.6.

Figure 2.6. A growing pattern



Terms of growing patterns are composed of constituent parts, which contain some element or elements. The quantity of elements in each constituent part may remain constant or may vary between terms. Variations between terms are structured, and consistent, that is the quantity of some component of the constituent elements increases or decreases at a uniform rate. For example in Figure 2.6, if the task is to explore the number of Christmas decorations on each Christmas tree, the single decoration on the top could be considered a constant. The number of decorations on the sides of the tree varies as the sequence progresses. The total number of decorations on a tree must include the constant plus a quantification of the number of other decorations depending on the position of the tree within the sequence.

Progressing from repeating patterns, children may have an opportunity to explore growing patterns such as the one presented in Figure 2.6. Warren and Cooper (2008) present three reasons for facilitating children in exploring growing patterns in primary school:

- They are visual representations of number patterns,
- They can be used as an informal introduction to the concept of a variable;
- They can be used to generate equivalent expressions (p. 113).

In order to utilise term numbers in this way, however, children must engage in a complex understanding of the 'term number'. For example, to construct new terms for a pattern, children must first identify the ordinal role of the term number. For some children, this may involve their coming to understand that such an object exists, that is a number which identifies where a term is placed within the sequence of terms in the pattern (Radford, 2000). In this context, the term number acts as a "dynamic general descriptor" of terms in relation to the sequence of terms which form the pattern. Having grasped the *ordinal role*, the child must then grasp the possibility of using this number within a computation, and perceive the number, therefore, within its *cardinal role*. Radford (2000) outlines a situation where a teacher seamlessly transitions from using a term number as an indicator of the term's position in the pattern, to a number which may be operated on within a computation. The teacher is competently using this number within a dual role but, as Radford describes, some of the children in her class struggle with the duality of purpose of the term number. Referring to the third term of the pattern presented in Figure 2.6, a child may need to identify the position of this term as third in the pattern, and then to associate this ordinal number '3' for use within a calculation to find the number of decorations on the associated tree. While for many children perceiving a number as performing both roles may not cause a perturbation in their thinking about patterning terms, for some children the duality may be a source of confusion (ibid.).

Within the Irish PSMC, there are no content objectives relating to growing patterns, other than sequences of numbers. While number patterns provide a context for consideration of relative rates of change, and generalisation, the PSMC does not draw attention to such aspects of their exploration, but maintains a narrow base for the thinking required of children. Details of the treatment of growing patterns within the PSMC will be outlined in the 'Generalisation Strategies' section of this chapter.

Functional thinking

Warren and Cooper (2008) consider that when children progress to identifying the relationship between a term and its position, they are developing the ability to think of the pattern as a function. For example, functional thinking would occur if a child identified from a sequence such as that presented in Figure 2.6 that the number of decorations depends in some way upon the position of the tree in the sequence, and also if the child considered how to calculate the number of decorations for a specific tree. In discussing findings from research with young adolescents, Warren and Cooper (2008) describe difficulties which children participating in their research encountered when attempting to develop this understanding of patterns as functions. They found that a lack of appropriate language inhibited the children in describing relationships and that children tended to fall back on additive strategies, focusing on a single variable within a pattern. Children may encounter difficulty in visualising a pattern spatially and this may present a challenge to them as they aim to isolate what is changing and the rate of change. Children also found it difficult to relate a term to its position. Furthermore, compiling a table of terms and values within which they could identify patterns seemed to increase the cognitive load of the task in hand (ibid.).

Warren and Cooper (2008) attributed many of the difficulties encountered by the children in their study to a lack of exposure to the language and methods involved in identifying a pattern explicitly. Given the absence of explicit patterning methodologies in the Irish curriculum, it is probable that many of the children in senior classes in Irish schools will be faced with similar challenges. Warren and Cooper did find, however, that children could apply an explicit approach to patterning when they had been facilitated in their development of the requisite language and methods through a teaching intervention. In terms of the task-based group interviews, it may be necessary to present children with a task, and to prompt the children to create a table if they have never encountered such a method. A note of caution is necessary here as Warren (2005) suggests that the utilisation of a table to record results and as an aid in the identification of a commonality in a pattern may in fact encourage children to focus on single variations rather than tracking how the pattern progresses as it extends. Warren notes in her research that this tendency to focus on a single variable may be entrenched in children and that the impact of the use of the table may not be profound but it is pertinent to remain vigilant to the potential.

Generalisation

Throughout this chapter I have drawn on research that identifies a core aspect of algebraic thinking as the construction of a general case for a pattern, or for a scenario which contains a generalisable phenomenon. Within formal algebra, students are required to apply a mastery of generalisation skills, to understand what it is to generalise and to manipulate expressions of generalisations while accepting their generalisability (Kaput et al., 2008). In primary school, there is a necessity for children to be in some way prepared to use mathematical generalisations, that is to express multiple events in one sentence, for example the commutative property of addition or multiplication. The traditional curricula prevalent in many countries delayed manipulation of generalisations until secondary school and great numbers of students failed to proceed in developing skills and habits of mind which could encompass abstraction (Arcavi, 2008; Kilpatrick & Izsák, 2008; Mason, 2008).

Additionally, generalisation and the skills involved in identifying generalisations support children in their development in many areas of mathematics. Schifter et al. (2008) found that where children were involved in stating and proving generalisations about the number system, such activities benefitted the development in mathematical thinking of all children, particularly in terms of their computational fluency. Children who are attaining at a high level in mathematics are challenged in their thinking when they are involved in such early algebra activities and children who struggle with mathematics are supported in gaining basic understanding of the operations. Schifter et al. (2008) also state that the practice of noticing a pattern in computation, stating the underlying generalisation and proceeding to prove it can arise naturally from children's early activities in arithmetic. Additionally, if children develop strategies for computation involving the commutative rule, covariation and inverse, the rote-learning of numerous facts is removed from the processes of addition and subtraction (ibid.).

Generic example

Central to an understanding of children's early algebraic thinking are concepts of "the particular" and "the general" (Mason and Pimm, 1984). Mason and Pimm identify three ways of talking about numbers, as specific numbers, generic numbers, and the general case. Specific numbers are identified without the use of symbols, or when symbols are used, there is no ambiguity about the value associated with the symbol, for example in the expression x+3=10, x can only represent 7. The general case is an expression of something that is true for all numbers, or in the solution of patterns, an expression which defines all terms of the pattern. Examples are

For all n ε N, 2n is an even number

The quantity of tiles needed for any term (n) in Pattern 1=2n+1

The generic example is an example wherein a specific number plays the role of the symbol n, "but one presented in such a way as to bring out its intended role as the carrier of the general" (ibid, p. 287). When solving for patterns, children may refer to far terms in order to express their generalisation, but their use of a numbered term is intended to describe the general case (ibid.; Radford, 2010).

Factual, contextual, and symbolic generalisations

Radford (2001) presents layers of children's algebraic generalisations which contain parallels to Mason and Pimm's trichotomy of specific, generic and general. Radford asserts that children's constructions of generalities may be factual, contextual or symbolic. Factual generalisations involve instantiating a general structure to specific terms, whereby children do not express a generalisation as applicable to all terms, but apply an "operational scheme" which allows them to calculate a value for particular terms (p. 82). Many of the children involved in this research project applied factual generalisations when they described the near and far terms for the patterns, as in the example of Emily for Pattern 3 (Appendix R) when she said "You'd need a hundred and twelve horizontal and then fifty-seven vertical". Emily applied her understanding of the 2n horizontal poles, and n vertical poles in order to calculate the number of poles for this far term. Equally she could apply this thinking to any term of this pattern, and could therefore be said to have factually generalised, even though she does not use abstract symbols in her expression, or refer to a general case.

Contextual generalisations, by comparison, involved the consideration of non-specific terms (ibid.). While contextual generalisations are not completely abstract, or general to all terms, they indicate a distancing from the specific, whereby children may make reference to "the next term" or to a generic term. As an example, Grace could be said to have constructed a contextual generalisation for Pattern 1 (Appendix R) when she said "whichever number it is at the top it will just be one more than it, and at the bottom".

Symbolic generalisations involve the abstract expression of disembodied mathematical objects, wherein children express the algebraic concepts with no reference to the method of their calculation, or to any specific term. Radford (2001) questions how children may develop the capacity to consider pattern terms in abstractly symbolic ways, and emphasises the complex developments required in the children's thinking in order for them to engage in this level of abstract expression. For children to express their thinking symbolically without reference to specific or situated instances, requires a different perspective on the mathematical objects involved. This higher order perspective must be accompanied by a "layer of discourse" appropriate to the description of the mathematical objects, in this case general terms for patterns, without alluding to any specificities, either in specific terms or in pointing towards thought processes involved in constructing the generality (p. 88). Thus, in planning to explore children's constructions of generality I considered it necessary to attend to the role played by symbols in how the children might communicate their thinking.

The role of symbols

In exploring the construction of general terms, it is pertinent to outline the position I adopted in relation to the role of abstract symbolism within children's early algebraic thinking. Brizuela and Ernest (2008) speak about the language of mathematical symbols, saying "language is a generative system through which children learn to represent spoken, written, and mental forms based on a common set of rules" (p. 274), and likewise children must learn to use the system of algebraic symbols to represent their thinking. As with all languages, the acquisition of the language of symbols requires opportunities to express oneself. Within the PSMC, variables are not included within content objectives until 5^{th} class, and it is probable that the children participating in my research may not have had opportunities to express mathematical ideas using symbols before taking part in the task-based interviews.

Kaput et al. (2008) suggest that along with the skills of generalising and expressing generalisations, as presented earlier in this chapter, algebraic reasoning comprises the use of "specialized systems of symbols to reason with the generalizations" (p. 21). Algebraic reasoning relates to children's ability to think logically about quantities (known or unknown) and the relationships between them. Carpenter and Levi (2000) define algebraic reasoning by identifying two central themes, namely "making generalizations and […] using symbols to represent mathematical ideas and to represent and solve problems" (p. 2).

Radford (2012) asserts that while an activity may not involve symbols in the expression of ideas, this does not necessarily erode the algebraic nature of the thought processes involved. Kaput et al. (2008) maintain however, that symbolisation is a core element of algebraic reasoning as it is intrinsic to generalisation. They assert that in the expression of a generalisation, one is speaking about multiple incidences without repetition, and that the use of symbols in so doing is efficient and purposeful. However, Kaput et al. present activities and processes where algebraic thinking is involved but without symbolisation. They term these activities and thought-processes as quasi-algebraic and include expressing generalisations verbally or with concrete objects. While Kaput at al. (2008) seem to be introducing flexibility in terminology by introducing the concept of quasi-algebraic, generalisation remains a key focus of learning and teaching activities. An activity is algebraic if it refers to a general case, regardless of how it is expressed and an activity is not algebraic if it involves an arithmetic solution even if expressed in symbols (Radford, 2011 & 2012; Kaput et al., 2008).

Within the algebra strand of the PSMC, content objectives are described from Third class[5] wherein children use a frame, or empty box, to represent an unknown. In all objectives, the frame denotes an unknown with one unique solution, rather than a variable, and the focus appears to be on the translation of number sentences into word sentences, and vice versa. While there are also patterning objectives for Third class, and subsequent classes, no connection is established between the two. This is a missed opportunity for children to encounter a context wherein variables are given meaning

[5] As outlined previously, in Ireland children attend primary school for eight years, typically from the age of four or five. The classes are referred to as Junior Infants, Senior Infants, and 1st through to 6th class, with children typically aged eight or nine years at the beginning of 3rd class.

through the varying quantities of elements in terms of a sequence (Lannin, Barker, and Townsend, 2006).

To summarise this section, I accept that children require opportunities to develop fluency in the use of symbols, as per the assertions regarding the linguistic aspect of abstract symbolism outlined by Brizuela and Ernest (2008), and also referred to by Radford (2001). Equally, I accept the distinction drawn between factual or contextual generalisations, which depend upon description of specific terms, and purely abstract generalisations expressed using disembodied symbols. Within this research project, I aimed to explore the children's constructions which use generic examples, or point to contextual aspects. I also aimed to facilitate children in expressing their thinking as generally as possible, to avoid placing an upper limit on my expectations of their algebraic thinking. In developing the task-based interviews for this research study I held the position that children's thinking could be considered algebraic without the use of a formal system of symbols. I planned to facilitate them in describing generality verbally. At times, the language that the children did use in their expressions of generality was clumsy, and I placed an emphasis on their justification of their thinking, in order for me to fully explore their understanding.

Justification of generalisations

Stacey (1989) conducted a study of the methods used by children in the age range 9 to 13 when presented with a linear generalisation problem. In analysing the children's responses, Stacey found that participants "grab at relationships and do not subject them to any critical thinking" (p. 163). Children involved in Stacey's study were satisfied to accept generalisations which were false even when the error in the generalisation was easily identified. Warren (2005) also found that children were inclined to change the terms in the given pattern in order to fit their generalisation, rather than to modify the generalisation which they had presented.

There is a need for children to develop a willingness to question their own approach and to justify the generalisation they present. The generalisation of a pattern or operational rule will not withstand the rigour which is prerequisite in mathematical thinking unless it is justifiable and justified. Lannin (2005) emphasises that "generalization cannot be separated from justification" (p. 235), and outlines a framework for children's development of rigorous justification skills. Children commence at a level where they use no justification, progress to a level where they appeal to a higher authority, and from this position advance to a level where the child is capable of demonstrating empirically why their generalisation should be held to be true. Later in this chapter, and in Chapter 3, I will outline the theoretical base for my assessment of children's algebraic thinking, which will include a consideration of how my research instrument was designed to motivate the children to justify their thinking.

Generalisation Strategies

Much research has been devoted to the approaches children take in seeking to construct general terms for shape patterns. Within this research study, I will draw from two overarching categorisations, how children situate a term within the structure of the pattern, and what aspects of the pattern children draw on in order to examine the underlying structure.

Relationships within pattern structure

Research exploring how children situate a term within the structure of a pattern has found that when children begin to examine patterns, their natural reaction is to reason recursively, meaning that they examine the mathematical relationship between consecutive terms in a sequence (Lannin, 2004). For example, looking again at Figure 2.6, one could determine that the number of decorations is increasing by 2 each step as the sequence progresses. The number of decorations on the tree in Step 5 would therefore be 11, 2 more than the number of decorations on the tree in Step 4. As children develop sophistication in their approach to patterning, they need to move beyond the recursive method of exploring a sequence. To gain an insight into a greater range of patterning and also a structural understanding beyond the most basic repeating pattern, children may benefit from an understanding that rules underlie patterns and that to expand some patterns efficiently, a rule must be identified. Lannin (2004) refers to an ‘explicit approach’ to the solution of a pattern, wherein a rule is identified for the relationship between a term and its position in the pattern. Seeking to clearly distinguish an explicit approach from recursive approaches, Carraher, Martinez and Schliemann (2008) identify as explicit those approaches that “treated [..one quantity..] as a function of [..another quantity..] or as an input–output function explicitly linking two variables” (p. 14). Referring again to the sequence of trees in Figure 2.6, one could calculate the number of decorations as ‘twice the Step number of the tree plus one’ allowing ready calculation of any tree in the sequence. Owen (1995) suggests building up a sequence element by element to facilitate children in realising that recursive methods are not always accurate.

It is advisable therefore that teaching activities and materials avoid an immersion in sequences which favour a recursive approach. Children need recourse to both explicit and recursive methods of solving patterns, and their thinking should be developed to include an ability to determine which method is appropriate in a particular situation (ibid.). In exploring children’s approaches to patterning in Taiwan, Ma (2007) found that many patterns evident in textbooks involve a constant difference between terms and may in this way reinforce reliance on a recursive method. Within the Irish PSMC, children in Fifth class are expected to identify and record rules for sequences which ideally should incorporate explicit rather than recursive thinking (Government of Ireland, 1999). However, the two exemplars given within the curriculum are of sequences with constant differences where the rules are to increase or decrease by a constant. This tendency towards sequences which are exclusively based upon a constant difference is reproduced in Irish textbooks. 'Mathemagic' is a popular series of textbooks in use in Irish schools, and within the Fifth class Mathemagic textbook, not only are all sequences either arithmetic or geometric in nature but the accompanying teaching notes instruct the children to compare each element to its predecessor and successor, and never suggest that there may be a relationship between the term and its position in the sequence (Barry, Manning, O'Neill & Roche, 2003). Kavanagh et al. (2015) found in an examination of teaching approaches in Irish classrooms that textbook use was pervasive in senior classes and suggested that teachers were possibly overly reliant on textbooks, even though many teachers pointed to flaws in the presentation of mathematics within the textbooks they used. If teachers are teaching sequences and patterning solely from textbooks such as Mathemagic 5, it is probable that many children in Fifth class will never have been encouraged to consider sequences other than recursively.

In assessing the algebraic thinking of children in terms of their approaches to patterning, it will be pertinent therefore to ensure that the tasks requiring an explicit approach are carefully designed to allow children to make some progress. Some children may have encountered explicit thinking in other contexts but have difficulty applying it to a patterning activity. In research involving children in 5th and 6th grade of elementary school in Taiwan, Ma (2007) found that the participants who adopted a geometric approach through their observations of an iconic representation could progress easily from recursive to explicit thinking when encouraged to do so. It may be possible for tasks, therefore, to facilitate children in realising that an approach other than recursive is sometimes required, by commencing with geometric representations, and building in such a way as to highlight that a recursive approach is inappropriate (Ma, 2007; Owen, 1995). If children are capable of thinking algebraically, and seek to generalise or identify a rule, the tasks should allow them to do so and reduce the limitations placed upon them by their lack of familiarity with an explicit approach to sequences. In the next section, I will highlight the role played by geometric elements of shape patterns.

Children are often requested to commence the exploration of a pattern by formulating a near generalisation, that is by identifying a near term (Warren, 2005; Stacey, 1989; Lannin, 2005). Near generalisations are achievable by building upon given information in a recursive fashion and utilising drawing or counting. The conclusion of a pattern based exercise is routinely the identification and application of a far generalisation, where children must identify a rule for the identification of a pattern term dependent on its position. In her research, involving children between the ages of 9 and 11, Stacey (1989) used the numbers 100 and 1000 considering them to be sufficiently large as to play "the role of generalised number" (p. 150).

Encountering patterning activities which require the derivation of a rule or generalisation may challenge children's thinking and initially the approaches they take may be flawed. As discussed previously in the patterning section of this chapter, many children are inclined to reason recursively unless facilitated in considering the benefits of an explicit approach to solving a pattern. Building on the research of Healy and Hoyles (1999), Stacey (1989), Swafford and Langrall (2000) and Lannin (2001), Lannin et al. (2006) developed the following framework of generalisation strategies:

Explicit:	The child constructs a rule which he can use to identify a term for any position in a pattern.
Whole-Object:	A term is used as a portion to identify a larger term by using multiples, e.g. the 10th term is calculated as twice the 5th term. The child may or may not compensate for inaccuracy in the resulting term, when applicable.
Chunking:	The child builds repeating units onto an identified term, e.g. for term 10 he adds 5 'units' onto term 5.
Recursive:	The child observes and uses a relationship between consecutive terms.

Lannin et al. (2006) view the strategies "along a continuum from recursive to explicit" and discuss factors which influence children's strategy selection, including factors relating to the task plus social and cognitive factors. Within an interview to assess algebraic thinking, it will be necessary to include a variety of tasks where participants are asked to identify a far generalisation. In this way, I hope to facilitate analysis of a participant's generalisation skills on different tasks and to allow for certain aspects of tasks which may encourage participants to adopt strategies other than explicit.

Barbosa (2011) encountered similar strategies in her study of the performance of children attending 6th-grade in solving shape patterning tasks. In order to assess the children, Barbosa (2011) extended the above framework from Lannin et al. (2006) to differentiate between a range of whole-object, chunking and recursive strategies which children in her research used, as presented in Table 2.1.

Table 2.1. Generalisation strategies framework (Barbosa, 2011, p. 2)

Strategy		*Description*
Counting (C)		Drawing a figure and counting the desired elements.
Whole-object	No adjustment (W1)	Considering a term of the sequence as unit and using multiples of that unit.
	Numeric adjustment (W2)	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on numeric properties.
	Visual adjustment (W3)	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on the context of the problem.
Difference	Recursive (D1)	Extending the sequence using the common difference, building on previous terms.
	Rate - no adjustment (D2)	Using the common difference as a multiplying factor without proceeding to a final adjustment.
	Rate – adjustment (D3)	Using the common difference as a multiplying factor and proceeding to an adjustment of the result.
Explicit (E)		Discovering a rule, based on the context of the problem, that allows the immediate calculation of any output value given the correspondent input value.
Guess and check (GC)		Guessing a rule by trying multiple input values to check its' validity.

Her findings included the following points which are salient to the consideration of a developmental pathway for generalisation in senior classes:

- Reverse thinking was challenging for many children even when they had some experience with patterning activities;
- Some children only used faulty strategies but this tendency declined as children became aware of the limitations of the strategies;

- Finding a functional relationship between variables was challenging for children and many generalised rules that worked in specific cases or repeatedly adopted recursive strategies;
- The ability to visualise from a context proved useful both in drawing to solve a near generalisation task and also in visualising the elements of the pattern and seeing the structure when solving for far generalisations (Barbosa, 2011).

The categorisations presented by Lannin et al. (2006), and Barbosa (2011) are presented as belonging on a continuum, rather than as discrete stages in children's development. Also, while recursive thinking may occur earlier than explicit for many children, and not require the same extent of intervention, it remains a valid strategy for the solution of patterning tasks. Watson et al. (2013) emphasise that explicit and recursive thinking should not be considered as hierarchical but complementary, and that children are supported in developing robust flexible reasoning skills when facilitated in engaging with both.

Figural and numerical aspects of pattern structure

In tandem with examining how children identify and express relationships between terms, or between terms and their position in the sequence, I also explored the aspects of the terms that children attended to. It is possible to draw a distinction between numerical and spatial (also referred to as geometric or figural) aspects of patterns, and of terms within patterns. In this section I review the literature pertaining to the dynamics of children's observations of spatial and numerical aspects.

Radford (2011) emphasises the need to marry both the spatial and the numerical structure, as children seek to 'grasp' a pattern when extending shape patterns. Mason, Graham and Johnston-Wilder (2005) draw attention to the assumption that children "see what the teacher sees" when engaging with patterning tasks, and caution that this may not always be the case. They emphasise that rather than considering what the children see, teachers should consider what the children are attending to. Moss and McNab (2011) state that children have a strong tendency to focus on the number of elements in a pattern, ignoring geometric properties of successive terms, and Ma (2007) found that making use of geometric aspects of patterns supported children in reasoning explicitly. It is useful therefore in an exploration of strategies that children use, to draw a distinction between strategies which involve attention to figural aspects of patterns, and strategies which do not.

In drawing attention to variation in the approaches that children adopt in exploring pattern structure, Becker and Rivera (2006) suggest that some children adopt a 'numerical' approach while others adopt a 'figural' approach. A figural approach involves "visual strategies in which the focus is on identifying invariant relationships from among the figural cues given" (p. 96). That is, when adopting a figural strategy, children interpret variables contextually, whereas in adopting a numerical strategy, children consider the numerical quantity of elements within a term as decontextualized. Children whose attention is focused on numerical aspects, without support from figural cues, often adopt strategies involving trial-and-error, and they may be seen to accept a lack of sense-making in their perception of the pattern.

Rivera and Becker (2011) point to the limitations of a numerical approach, where children in their research who adopted numerical strategies experienced more challenge in justifying their generalisations correctly. Also, within algebraic thinking, shape patterning has relevance in the context it provides for children to identify a commonality, and to apply it to all terms in the pattern (Radford, 2006). Equally, drawing on findings from Mason, Graham and Johnston-Wilder (2005), Rivera and Becker (2011) state that perceiving some properties of terms in a pattern and identifying those which are changing and those which are not is "necessary and fundamental in generalisation" (p. 329). Numerical generalisation, however, requires only a "surface grasp of a commonality" and as such does not engage children in thinking in a meaningful way about what is changing, and what is staying the same, or about how the generalisation applies to all terms in the pattern (p. 356).

In their research involving middle school students in a three year study, Rivera and Becker (2011) found that many children shifted from figural thinking to numerical thinking during the middle part of their research project, returning to figural thinking in Year 3 of the study. Rivera and Becker explained this shift by pointing to the use of an established numerical strategy, wherein children derived a table of input and output values for the figural patterns, and used the table to determine the rate of change and any constant element. The children found this method "predictive and methodical" but as mentioned above, Rivera and Becker emphasised that the algebraic thinking involved on the part of the children was limited. Rivera and Becker outlined the difficulty they felt the children were encountering, wherein the use of a figural strategy involved "cognitive perceptual distancing" in order to

1. figurally apprehend and capture invariance in an algebraically useful manner;
2. selectively attend to aspects of sameness and difference among figural stages and;
3. create a figural schema or a mental image of a consistent generic case and then transform the schema or image into symbolic form (p. 356).

Rivera and Becker define the steps undertaken by children in adopting a figural strategy to generalise from a shape pattern. As outlined earlier in this chapter, the third step in this process involves the creation of a generic case, that is, a factual generalisation of the pattern. If one holds to the position that it is not necessary to express a generalisation using abstract symbols, Rivera and Becker's elements of figural generalisation are still valid, where 'symbolic form' is taken to include children's verbal definitions and descriptions.

Summary

The preceding sections of this chapter drew attention to theory relating to algebra, patterning and generalisation. The following sections explore the treatment of algebra within the Irish Primary School Mathematics Curriculum, and present a framework of growth points for patterning. Before proceeding, it may be appropriate to synthesise some key theory presented in the chapter to this point.

In supporting the development of algebraic thinking among children in primary school, the key content domains are generalised arithmetic, expressions and functional thinking (Blanton, Brizuela & Stephens, 2016). Study within these areas will involve the acquisition of skills, but also, crucially, children should engage in thinking algebraically, where algebraic thinking involves the application of observations of change and relationships in order to explore and make use of structure. For children to develop algebraic thinking, it is necessary for them to be facilitated in identifying rules for patterns, considering far generalisations, and operating upon unknowns. The means by which the children express their thinking at this stage in their education is not of relevance, the focus must be on the thinking and the children's developing strength in reasoning algebraically. Patterning plays a very significant role in the developing algebraic thinking of children in primary school and shape patterns provide a context for children to both construct and justify an algebraic generalisation "within the means available to the learner" (Rivera and Becker, 2011, p. 327). When presented with shape patterns, and asked to consider near, far and general terms, children may draw on varying observations and strategies in their constructions of terms. Such observations include attending to figural and numerical aspects, and also identifying various relationships within the pattern structure. Building upon, and further refining their observations, children may seek to construct general terms by applying explicit, recursive, whole object or counting strategies.

Algebra in the Context of the Irish Education System

Irish Primary Curriculum

As one of the five strands of the Irish PSMC there are learning objectives within the algebra strand from when children commence school and for every subsequent year of primary school. In this section I will examine what constitutes the formal algebra curriculum as it is prescribed to teachers who teach in Irish primary schools.

The broad objectives within the Algebra Strand state that, allowing for varying abilities and circumstances, children should be enabled to:

> explore, perceive, use and appreciate patterns and relationships in numbers; identify positive and negative integers on the number line; understand the concept of a variable, and substitute values for variables in simple formulae, expressions, and equations; translate verbal problems into algebraic expressions; acquire an understanding of properties and rules concerning algebraic expressions; solve simple linear equations; use acquired concepts, skills and processes in problem-solving (Government of Ireland, 1999, p. 13).

Along with the broad objectives are detailed content objectives. The content of algebra study, as intended under this curriculum, includes the identification and extension of pattern and sequences and the application of patterns as an aid to computation. The concept of an unknown quantity within a mathematical expression is explored and this exploration is extended to include variables in Sixth class. It is prescribed that children learn to translate number sentences into word sentences and vice versa, including sentences which contain an unknown quantity. Children should solve such sentences from Third class. In Fifth and Sixth class, negative numbers are introduced when children should learn to identify them on the number line and add them. Finally, in Fifth and Sixth class, children are required to know the basic rules of mathematical operations, their priorities and the use of brackets.

As the foci of my research are patterning and generalisation, and generalisation is not specifically mentioned in the PSMC, I will discuss the content objectives which relate to patterning. The patterning content objectives included under the Algebra strand of the

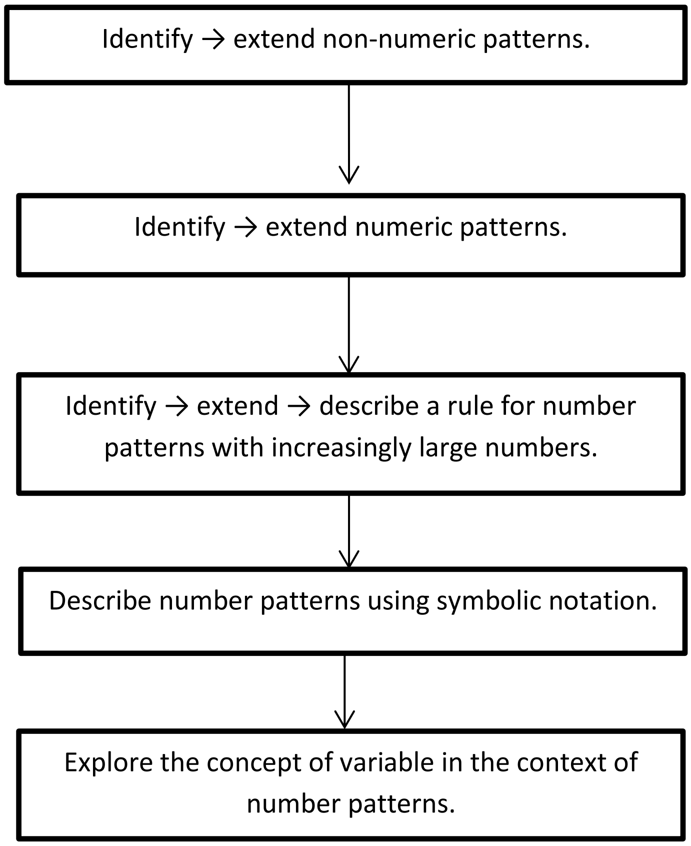
Irish primary school curriculum are outlined in Table 2.2. Within the patterning content objectives outlined in Table 2.2, there are objectives concerned with generalisation of arithmetic in the identification and application of the properties of operations. Of more relevance to my research are the patterning objectives which could facilitate children in developing their ability to explore, identify, express and apply generalisations, in a broader context, and not limited to the area of arithmetic.

Table 2.2. Algebra strand content objectives relating to patterning, in the Irish primary school curriculum (Government of Ireland, 1999)

Class	Content Objectives
Junior Infants	Identify copy and extend patterns in colour, shape and size.
Senior Infants	Identify copy and extend patterns in colour, shape, size and number (3-4 elements);
	Discover different arrays of the same number;
	Recognise patterns and predict subsequent numbers.
First class	Recognise [numeric] patterns, including odd and even numbers;
	Explore and use patterns in addition facts.
Second class	As for 1st class but progressing to the prediction of subsequent numbers in the pattern recognition objective.
Third class	Explore, recognise and record patterns in numbers, 0-999;
	Explore, extend and describe (explain rule for) sequences;
	Use patterns as an aid in the memorisation of number facts.
Fourth class	As for 3rd class but progressing to include numbers from 1000-9999.
Fifth class	Explore and discuss simple properties and rules about brackets and priority of operations;
	Identify relationships and record verbal and simple symbolic rules for number patterns.
Sixth class	Know simple properties and rules about brackets and priority of operations;
	Identify relationships and record symbolic rules for number patterns;
	Explore the concept of a variable in the context of simple patterns, tables, and simple formulae and substitute values for variables.

Examining the patterning content objectives there is a developmental sequence from Junior Infants to Sixth class as summarised in Figure 2.7.

Figure 2.7. The developmental progression in patterning described through the curricular content objectives, from Junior Infants to Sixth class



It is worth mentioning that within the Irish Mathematics curriculum, algebra is not referred to under any of the other strands either among the content objectives or the teaching guidelines (Government of Ireland, 1999). There is an attempt within the mathematics curriculum to guide teachers towards the possibility for both integration of mathematics with other subject areas and also linkage of strands and strand units. Specifically, each strand unit contains suggestions for subject area(s) which would be relevant for integration and many content objectives contain suggestions for linkage with other strand units. Within the algebra strand there is no suggestion for linkage or integration and there is no mention of linkage with algebra under any other strand unit of the curriculum. The Irish curriculum seems to endeavour to cover fundamental content which will enable children to engage with algebra in the secondary school, and also to introduce the concept of number pattern, and the underlying structure of number. These are vital elements of any instruction in algebra, but teaching them in isolation can be seen as more arduous and less efficient than teaching algebra through an integrated curriculum (Carpenter et al., 2005; Blanton & Kaput, 2000).

The patterning content objectives of the Irish curriculum in Infant classes are supported by the position of Threlfall (1999) and Owen (1995) as discussed in the patterning section of this chapter. The viewpoint expressed in Radford (2011) may highlight a deficiency in the exclusion of far terms under the suggested activities accompanying content objectives. Interrogating the suggested activities within the curriculum, none of the non-numeric patterns are growing and no reference is made to near or far terms. The numerical pattern work suggested for Infant classes is limited to identifying missing numbers in forward and backward counting sequences. In First and Second class the patterning activities suggested are comprised entirely of group counting and there is no mention of non-numeric patterns. Within the recommended activities, there is no mention of near or far terms but rather the focus is on the prediction of subsequent numbers in a pattern (ibid.).

In this context, it may be said that some of the number pattern activity suggested in the Irish curriculum, neither requires nor promotes algebraic thinking. In First and Second class, children are engaged in extending and repeating linear patterns and the curriculum advises that they should be enabled to explore pattern as a support in their growing computational skills, e.g. 2+5=7; 2+15=17; etc. Children are engaged in observing patterns in odd and even numbers and in multiples of 2, 3, 4, 5 and 10. Children are asked to explain a rule for patterns but there is no mention in the curriculum of applying this rule to a general term or a far term. Importantly, there is also no justification in the curriculum for this patterning work and I would voice a concern that teachers see such activities as a support to arithmetic which becomes almost trivial in nature without a discussion regarding generalisation. Within the broad objectives of the curriculum there is the requirement that children should be facilitated in developing the ability to “reason, investigate and hypothesize with patterns and relationships in mathematics” (Government of Ireland, 1999, p. 12). This objective does not seem present in the breakdown of content objectives and recommended activities.

Along with generalisation, it seems problematic that there is no mention of an explicit approach to pattern-solving. When children begin to examine patterns, their natural tendency may be to reason recursively (Lannin, 2004). In the development of a child’s thinking, it may be necessary for a teacher to encourage the child to consider an ‘explicit approach’, whereby the child identifies a rule for the relationship between a term and its position in the pattern. Within the curriculum objectives for algebra and patterning, a recursive approach is favoured in the presentation of both learning objectives and associated activities, while explicit thinking is not outlined as an approach to pattern solving. The following are the content objectives with recommended sample activities for 5th and 6th Class, the two most senior years of primary school.

5th Class:

identify relationships and record verbal and simple symbolic rules for number patterns

identify and discuss rules for simple number sequences

2.0, 3.5, 5.0, 6.5 … i.e. sequence increases by adding 1.5

81, 27, 9 … decreases by dividing by 3

1, 4, 9, 16, 25, 36 …

6th Class:

identify relationships and record symbolic rules for number patterns

deduce and record rules for given number patterns

2, 6, 12, 20, 30 …

4:1, 8:2, 16:4 …

While mention of a ‘rule’ may in some contexts point to an explicit approach, the suggested solutions given for the 5th Class examples are recursive. No suggested solutions are included for the 6th Class examples, but recursive solutions would be more suitable for both, as there is a relatively easily identifiable relationship between consecutive terms in both, while an explicit solution is far more complex, and unwieldy. As discussed in the ‘Generalisation Strategies’ section of this chapter, textbook use is pervasive in Ireland, and within textbooks the interpretation of curriculum content objectives for patterning may be limited to recursive approaches, as typified by Mathemagic 5.

Conclusion

Focusing on the algebraic skill of generalising, Mason (2009) states that “young children are able to generalise, because without it they could not function in the world and certainly could not grasp language” (p. 159). When children enter Irish primary schools they bring with them nascent skills in generalisation and it is the role of the school to nurture and develop such skills so that by the time children leave primary school they possess the competences necessary for engagement in abstract symbol manipulation. The Algebra strand of the Irish primary mathematics curriculum is very rich in patterning activity but the algebraization of this activity is very much the responsibility of the classroom teacher, and thus possibly inconsistent. Corcoran (2005a) researched the mathematical knowledge and skills of a cohort of 71 preservice teachers and found that only 42% of the participants could correctly answer an item which involved algebra. While the study sample and number of items were too small to be generalisable, Corcoran expressed concern about the mathematical literacy of these preservice teachers as overly procedural. Limited mathematical literacy and a procedural approach to mathematics are unlikely to lend themselves well to an independently algebraic approach to patterning activities, and I would voice concern that in many classrooms the patterning activity of children may involve little algebraic thinking.

Many recommendations of research literature in the area of early algebra are fulfilled within the Irish curriculum. Instruction commences early, and patterning is considered as a support to computation throughout primary school. However, there is no mention made of generalisation or identifying a far term in a pattern. While mention is made of formulating rules for patterns, all the pattern examples provided lend themselves most readily to recursive solutions. Including a focus on generalisation, far terms and explicit pattern solving would benefit children in their development of algebraic thinking, and in overcoming the challenges of formal algebra.

Developmental Pathway in Patterning

In developing a framework to plot a possible developmental pathway for patterning, I will hold to the theory expressed by Mason (2008) and Hewitt (2009) that very young children possess skills which comprise algebraic thinking. For example, research has shown that young children are capable of observing and applying some algebraic thinking skills in mathematical contexts while not necessarily possessing the ability to express them as a general rule or to apply them to all situations (Schifter et al., 2008). However while plotting the developmental pathway I remained cognisant of the statement of Radford (2012) that algebra is a cultural construct and children's skill development will benefit from facilitation by the children's educational environment (also Kieran, 2011). Therefore, in addition to my assumption of children's inherent skills, the development of the framework was guided by an understanding that for some children there are aspects of algebraic thinking that may require intervention for optimal development.

One possible conceptualisation of developmental pathways, outlined by Clements and Sarama (2009), includes an age for which each developmental progression is appropriate, and is based upon a theory of a learning process as 'incremental' where each process follows from the previous level in a hierarchical fashion. Rather than assuming a linear pathway where one learning goal follows from another when children are developmentally ready, a framework of growth points may include "cognitive structures" or "meaningful chunks of information" as identifiable developmental progressions in children's thinking (Ronda, 2004). Ronda outlines how the meaningful chunks of information in her study included key strategies and knowledge relating to a concept and reflected a focus on conceptual rather than procedural understanding. A broader view of growth points may allow for a range of progressions in children's learning, where some children may be expected to take a circuitous route to skill development, some may skip interim steps, and indeed some children may regress in order to surge forwards (Fosnot and Dolk, 2001; Barbosa, 2011). A learning trajectory designed to include all learners could not be linear or definite because until a child is presented with a concept, there is little way of being sure of how they will construct understanding (ibid.; Simon, 1995).

Developmental pathways support the learning and teaching of mathematics through the information they afford to policy makers, and teachers. Pathways have the potential to inform assessment and planning for learning, and may be rich resources for supporting teachers in understanding how children develop understanding of a concept, as they explicate the steps from initial introduction to the concept through to robust understanding (Clarke, 2001). The content and presentation of pathways reflect specific understandings of mathematical proficiency. In Chapter 1, I presented my perspective of mathematical proficiency which incorporates multiple elements, and is developed through rich mathematical experiences (see Figure 1.1). Dunphy, Dooley and Shiel (2014) state that pathways which reflect sociocultural perspectives must be a) provisional in accommodating the varying learning paths children may present with, b) not linked to age as to do so may limit some children and be inappropriately challenging for others, and c) emerge from "engagement in mathematical-rich activity with children reasoning in, and contributing to, the learning/teaching situation" (p. 11). The following sections outline a developmental pathway in patterning, and seek to accommodate the recommendations of Dunphy, Dooley and Shiel (2014) along with the literature pertaining to children's algebraic thinking, focusing on the findings of Lannin et al. (2006), Barbosa (2011) and Rivera and Becker (2011).

A framework of growth points in patterning

In Chapter 1 I expounded my decision to adopt a framework of growth points as a developmental pathway in my exploration of children's thinking. In this section I present a potential developmental pathway in patterning, conceptualised as a framework of five growth points. The growth points are presented as flexible with an assumption that many children may skip some growth points, or spend considerably more time on progressing from one than from another. The growth points are not linked to ages, and there is an assumption that progression between growth points will be best facilitated by rich mathematical experiences that draw on children's thinking rather than exposition of procedures, followed by practice through repeated application.

This framework synthesises the theories relating to algebraic thinking outlined above, along with Rivera and Becker's framework of constructive and deconstructive generalisation, which is addressed in the upcoming section *Achievement of growth point 3*. An overview of the framework of growth points is presented in Table 2.3, with the criteria for achievement of growth points 1, 2, and 3 deconstructed into constituent elements in Tables 2.4 to 2. 6. The purpose of the framework within this study was both theoretical and methodological, as it informed my understanding of children's algebraic thinking, supported the development of the task-based group interviews, informed my prompts within interviews and facilitated me in considering the extent of children's progress with the patterning tasks.

Each growth-point is broadly envisaged and while the framework aims to track a developmental pathway for primary school, there are no age ranges or class levels automatically associated with growth-points. There are inconsistencies in research findings regarding the development of algebraic thinking in Irish classrooms (Shiel et al., 2014; Government of Ireland, 2005), and I aim in the framework to allow for a broad variation in children's prior engagement with algebraic thinking.

Table 2.3. Overview of a framework of growth points in patterning

Growth Point	Characteristics
GP 0:Pre-formal pattern	Children's understanding of "pattern" is informal and possibly incomplete. Children demonstrate affinity for and recognition of order and structure, without formally describing a pattern or identifying a repeating term.
GP 1: Informal pattern	Children can identify a commonality and demonstrate understanding of pattern by copying, extending, inputting missing term, in shape, numeric, repeating and growing patterns. Children can describe some aspects of the structure of patterns.
GP 2: Formal pattern	Children can describe a pattern verbally. Children can offer a possible near (not next) term with reasoning.
GP 3: Generalisation	Children can correctly identify a near term. Children can describe a pattern explicitly. Children can offer a possible far term with reasoning.
GP 4: Abstract generalisation	Children can describe a pattern explicitly, describe the rule as an expression in symbolic notation and utilise the expression in order to generate a far term.

The first three Growth Points are entitled Pre-formal Pattern, Informal Pattern and Formal Pattern. While there is no implication that the only skills relevant to these growth points are those of pattern solving, much discussion regarding extending algebra to the early years of schooling involves patterns and patterning, as discussed in the 'Patterning' section of this chapter.

Growth Point 0 and achieving Growth Point 1

On entering primary school, some young children would have already moved beyond GP0 but some may not. In the research available regarding the early algebraic activity of primary school children, there is much focus on patterning (Threlfall, 1999; Clements and Sarama, 2009a). There is a range of fine-grained sub trajectories available in print, therefore, which outline the composite skills involved in early patterning, and the various approaches and strategies that children take. I incorporated the trajectories of Rustigian (1976), Vitz and Todd (1969) and Clements and Sarama (2009a) in compiling the possible developmental pathway in repeating patterns presented in Table 2.4, and the original trajectories are included in Appendix G. In Table 2.4 I aim to integrate these interim trajectories as they may inform the monitoring of children's progress towards GP1, Informal Pattern. In order to achieve GP1, children must have an understanding of pattern as growing or repeating, and also be able to consistently identify a commonality in a repeating pattern.

Table 2.4. Achieving growth point 1

Growth Point	Characteristics
GP 1: Informal pattern	Overview: Child can identify a commonality and demonstrate understanding of pattern by copying, extending, inputting missing term, in shape, numeric, repeating patterns. GP1.1: Child can copy pattern of the form ABAB GP1.2: Child can extend pattern of the form ABAB. GP1.3: Child can extend pattern of the form AABBCAABBC and recognise that patterns contain a unit of repeat GP1.4: Child can create a pattern with a unit of repeat. GP1.5: Child can extend a repeating shape pattern. GP1.6: Child can extend a growing pattern of the form A, AA, AAA and identify a missing term. GP1.7: Child can extend a linear numeric sequence of the form 2, 4, 6, etc. and identify a missing term.

Achieving Growth Point 2: Formal Pattern

In order to achieve GP2 children must formalise their understanding of pattern and sequences. While the ability to consistently solve a pattern explicitly is not contained within GP2, children may be considering any relationships between terms and their positions. In solving patterning tasks, children achieve greater success in near generalisations than in far generalisations (Barbosa, 2011). In achieving GP2, children must develop an ability to solve a pattern by identifying a near term. Warren and Cooper (2008) discussed the challenges encountered by children in solving a pattern involving two variables. Radford (2012) highlighted the difficulty encountered by children at this stage in linking numeric and spatial aspects of such a sequence. Incorporating the research of Warren and Cooper (2008) and Radford (2012), the interim trajectory which will suggest a developmental pathway between GP1 and GP2 will involve pattern solving where the cognitive load is gradually increased, as depicted in Table 2.5. The extent to which a child's skills have developed will be indicated by her success on increasingly challenging patterns.

Table 2.5. Achieving growth point 2

Growth Point	Characteristics
GP 2: Formal pattern	Overview: Child can describe a growing pattern verbally. Child can offer a possible near (not next) term with reasoning. GP 2.1 Child can identify next term in a growing shape pattern maintaining shape or quantity but not both. GP 2.2 Child can identify next term in a growing shape pattern maintaining both shape and quantity.

Achieving Growth Point 3: Generalisation

In the section of this chapter entitled 'Generalisation Strategies' I outlined the tendency of young children to reason recursively about patterns, and the potential benefits to children's thinking of encouragement to consider an explicit approach (Lannin, 2004). In order to achieve GP3 children must move from a recursive to an explicit approach to pattern solving. Lannin et al. (2006) outline a continuum of strategies which children may progress through as they develop the skill of explicit pattern solving, entitled Recursive, Chunking, Whole-Object and Explicit. Barbosa (2011) expanded upon the framework of Lannin et al. (2006) and found that participants in her research adopted the strategies outlined in Table 2.1 (earlier in this chapter). Rivera and Becker (2011) investigated the work of participants in generalising from linear figural patterns and identified three distinct approaches:

> Deconstructive generalisation (DG) : identifying a rule by commencing with a multiplicative strategy and identifying overlapping between variables in order to refine the rule
>
> Nonstandard constructive generalisation (CNG): identifying a rule by noticing and separating the elements within the structure of the pattern, but resulting in a non-standard generalisation which requires simplification.
>
> Constructive generalisation (CG) : identifying a rule by noticing and separating the elements within the structure of the pattern, but resulting in a generalisation which requires no simplification (Rivera & Becker, 2011).

Table 2.6 contains a framework for analysis of generalisation strategies based upon the framework developed by Barbosa (2011) and incorporating definitions of approaches to generalisation suggested by Rivera and Becker (2011).

Table 2.6. Achieving growth point 3

Growth Point	*Characteristics*
GP 3: Generalisation	Overview: Child can correctly identify a near term. Child can describe a pattern explicitly. Child can offer a possible far term with reasoning.
	GP 3.1 Counting
	GP 3.2 Whole-object (WO) no adjustment
	GP 3.3 WO - Numeric adjustment
	GP 3.4 WO - Visual adjustment
	GP 3.5 Recursive
	GP 3.6 Rate – using the common difference without applying a final adjustment
	GP 3.7 Rate – using the common difference, but applying a final adjustment
	GP 3.5 Explicit DG
	GP 3.6 Explicit CNG
	GP 3.7 Explicit CG

Barbosa (2011) also found in her research that while the use of a recursive strategy precedes correct calculation of a near generalisation, there is the potential for children to skip this growth point and succeed at points further along the trajectory. Also, Watson et al. (2013) emphasise that recursive thinking may be the appropriate approach to some patterns. The work of Barbosa (2011), and Watson et al. (2013) are further evidence that while developmental pathways may be of theoretical benefit, their use must be predicated upon an understanding of development which allows for myriad pathways.

Achieving Growth Point 4: Abstract Generalisation

In achieving GP4 a child should demonstrate many of the range of skills outlined by Kieran (2004) as encompassed by an “algebraic way of thinking” (p. 140). Such skills include the ability to work with problems and numerical expressions without feeling obliged to solve them. To achieve GP4, children should also be competent in dealing with abstract symbols, and be comfortable in “(i) working with letters that may at times be unknowns, variables, or parameters; (ii) accepting unclosed literal expressions as responses; (iii) comparing expressions for equivalence based on properties rather than on numerical evaluation” (Kieran, 2004, p. 140). To summarise, in achieving GP4 children will demonstrate competence in algebraic thinking which will enable them to engage with abstract symbol systems and expressions; and also to utilise algebraic notation and thinking in other areas of mathematics.

Conclusion

The framework of growth points presented in this section has not been extensively researched with children, and is by no means conclusive. It is based upon a review of the relevant literature and I have utilised it to develop the interview presented in the next section. This framework forms a basis for the identification of relevant tasks and also for the scripting of the interview. It is important to note that while the framework presents children’s thinking developing in a linear fashion, there is an understanding that not all children follow identical linear pathways in their learning. The tasks underpinning the interview and the interview schedule are designed to investigate children's thinking, allowing themes to emerge which are significant to participants' views of generalisation in mathematics.

The framework incorporates a number of approaches to algebraic thinking, but does not reflect whether children adopt numerical or figural strategies in seeking to construct general terms. Rivera and Becker (2011) found that many of the children involved in their three-year longitudinal study used figural thinking in Year 1, adopted more numerical approaches in Year 2, and returned to figural thinking in Year 3. It may be difficult therefore to consider either numerical or figural modes of generalisation as developmentally progressing in a way which would be meaningfully represented in a developmental pathway.

Assessing Algebraic Thinking

In this section, I will discuss essential aspects of the assessment of algebraic thinking, and their implications for my research. I begin by considering broadly what it is to assess children's mathematics. Having considered the assessment of mathematical thinking, I narrow my focus to the assessment of algebraic thinking, where I draw attention to the nature of the development of algebraic thinking among young children, and the relevance for assessment of the constituent elements of algebraic thinking as highlighted by Cai and Knuth (2011a) and Kieran (2007). Lastly in this section I present the case for diligence in the assessment of algebraic thinking in order to ensure that the assessment is capturing children's *algebraic* thinking, rather than alternative approaches which are not algebraic.

In order to design an assessment of children's mathematics, it is imperative to have a clear idea of what constitutes mathematical proficiency. Milgram (2007) discusses the difficulty encountered in attempting to clinically define "mathematics" and suggests instead a description of the most important characteristics of mathematics, namely "precision" and "stating well-posed problems and solving them" (p. 33). In the same volume, Schoenfeld (2007) suggests that the key element of proficiency in a subject is the ability "to use it in the appropriate circumstances" (p. 59). While Schoenfeld and Milgram adopt differing perspectives on defining mathematical proficiency, their assertions complement each other. Thus, underlying this research is a theory of assessment as a measure of a participant's ability to apply mathematical knowledge and skills appropriately in solving new problems while always retaining a focus on precision.

Kaput (1998) discusses the development of algebraic thinking and considers the need to nurture and encourage the "roots of algebraic reasoning" over the primary school years. Mason (2008) identifies elements of algebraic thinking which are evident in very young children as imagining and expressing, focusing and de-focusing, specializing and generalising, conjecturing and convincing, classifying and characterizing. The immature skills of young children as highlighted by Mason (2008) develop over a length of time into strong broadly applicable skills of algebraic thinking. To assess algebraic thinking during this time requires an understanding that the skills under assessment are in development. In discussing the tools and artefacts of assessment, Smagorinsky (1995) warns against presuming that evidence of children's thinking represents a "cystallized, fully formed state of development independent of the artefact's cultural significance and the means through which the learner has appropriated an understanding of how to produce it" (p. 199). In designing an assessment, it may be preferable to engage participants in non-routine tasks which are not derivatives of a rote-learned approach.

Children, whose skills are emergent, may present with a variety of skills while complete solution of a problem remains beyond the range of their ability (Radford, 2012). When operating within their Zone of Proximal Development (ZPD), children are capable of demonstrating skills which they are currently developing (Vygotsky, 1978). In assessing algebraic thinking therefore, an assessment instrument is required which will not only facilitate children in engaging with tasks within the highest cognitive ranges of the ZPD but also facilitate the researcher in observing, as far as is possible, the mental processes underlying the approach each participant adopts. In this section, I aim to discuss the characteristics of the task-based group interview and how it may be suited to the assessment of young children's algebraic thinking.

The development of algebraic thinking involves more than the rote-learning of routine algorithms or processes. To reason algebraically, it is necessary for children to develop appropriate strategies and habits of mind (Cai & Knuth, 2011a). Kieran (2007) discusses the shift in thinking which is required for a child to progress from a purely arithmetical approach to developing algebraic thinking. Among other skills, she suggests that it is necessary to focus on relationships and not simply on the calculation of an answer. In considering methods for assessing children's algebraic thinking, it is necessary to develop a strategy for assessing such constituent elements as foci, skills and habits of mind. The task-based group interview creates an opportunity to assess the reasoning which underlies decisions a participant makes in the solution of tasks (Goldin, 2000).

There is an inherent challenge in skills-based assessments of algebraic thinking, particularly among children. It is not sufficient to score participants on the basis of correct or incorrect answers, but rather assessments are required to incorporate questions which afford insight into how the participant is thinking. Participants may arrive at a correct answer in an item that purports to assess algebra without applying algebraic thinking. For example, Osta and Labban (2007) found that there was a need to alter the test question in their study of Seventh Graders' algebraic problem-solving strategies. In the solution of a numerical sentence with an unknown, children reverted, when possible, to a trial-and-error application of arithmetic, rather than adopting an algebraic approach. In interpreting the findings of research into Irish primary school teachers' mathematical knowledge for teaching Delaney (2010) expressed concern regarding findings in the area of algebra. There existed a possibility that in solving a question designed to assess algebraic skills, some teachers may have adopted an arithmetical approach which would have been sufficient to solve the problem correctly. Equally, as suggested by Van de Walle (2004), in problem-solving items which are not designed to assess algebraic thinking specifically, some children may utilise an algebraic approach but their doing so may not be identified by the assessment.

An alternative to a standardised assessment is that of an interview. Ginsburg, Kossan, Schwartz, and Swanson (1983) assert that "to establish different aspects of competence, it is useful to use flexible, nonstandardized procedures" (p. 14). The task-based group interview method affords a researcher insight into the "complex cognitions associated with learning mathematics" along with the means by which children approach problems, and the relationships between elements such as affect, cognition, and learning (Goldin, 2000). In the following section I will discuss the task-based group interview and why I consider it to be a more appropriate form of assessment of algebraic thinking.

Assessment through interviewing

Piaget (1929) presents a view that assessments with a set protocol of questions are destined to skew the impression given of an individual child's ability. Piaget suggests that within assessment there are occasions when a fixed questionnaire is not an appropriate instrument as it may yield insufficient information regarding the internal reasoning of the participant. As mentioned above, in solving a task which is purporting to assess algebraic thinking, a child may use trial and error or purely computational strategies (Osta & Labban, 2007; Delaney 2010). Piaget's clinical method "claims to unite what is most expedient in the methods of test and of direct observation, while avoiding their respective disadvantages" (p. 19). While more often associated with individual interviewing, a clinical method of assessment is in essence an interview scenario where the researcher allows the child to guide the direction of the assessment, while constantly maintaining the focus on the area of research. As such, many of the affordances, constraints and cautions highlighted by Piaget are equally relevant to task-based group interviewing where the focus is on why the children responded to tasks in the way they did.

Vygotsky (1978) discusses the ZPD within the context of assessment and advises examiners to remain cognisant of the exact focus of assessment and of what assessments inform us about children. Without support from a teacher or more able peer children may only complete tasks for which they have already acquired the requisite knowledge and skills. The ZPD deals with the knowledge and skills which the children are in the process of acquiring, which may be as yet not fully formed but are in gestation (ibid.). As part of the task-based group interview design, mediation will be provided through peer interactions, concrete manipulatives, drawing tools, and researcher prompts in order to scaffold children in achieving at the highest possible level, without interfering in their thinking to the extent that the demands upon them become less cognitively challenging than is merited by the task. Vygotsky (1978) asserts that there is a distinction between tasks of which the child is capable with support and those which are beyond his/her ZPD. It may be necessary for task design, researcher-pupil interaction and analysis to reflect the continuum of independent success -> supported success-> unfulfilled task. In analysing children's responses to the tasks of the task-based group interview, attention should be paid to the level of mediation required as children work on a task, whether they progress to complete the task, or find it too challenging and do not succeed in progressing.

Task-based interviews

> Structured, task-based interviews for the study of mathematical behaviour involve minimally a subject (the problem solver) and an interviewer (the clinician), interacting in relation to one or more tasks (questions, problems, or activities) introduced to the subject by the clinician in a pre-planned way (Goldin, 2000, p. 519.)

Goldin emphasises that task-based interviews, involving individuals or groups, have become an essential tool within mathematics education research, as the goal of mathematics education has moved from the transmission of disconnected facts, to the development of children's rich understanding and "internal constructions of mathematical meaning" (p. 524). In order to explore children's complex understandings, and to observe their approaches to the solution of problems, it is necessary to adopt a research approach designed specifically for this purpose. In this way, it may be possible for educators and policy makers to assess whether the application of progressive approaches within classrooms are working to develop children's mathematical understanding and robust problem-solving skills.

Goldin (2000) asserts that the use of task-based interviews, conducted with individuals or groups, does not necessitate a lessening of scientific rigour within a research study. Rather he discusses the rich findings which are possible from such research, wherein children are prompted to describe the reasons for their answers. There is a need for caution, however, as rigour is necessary to ensure that a research study underpinned by task-based interviews, is to some extent replicable. The researcher must be explicit about the elements of the research which were within his/her control, and those which were not. For example, the tasks, their presentation, the duration of assessment, allowable prompts and suggestions, materials provided, and the assessment environment should be prepared with diligence and foresight. The children's interpretation of the tasks, however, and their interaction with the environment will inevitably produce variations between the experiences for individual children of engaging with the research. This does not invalidate task-based interview studies, but rather validity is bolstered by paying explicit attention to aspects of the research such as the children's interpretation, and also the researcher's inferences. When the interviews have been completed, and the researcher undertakes analysis of the children's interactions within the interview setting, the validity of the research is strengthened by explicit description of the inferences made when the researcher pieces together representations of the children's thinking from the verbal utterances, gestures, drawings and any other data available (ibid.).

Goldin (2000) proposes ten principles for the design of interviews, which should aid the researcher in his/her search for rigour and validity. The recommended principles are as follows:

> Design task-based interviews to address advance research questions.
> Choose tasks that are accessible to the subjects.
> Choose tasks that embody rich representational structures.
> Develop explicitly described interviews and establish criteria for major contingencies.
> Encourage free problem solving.
> Maximize interaction with the external learning environment.
> Decide what will be recorded and record as much of it as possible.
> Train the clinicians and pilot-test the interview.
> Design to be alert to new or unforeseen possibilities.
> Compromise when appropriate (p. 542).

In Chapter 3 I will address the relevance of these ten principles to my research, as I outline the steps I took to fulfil those which were applicable.

Group interviews

As discussed in the section entitled "Assessment through interview", children should be supported in operating within their ZPD in order to assess their algebraic thinking which is emergent. If the aim of an adult-child interaction is to work within the child's ZPD attention should be paid to the balance of autonomy within the researcher-child dynamic (Jordan, 2004). In research on clinical interviewing of individual children Jordan stated that in order for teachers to gain insight into how young children are thinking, it is necessary that there is a co-construction of understanding between two parties. Co-construction occurs when adult and child are "interpreting and understanding activities and observations as they interact with each other" (p. 33). It is necessary therefore that the activity be meaningful and that the child and adult are engaged in interpreting information in the process of acquiring information. Not only must the adult remain vigilant to the pre-existing understanding of the child but also she must maintain a position of shared autonomy in the activity at hand. Jordan explains that co-construction depends on the extent to which a shared understanding is developed, and that this in turn depends upon the metaphorical distance between researcher and participant and on how power is shared between them.

In my research, where I am presenting mathematical tasks to children aged nine or ten years, it would not be possible to attain a sense of equal power-sharing between myself, as an adult stranger who brought the tasks with her, and a child to whom the tasks are novel, and the setting relatively formal. Jordan also recommends a questioning paradigm which involves questions to which the adult does not have ready access to the answer, where silence is allowed and the child's lead is followed. In presenting tasks to very young children, it may be possible to create a mutual sense of exploration, where the adult posits "I wonder how we could solve this task". The children involved in my research are beyond an age when they would accept that the level of novelty or uncertainty experienced by researcher and child were equivalent. It was necessary therefore to facilitate a situation where the children approached the task alongside others who were experiencing relatively similar degrees of uncertainty, so that they could engage in high quality mathematical discourse, involving proposal of ideas, asking questions, and justification of their thinking, as will be expanded upon in the next paragraph.

Mercer and Littleton (2007) present a three-part typology of talk demonstrated by children in group-work situations: 'disputational talk, 'cumulative talk' and 'exploratory talk'. When children are engaging in disputational talk, ideas are not shared, and children are not using their peer's thinking as a resource from which to build their own understanding. During exchanges of cumulative talk children are uncritical of each other's ideas, and construct common understandings, which are not necessarily justified. When children participate in exploratory talk they engage "critically but constructively with each other's ideas" (p. 59). Ideas are challenged, and therefore children are compelled to justify their thinking. Alternatives are offered rather than all group-members aiming for consensus. In this way, children may build upon and learn from each other's thinking, and be thus facilitated in working within their ZPD. As will be discussed in the Research Methods Chapter, it is not inevitable that children when working together will support each other to achieve a level of collaboration facilitative of thinking within the highest cognitive ranges of their ZPD. The methodological approaches I adopted in order to support such high-level collaboration are outlined in Chapter 4.

It is pertinent at this point to consider the relevant experience the children may have had in developing the necessary skills inherent in using 'exploratory talk' during discussions (Mercer and Littleton, 2007). Without opportunities to develop shared practices such as justifying answers, and building upon each other's suggestions, children may collaborate at very superficial levels when working together on mathematical tasks. Nic Mhuirí (2014) researched the patterns of mathematical talk occurring in some Irish classrooms, and found that the incidence of exploratory talk was extremely low. Supporting the findings of Nic Mhuirí, Kavanagh et al. (2015) also cited low levels of group discussion during mathematics lessons, where a teaching approach typified by being textbook based, and centred around the teacher, predominated in Sixth Class.

Therefore, in planning the delivery of the task-based group interviews, I paid close attention to concerns relating to children's participation, and collaboration, while accepting that some children may not engage in robust exploratory talk, and may therefore not be facilitated in engaging at the highest cognitive ranges of their ZPD.

The role of mediation

Along with peer interactions, many other factors impact on children's thinking within a task-based group interview assessment. While the task-based group interview underpinning my research will be an assessment rather than a teaching activity, the tasks presented to the children should be sufficiently challenging so as to facilitate the child's learning. Indeed, Schoenfeld (2007) proposes that participation in an assessment should always involve learning. Smagorinsky (1995) concurs and adds that rather than viewing a researcher as potentially contaminating an assessment by her presence, it is more beneficial to view a researcher as a mediator in the participant's operating at the upper end of his ZPD. Mediation, and the role of mediators, are central to this aim of supporting children in operating within the upper cognitive ranges of their ZPDs. The mediating agents I will attend to in this section are my input as interview facilitator, the tasks, the concrete manipulatives, and the children's drawings.

Mediation by the facilitator

Goos, Galbraith, and Renshaw (2002) highlight the collaborative ZPDs constructed when children work with "peers of comparable expertise", and emphasises the need to focus attention on the delicate role of the facilitator (p. 193). It may be constructive for the facilitator to support the children's engagement, and to avoid the children becoming distracted or confused in a manner that would not be productive. However, there exists the possibility that the facilitator's intervention may confuse children if she misinterprets their thinking, or misjudges what may be appropriate. Also there is a balance to be achieved between supporting children's progress and allowing children to figure out the mathematics collaboratively as a group, without intervention (ibid.). Goos (2004) presents an example of a teacher whose practice may be generalised as a template in order to guide facilitators of collaborative group work. The teacher in question presented a rich problem; elicited the children's opinions; withheld advice to "maintain an authentic state of uncertainty"; and intervened in order to encourage children to test their theories, or justify their thinking for their peers (p. 282). In addition, when considering the role of facilitator intervention Goldin (2000) refers to 'hints' and 'suggestions' that would aim to guide children past obstacles when strictly necessary. Building upon this view of a facilitator as one who should limit mathematical advice, but support collaboration, I will expand upon the actions I took as facilitator when outlining my Research Methods in Chapter 4. In describing my interventions during the interviews, I refer to all such hints, suggestions and requests for justification as 'prompts'. Prompts I contributed to the children's discussions took the form of both questions and statements, and examples are included in Appendix I.

Mediation through the tasks

In considering the role played by tasks within a task-based group interview, I would suggest that tasks play two key overlapping roles, which merit attention individually. Firstly, tasks may facilitate or inhibit engagement, and secondly the context of the tasks is key to revealing the children's understanding.

Knowledge is meaningful within the "social practices" where it develops (Matos, 2010). Assessment of mathematical thinking involves investigating children's application of their existing knowledge, and their propensity or preparedness to assimilate and apply new understandings during the period of assessment. This assessment must acknowledge, and incorporate the situated and socio-cultural nature of knowledge through attending to any socio-cultural aspects of the tasks presented. In particular, it is pertinent to consider the ethnic or socio-economic factors which may undermine children's participation. Smagorinsky (1995) discusses the notion of *telos*, meaning an optimal pathway for development. A sense of telos is underpinned by a myriad of "unexamined cultural assumptions about the ways in which people have historically developed" (p. 194). The concept of telos allows us to consider and discuss the cultural underpinnings of an assessment protocol. In accepting telos as socially constructed, inevitably assessments include elements which are more readily identifiable by children of certain social origins. As an example, Moll and Greenburg (1992) assessed Latino students who had been identified by their schools as underperforming. The research found that when the assessment was founded upon the ZPD, utilising mediational tools which were appropriate to the students involved, the potential of the students was far more analogous to that of their peers.

Frobisher and Threlfall (1999) present the possibilities of under assessment due to children's abilities in expressive language, as they highlight concerns regarding the language used in wording tasks; the context of the tasks and cognitive demands on the child which are extraneous to the measured competence. Shannon (2007) also demonstrates how the context of problems can mitigate against students. While a problem with no context may appear abstract, some contexts are in reality misleading and do not support the participants' thinking. Shannon outlines the example of the Four-Card problem, which is a problem presented with two contexts, where the underlying mathematics of the problem remains the same within both contexts. Drawing from the work of Johnson-Laird, Legrenzi, and Sonino Legrenzi (1972) Shannon presents overall findings from a collection of settings, which demonstrate that one iteration of the problem is significantly more accessible to many people than the other, and asserts that in solving a problem people usually "rely" upon contextual factors rather than objectively mathematising the constituent elements of the problem presented (p. 178).

Considering the findings of Moll and Greenburg (1992), Shannon (2007) and Frobisher and Threlfall (1999), I planned to choose mathematical tasks which would not exclude children due to variations in socio-economic backgrounds, would not require extensive reading, and would not complicate the underlying mathematics unnecessarily. For this reason the first two patterns I chose did not have a real-life context, and the real-life context for the third was a series of fences, as I believe that a fence is an item sufficiently commonplace to be familiar to all children. I did not include any contextual needs for fences of different sizes.

The second aspect of tasks which I wish to draw attention to in this section, is their role in supporting children in demonstrating their thinking as they consider the problem, and seek to find a solution. A number of factors are vital in designing tasks which truly indicate the ability of children in the area of algebraic reasoning. Ronda (2004) developed a framework of growth-points in students' developing understanding of functions. Building upon the descriptions of assessment tasks that support the construction of understanding as outlined by Hiebert et al. (1997), Ronda (2004) established principles of good task design, namely:

> Principle 1: Contexts familiar to the students should be used.
>
> Principle 2: Tasks should be formulated in such a way that what made tasks problematic would be the mathematics rather than aspects of the situation.
>
> Principle 3: The tasks should encourage students to use their natural strategies and use skills and knowledge they already possess.
>
> Principle 4: The task could be solved in different ways and would encourage students to use the strategy that would highlight the depth of their understanding of the concept involved. (ibid., p. 71).

I have attended to Principle 1 of Ronda's framework earlier in this section, and highlighted implications of Principle 2 in the section entitled "Assessment through interviewing". Principle 3 points to the need to avoid rote-learnt solutions, or setting tasks that lead children towards practised approaches. The patterning tasks that I used were sufficiently novel for the children, so as to satisfy this principle. Overlapping with Principle 3, Principle 4, if satisfied, should ensure that the cohort of children demonstrated a range of approaches, as relevant to their personal perspectives on the problems presented. To this end, it would be necessary that the problems be sufficiently problematic, and that the children needed to reason about the problem rather than arriving at a solution rapidly (Hiebert et al., 1997). At the commencement of each interview, I encouraged children to share their ideas with their peers, and emphasised that a variety of opinions are valuable in how the group thinks about the patterns. In this way I endeavoured to fulfil Principle 4 of the principles of task design as presented by Ronda (2004).

Mediation through the concrete materials provided

Matos (2010) states that knowledge develops through actions on material objects, in socio-cultural settings. For children to demonstrate emerging understanding during a task-based group interview, acting upon material objects would be an essential element of the scenario. The views expressed by Matos (2010) concur with the recommendations taken from the work of Vygotsky, and mentioned above in relation to facilitating children in demonstrating thinking within their ZPD. In particular, Vygotsky (1978) stressed that “children solve practical tasks with the help of their speech, as well as their eyes and hands”, suggesting that an absence of manipulatives may limit children’s potential in the solution of problems, and thus skew any assessment of their thinking. Facilitating children in constructing imagined patterning terms, may support them in testing hypotheses, and thus confirming their understanding of the structure of patterns.

Mediation through drawing

In discussing the development of children’s thinking, Vygotsky (1978) highlights the broadening of freedom of children’s operations as their speech develops. They are no longer dependent upon manipulation of the tools at hand, but can "include stimuli that do not lie within the immediate visual field" (p. 26). Using language the child may plan a solution, and may prepare stimuli to support achievement. Such stimuli may include jottings or drawings to supplement, and complement, the child's speech and any physical tools. Equally many of the points made in the preceding section with regard to concrete manipulatives are equally relevant to children's opportunities to draw and sketch their constructions. In particular, Kyttala and Lehto (2008) specifically refer to drawing as supportive if children are at risk of struggling due to the limitations of their working memory.

Summation

To summarise, the mediation provided by the facilitator, each child's peers, the tasks at hand, and concrete materials, or drawing tools are vital components of a task-based group interview where the aim is to facilitate children in demonstrating their potential, and stretching their thinking. It may not be appropriate to suggest therefore that the thinking demonstrated by children on the tasks presented to them would be generally applicable by them on all shape patterning tasks, particularly within a different context with different, or fewer mediating factors.

Conclusion

Smagorinsky (1995) discusses Vygotsky's assertion that assessment when positioned after learning produces very limited and often misleading information regarding a participant's ability. In order to ascertain current skill-level within a field, it is preferable to engage the participant in a learning activity positioned within the upper cognitive range of his ZPD. Also, skills-based assessments of mathematical proficiency are optimised when the researcher has some access to the thought processes underlying the participant's solutions to tasks. Administering a task-based group interview requires the researcher to apply her/his personal judgement and interpretation during the interview and rigour is required in both design and administration to maximise validity, which I will discuss further in the Research Methods Chapter. Analysis of the data gathered can be both difficult and time-consuming. The interview, however, affords participants an opportunity to engage in tasks at the highest cognitive range of the ZPD and also allows the researcher insight into the thinking underlying a participant's solutions to a task. The task-based group interview offers a model of assessment which will allow me to investigate the algebraic thinking skills of participants in my research.

Conclusion and Clarification of the Research Question

In my research I am planning to investigate the algebraic thinking of children attending Fourth class in primary school in Ireland. Central to algebraic thinking in primary school are the skills involved in generalising. Patterning activities provide a context wherein children may identify, predict, generalise and communicate their generalisations. In international studies into children's algebraic thinking in primary school, there are extensive research findings available for children in the age range 8-10 (Radford, Warren, Rivera). Many of the studies regarding generalisation select shape patterns as tasks which may facilitate investigation of generalisation skills while appropriate for this age range. Amit and Neria (2008) explain the use of shape patterns in their research of children's algebraic thinking by suggesting that "these problems are accessible to young students on the one hand, while being loaded with mathematical generalizations on the other" (p. 111). Many of the shape patterns used in the research of Radford, Warren, Lannin, Barbosa and Rivera involve multiplicative thinking. As children in Irish primary schools begin formal multiplication in Third class, I will select my sample of participants from Fourth class where multiplicative thinking embedded in tasks may not present a barrier to most children's engagement with tasks.

There has been a movement in how mathematics is perceived as a subject to be taught and learned, from a collection of disconnected facts to an interconnected web of skills and knowledge. Such progress has implications for how a researcher assesses and explores children's thinking, as interest lies on the underlying thinking rather than rote-learned facts or procedures. This change of focus is particularly relevant to algebraic thinking where the extent of children's thinking which is algebraic may be under or over-rated, and it is challenging to ascertain from a written answer the approach a child took in achieving it. The questions I present as pertinent for exploration are thus:

Question One

What strategies do children in Fourth class employ in seeking to construct general terms from shape patterns?

Question Two

What factors impact on the strategies the children adopt, and the success they experience in their constructions of general terms?

CHAPTER 3: RESEARCH METHODOLOGY

Introduction

The purpose of this research study was to explore the strategies children in Fourth class adopted when asked to construct a general term for shape patterns, and also the factors which impacted on their thinking. In this chapter I outline the methodological decisions I made when planning and undertaking the many elements of this research study. Adhering to the recommendation of Van Manen (1990) that one should distinguish between research methodology and research method, this chapter describes the theoretical basis for the practical steps outlined in Chapter 4: Research Methods. Creswell (2013) emphasises that the process of research flows from "philosophical assumptions, to interpretive lens, and on to the procedures involved in studying social or human problems" (p. 44). In this chapter, I present a detailed research design and conceptual framework which include explication of my epistemological assumptions, and the hermeneutic phenomenological approach through which I planned and implemented my research. Following the presentation of my research design and conceptual framework, in Chapter 4 I present the ethical decisions I made before undertaking this research with children, and proceed to outline the decisions I made which underpinned my sampling, group composition, the promotion of collaboration during group work, and the tasks I presented to the children.

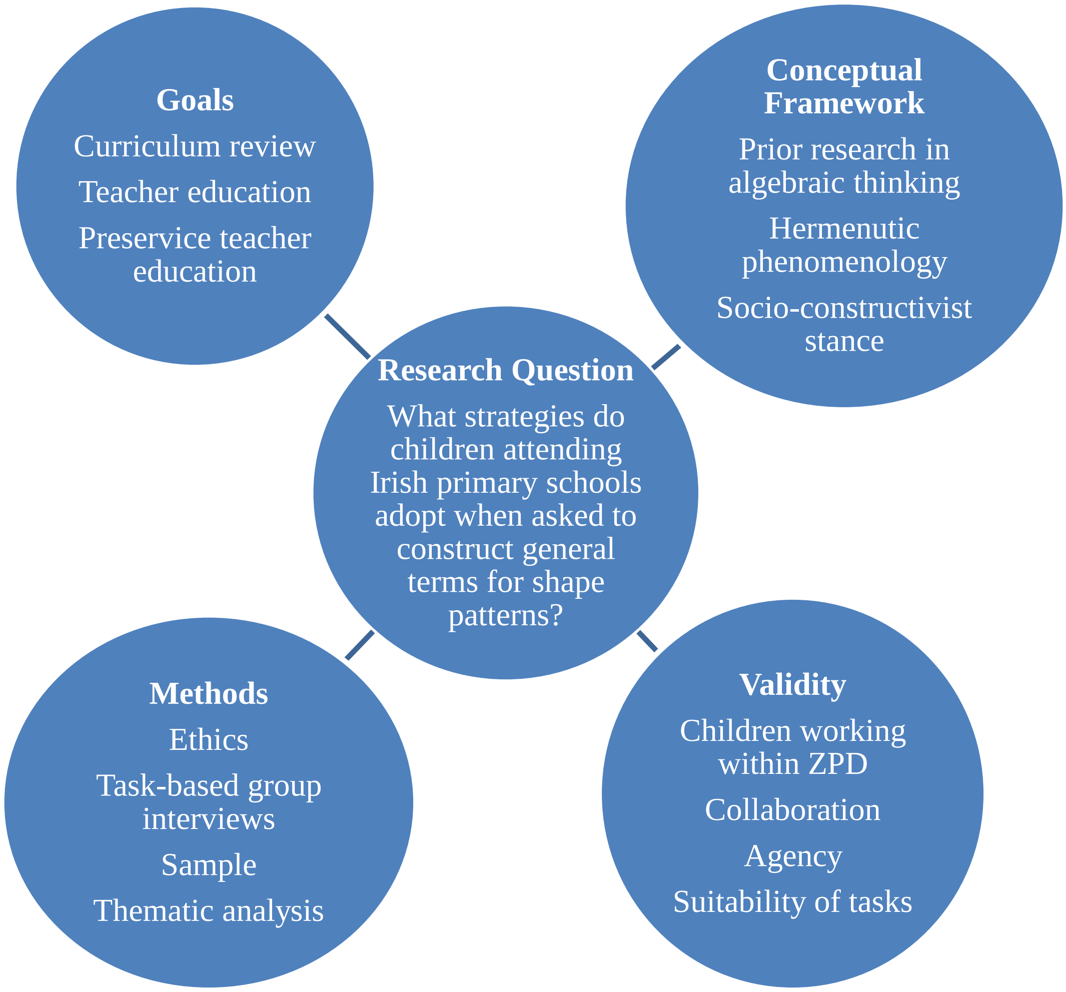
Details of study

Preceding my explication of the conceptual framework which underpinned my research, in this section I present an overview of my study. I conducted a pilot study in January 2014 which afforded me the opportunity to interrogate my research approach. Building upon my observations from this pilot I conducted a second pilot study in September 2015, and the principle data collection in November 2015. Data collection involved task-based group interviews with eleven groups, each of which had three or four participants. Data collection consisted of seven interviews which took place in School A in October 2014, and a further four interviews in School B in November 2014. Relevant school details are presented in Chapter 4, along with details of the children who participated. All children were attending Fourth class, and had a mean age of 9.83 years at the time of the interviews. Most interviews were video recorded, and two were audio recorded as some children did not give permission for video recording. To facilitate strong child agency during the interviews, I encouraged the children to share and explain their ideas with each other, and to explore differences between their approaches without verification from me (Howe et al., 2007). Four interviews were selected for in-depth analysis, where the responses of the four selected groups included a broad range of strategies and approaches to the patterning tasks presented. In Chapter 4 I furnish further details regarding my sampling frame, steps taken to encourage engagement and the choice of interviews for in-depth analysis.

Research Design

In planning my research design, I utilised an interactive model (Maxwell, 2012). Maxwell (2012) presents the view that qualitative research requires a flexible design which is inductive in nature. Such a design reflects not only the individual components such as the rationale for the research, the research question, and data collection methods, but also must attend to the interactions between components. Maxwell (2012) suggests the inclusion of five core components, namely research goals, a conceptual framework, research questions, methods and validity. Maxwell recommends developing a typological version of the model, wherein features are collapsed within categories. In developing a research design for my study I attended to such recommendations and using Maxwell's template, I developed the research design presented in Figure 3.1. In Appendix H, I have included the detailed model I used as the basis for the typological version presented here.

Figure 3.1. The design underpinning the research



In this chapter and in Chapter 4 I unpack aspects of this research design, including the use of task-based group interviews, the composition of such groups, the children who participated, decisions I made pertaining to ethics, and steps I took to facilitate collaboration between the children during their group interviews. For clarity Table 3.1 connects the elements of the design with the chapter sections. Goals of the research are considered in Chapter 1, and theoretical elements are explored in Chapter 2.

Table 3.1. Elements of the research design and their location within the thesis

Research Design Element	*Chapter*	*Section*
Rationale:		
Curriculum review	1: Introduction	
Teacher education	1: Introduction	
Preservice teacher education	1: Introduction	
Conceptual Framework		
Prior research	2: Literature Review	
Hermeneutic phenomenological	3: Research Methodology	Hermeneutic phenomenological research methods
Sociocultural stance	3: Research Methodology	Sociocultural stance
Research Methods		
Ethics	4: Research Methods	Ethics
Task-based group interviews	3: Research Methodology	Methodological decisions arising from the first pilot study
Sample	4: Research Methods	Children: Sampling frame and sample selected
Thematic analysis	4: Research Methods	Thematic analysis
Validity concerns		
Children working within ZPD	3: Research Methodology	Methodological decisions arising from the first pilot study
Collaboration	4: Research Methods	Promoting collaboration
Agency	4: Research Methods	Child agency
Suitability of Tasks	4: Research Methods	Tasks

Conceptual Framework

In this section I present my research approach and the epistemological underpinnings of my research. I also present the methodological reasons behind my use of task-based group interviews and the interview schedule. I frame the presentation of these elements within the narrative of my learnings from my first pilot study, as my findings from this study prompted me to explicitly evaluate my beliefs regarding children’s agency in mathematics and the conditions necessary for children to engage with mathematical thinking within their Zone of Proximal Development.

Hermeneutic phenomenological research methods

The focus of this section is the methodology which underpinned my research, both in design and implementation. In seeking a research design which could allow me to explore the strategies adopted by children in their construction of general terms for shape patterns, I was conscious that my role both as researcher and facilitator would be central to the children's interactions with the mathematics. I sought to frame my research within a paradigm which would acknowledge and accommodate my centrality to the research, and my overlapping roles of facilitator and researcher. Equally I sought a research approach which could afford me a broad perspective, to research both the children's constructions, and any factors that may have contributed to their thinking.

The research approach I considered most appropriate was phenomenology, as phenomenological research methods make use of the experience and intuition of the researcher, as s/he makes sense of the natural phenomenon s/he explores in all of its complexity (Groenewald, 2004). A phenomenological approach to research seeks to explore a phenomenon where it occurs, and acknowledges the many factors which influence how the phenomenon plays out within the given setting (Creswell, 2013). In exploring the strategies adopted by children in seeking to construct general terms for shape patterns, I not only analysed the children's actions, but I also sought to analyse the contributing factors that impacted on the strategies children employed.

The typical features included in phenomenological studies are a focus on a single concept, or phenomenon; the involvement of a group of individuals; a distance between the researcher and the phenomenon; and a focus on why aspects of the phenomenon occurred as they did (Creswell, 2013). In my research the strategies adopted by children in constructing general terms for shape patterns constituted the single phenomenon of focus. My research involved collecting and analysing the perspectives of groups of children, and due to my role as authoritative stranger I was not a member of the children's groups, but I was "partly set aside" from the phenomenon allowing me to focus on the responses of the children (ibid., p. 78). As mentioned above, reasons contributing to why the children responded to the patterning tasks in the ways they did are an important element of my research.

Somekh and Lewin (2011) define hermeneutics as "the process of interpretation", with a focus on how people experience and make sense of the world (p. 34). Describing arguments within the philosophical field of hermeneutics, Van Manen (1990) posits that it is possible to draw a distinction between descriptive phenomenology and hermeneutic phenomenology but asserts that interpretation is central to an exploration of the phenomenon in question. Within hermeneutic phenomenology however the emphasis is on interpretation, rather than description. The decisions I made in interpreting children's verbal, gestural and written utterances played a key role in my analysis and findings, and I consider it pertinent therefore to distinguish from a descriptive account by referring to my research approach as a hermeneutical phenomenological research design. The approaches I adopted to the interpretation of children's thinking are outlined later in this chapter.

Within human science research where the data collection and analytical methods are qualitative, there is a need for an understanding of 'knowledge' which is both rigorous and complex, to allow for the multi-faceted nature of lived experience. Van Manen (1990) asserts the following:

> To do hermeneutic phenomenology is to attempt to accomplish the impossible: to construct a full interpretive description of some aspect of the lifeworld, and yet to remain aware that lived life is always more complex than any explication of meaning can reveal (p. 18).

As I sought to develop knowledge of children's thinking, I therefore required a research instrument which involved interaction with participants, rather than detached data collection and analysis. In order to explore the strategies children may adopt in seeking to generalise, I selected a task-based group interview as a research instrument. In using a task-based group interview, there was the opportunity for participants to interact with each other, the concrete materials provided, and inputs from me as interviewer, in ways which might have deepened their responses to the task questions. Task-specific interview questions were presented on a worksheet, and I asked further questions to probe children's thinking, and to encourage children to compare and justify their responses. The written interview questions were designed to motivate children to firstly explore the patterns, and to proceed to the construction of general terms. The verbal interview questions, and prompts, were designed to best facilitate the children in engaging with the patterns, and in articulating their thinking in response to the tasks (see Appendix I for examples of questions asked verbally during the task-based group interviews). Further detail on my reasoning behind the choice of a group setting, and the steps I took to promote children's engagement are presented later in this chapter.

Goldin (2000) advises that "by analysing *verbal and nonverbal* behaviour or interactions, the researcher hopes to make inferences about the mathematical thinking, learning or problem-solving of the subjects" (p. 518, my emphasis). In seeking to explore children's mathematical constructions, I was conscious throughout that my inferences from children's comments were approximations of their true meaning. As Van Manen (1990) attests "a *good* phenomenological description is an *adequate* elucidation of some aspect of the lifeworld" (p. 27, my emphasis) and while I sought to unpick as best I could how children thought about the mathematical tasks, I posit that it is not possible to feel a sense of completion, or closure, in relation to the children's thinking, but rather that interpretation is ongoing (Postelnicu & Postelnicu, 2013).

Sociocultural stance

When outlining the methodological decisions I made, it is pertinent to make explicit my epistemological stance, and how it might have impacted on my interpretation of children’s thinking. My research is underpinned by a sociocultural view of mathematical understanding as socially constructed, and situated. Equally, the knowledge base which informed both design of the interviews, and how they were conducted, included an outline of developmental pathways in algebraic thinking. The plotting of developmental pathways assumes a constructivist position wherein children’s thinking is individualised. I accept the position that sociocultural and constructivist stances are complementary, rather than in opposition, and that each contributes to my understanding of how children think, and how one may assess children’s understanding (Dunphy, Dooley & Shiel, 2014). In this section, I expand upon my interpretation of these two perspectives, and outline in what ways they were relevant to my research. Additionally, during analysis of my first pilot study, and when I presented my findings at conference and at my Upgrade Viva, I found the framework underpinning the pilot study to be disconsonant with my understanding of how children learn. Therefore in this section I explicate how I built upon the experience of my first pilot study in order to develop the conceptual framework of this research study.

Traditionally in many classrooms, mathematics was viewed as an individual pursuit, where children worked alone, using paper and pencil, to answer questions and complete tasks. More recently mathematics education has been considered to be enhanced by collaboration, and opportunities to work together towards shared goals, as sociocultural theorists claim that “social experience of language use shapes individual cognition” (Mercer, Wegerif & Dawes, 1999, p.96). When conducting the research presented in this thesis, I held the position that children’s understanding of mathematics “is formed by the social interactions and communicative practices” that occur when they work with others, and that their mathematical reasoning is best supported by opportunities for them to discuss their thinking, and to work collaboratively (Mueller et al., 2012, p. 371). Later in the chapter, I outline steps I took to support children in collaborating, by encouraging them to share their ideas, and by drawing their attention to each other's thinking.

A constructivist philosophy of mathematics education implies that each child constructs his/her knowledge individually based upon his/her prior understanding, as opposed to passively receiving knowledge in parallel with others (Von Glasersfeld, 1995, p. 18). The prior understandings and conceptions held by a child determine how s/he understands new concepts with which s/he is presented (Desforges & Bristow, 1994, p. 215). In task-based research each child may understand the context of a task differently, bring a different interpretation to what s/he is asked to do and also think differently in arriving at the response (Ginsburg, 1997). A task-based group interview provides an opportunity to explore the understanding that each child brings when s/he responds to a task, as children question each other's reasoning behind responses. It also allows children to deepen their understanding of the tasks by building upon the thinking of others, and thus facilitates them in proferring richer responses.

As discussed in Chapter 2, it is useful for research to situate the understanding of children's thinking along a developmental pathway, in order to consider what the child has achieved in answering as s/he did (Pearn, 2005). This may seem implausible if one adheres to a strictly constructivist perspective wherein children present with a range of understandings and thus follow a range of pathways in their learning. There are however many external influences on the child's perception of the world and on his/her individual cognitive schemas upon which he constructs new knowledge (Ernest, 1994). While constructivism answers many questions within mathematics education, it also creates quandaries, as in how to account for the social dimension of the child's learning and what elements of the social dimension must be considered. A developmental pathway for mathematics incorporates both an evolving individual mathematical 'construction' and also an evolving prowess in adhering to socio-mathematical norms (Yackel & Cobb, 1996; Lerman, 2000).

Often, traditional assessment techniques may 'test' how closely children's thinking comes to what the teacher or researcher perceives to be correct. The interviews I conducted afforded me an opportunity to ask open questions which probed what, and how, the children thought about the generalisation of patterns. Outlining a possible developmental pathway supported my understanding of the range of understandings that participants may bring with them, while not constraining, or setting unnecessary limits on my expectations. Ginsburg (2009) presents a case for the use of a developmental pathway in such a way, as a knowledge base from which the interviewer approaches the interview. Interviews can allow access to how children think without predetermined limits, and developmental pathways may "provide useful background for understanding the individual child" (p. 117). The structuring of the developmental pathways informed my understanding of how children's algebraic thinking may develop, but I aimed at all times to apply my understandings from an evolving perspective, whereby elements of my understanding of the pathways may change during the data collection, or analysis stages of my research. For example some children's thinking may not be represented within any pathway, and some children may demonstrate thinking relevant to more than one growth point. The developmental pathway also helped me to situate my findings in the international research into children's developing algebraic thinking, including the work of Lannin (2004, 2006), Rivera and Becker (2011), Warren and Cooper (2008).

Methodological decisions arising from the first pilot study

Sampson (2004) explores the role of pilot studies in establishing research validity, and in identifying potential pitfalls in a research study. She emphasises that pilot studies not only allow researchers to refine research instruments, but also may motivate some to reconsider the planned research approach. In seeking to be thorough in how I conducted my research, I planned and piloted a clinical interview designed to assess algebraic thinking in January 2014. Following analysis of this pilot interview, I altered my focus from individual to group interview, included the use of concrete materials, and drawings, and reduced my role during the children's interactions with the tasks, following which I repiloted a task-based group interview before I collected further data. My reasoning behind these decisions is explicated fully in this section, and for clarity Table 3.2 contains an overview of both pilot studies and the principal data collection of this research, including dates and pertinent details.

Table 3.2. Pilot studies and data collection

Research Stage	*Date*	*Details*
First pilot study	January 2014	Individual interviews with 15 children. Interview tasks encompassed multiple aspects of algebraic thinking, for example equivalence, variable use, patterning and generalisation.
Second pilot study	September 2015	Group interviews with 13 children. Interview tasks focused on the construction of general terms for shape patterns.
Data collection	November 2015	Group interviews with 42 children. Interview tasks focused on the construction of general terms for shape patterns.

Details of the first pilot study

In January 2014 I conducted a pilot study with 15 children from one primary school, selected for convenience and reasonable distance from both my home and St Patrick's College. Four children were selected from each of Junior Infants, First class, Fourth class and Sixth class by random number generation. One child was absent during the weeks of the interviews, and 15 children participated. Children's names and class levels are presented in Table 3.3.

Table 3.3. Children who participated in the first pilot study

Class	*Children's pseudonyms*
Junior Infants	Dan, Aoibhinn, Paul
First class	Lionel, Holly, Michael, Rory
Fourth class	Natasha, Bella, Tara, Nikki[6]
Sixth class	Lisa, Aoife, Rachel, Sarah

I interviewed the children individually and interviews included tasks chosen to assess a broad range of algebraic thinking including equations, equivalence, variable use, patterning and the properties of computation. Following my collection of the data, I analysed the responses of the children and I presented my findings at a conference of the British Society for Research into Learning Mathematics (BSRLM), and at my Upgrade Viva. I submitted two papers to BSRLM, one for the informal conference proceedings and a second short paper for the Research in Mathematics Education journal. The feedback I received from Dr Dolores Corcoran, who examined my Upgrade Viva, and also from reviewers, editors and delegates at conference, guided the alterations I made to the focus and methodology of my research on the basis of this preliminary pilot study. The most significant alterations were my decisions to use task-based group interviewing as opposed to individual clinical interviewing, and to focus on generalisation from patterning in Fourth class rather than a broader examination of multiple elements of algebraic thinking throughout the primary class levels. The reasoning behind both decisions is presented in the following sections of this chapter.

Details of the second pilot study

Due to the significant changes I deemed appropriate to the structure and content of my data collection instrument, I planned a second pilot study in order to run through the

[6] The participants from Fourth and Sixth class are all girls because the school was of a structure typical of some Irish primary schools whereby all children attend together until Second class, after which boys move to an all boys' school and the girls remain to complete Third through to Sixth class.

revised interview schedule and format before data collection. In September 2014 I conducted this second pilot study with 13 children from one primary school, selected again for convenience and reasonable distance from both my home and St Patrick's College. 16 children were selected from Fourth class by random number generation. One child did not assent to take part, one child's parents did not give consent for his participation, one child was absent during the weeks of the interviews, and 13 children participated. Children's names are presented in Table 3.4.

Table 3.4. Children who participated in the second pilot study

Class	*Children's pseudonyms*
Group 1	Amy, James, Cristiano, Marie-Thesese
Group 2	Joan, Kelsy, Seán, John
Group 3	Zoe, Mia, Alice
Group 4	Tina, Jimmy

The children were presented with the three patterns on three group worksheets (Appendix L), and each child also had an individual worksheet, where they could record their thinking to the questions presented on the group worksheet. The children were provided with blank paper, pencils, coloured tiles and match-sticks. I introduced the session by encouraging the children to use the materials on the table, and to share their thinking with each other. I also suggested that each of them could work by him/herself if s/he chose to do so. I found on review of my field-notes (Appendix W), and the video footage, that at times there was very limited interaction between the children. I endeavoured therefore to increase collaboration by removing the individual worksheets, increasing my emphasis on collaboration when introducing the session and removing the suggestion of working alone, and avoiding groups with only two participants. While the modifications I made to my research instrument due to this second pilot study were not as significant as those from the first pilot study, I felt that the fine-tuning I engaged in with regard to collaboration was appropriate, and contributed to the rich data I collected from children's interactions during the main study.

Task-based group interviewing

Children, whose skills are emergent, may present with a variety of skills while complete solution of a problem remains beyond the range of their ability (Radford, 2012). When operating within their Zone of Proximal Development (ZPD), children are capable of demonstrating skills which they are currently developing (Vygotsky, 1978). In utilising an interview as a research instrument, I was aiming to support children in operating within the highest cognitive ranges of their ZPD, as they engaged with novel tasks, which required reasoning with which they may have been unfamiliar. When I commenced my research I held a radical constructivist view of mathematical understanding as individual and measurable, and I chose therefore to use individual clinical interviews as my research instrument. In this section I outline the reasons why I moved away from this position, and towards the use of task-based group interviews in redesigning my research to reflect my growing appreciation of the sociocultural understanding of knowledge as situated and socially distributed.

Jordan (2004) suggests that in order to support children in this way, it is necessary to develop a co-construction of understanding, whereby child and researcher works together to solve a problem. During my first pilot, I interviewed children individually. I trusted that my presence as interviewer during an individual clinical interview could facilitate children in working within their ZPD, if my focus was on establishing a co-construction of understanding. During the first pilot interviews however, many participants were reticent, often saying 'I don't know' even to questions about how they were thinking or why they responded as they did. While I aimed to discuss the tasks with them, rather than simply asking questions, no participant spontaneously suggested something and no participant asked a question. I was conscious that some children were not completely comfortable being interviewed by me. For example, one child in First class, Rory appeared anxious and startled at every noise during his interview, and both Tara and Bella from Fourth class demonstrated body language, such as lip chewing, which may indicate discomfort.

Of particular relevance to my use of interview for data collection was the response of children to the open-ended questions which formed the clinical interview. Ginsburg states that open-ended questions "are far more likely to provide insights into the child's thought" as they allow "the expression of personal ways of thinking" (Ginsburg, 1997, p. 126). During the interviews I asked the children many open-ended questions, such as "what can you tell me about this?", "describe this pattern"; "how did you decide what to draw?" During my pilot study, I recorded many non-responses to these open-ended questions, and also responses which indicated that the children did not feel that they could answer. Table 3.5 summarises the incidence of these questions among the responses of the four girls from Fourth class, for the section of the clinical interviews during which a shape pattern was explored.

Table 3.5. A count of the instances of open questions, and non-responses or participants indicating that they didn't know how to answer

	Tara	*Nikki*	*Natasha*	*Bella*
Open questions	8	13	7	10
I don't know/nonresponse	4	5	3	0
Duration	5.5 minutes	7 minutes	4.5 minutes	6 minutes

I would contend therefore that a co-construction of understanding was not developed to a meaningful extent with many of the participants. Ginsburg (1997) cautions that strangers may not have a sufficiently warm relationship with children to truly engage in a clinical interview with them, and he outlines the value of spending informal time with children prior to engaging them in a clinical interview, and also in adopting an informal approach to the interview itself. Ginsburg suggests that such an approach may encourage a sense of shared participation between researcher and child and also remove the sense that the researcher is a teacher, who may require a certain type of behaviour and possibly a certain type of answer to a mathematics task. From a practical point of view, spending informal time with the children involved in my research was not feasible, as it would have required a significant time commitment on my part and that of the school. I was depending upon the good will of schools to allow me access to children for interviews during school hours, and as advised by Creswell (2013), I was committed to minimising disruption to the business of the school, rather than seeking to spend time with the children in a setting with a fabricated sense of informality, in order to develop a sense of familiarity. While I aimed to adopt an informal approach in administering the interview, the setting was quite formal, as the interviews took place in a classroom in the children's school which the principal made available to me.

Additionally, Dunphy (2005) discusses the balance of power that exists between researcher and participant during the course of a clinical interview. She suggests that while the interviewer has 'control' over the interview in that s/he has designed the schedule and organised the meeting, the participant's responses govern the direction of the questions and the content of the discussion. "The clinical interview encourages and indeed depends for its success on such agency" (ibid. p. 86). I was concerned, during the individual clinical interviews I conducted during my first pilot, that some children could not exercise such agency with a stranger researcher.

Equally, in order to support co-construction and children working within their ZPD Jordan (2004) emphasises the need for adult and child to work together towards the achievement of a task. Within a task-based interview setting, involving children between the ages of nine and eleven, it is questionable whether the participant could feel that s/he and the researcher are working together towards information that neither of them knows at the outset. Also, in establishing a co-construction of understanding, Jordan emphasises the need for full 'intersubjectivity' where both researcher and participant are "considered to be experts in the topics of discussion". Given the novelty of the tasks involved in the interview underpinning this research, such intersubjectivity might have been impossible to achieve, as the level of expertise held by me presenting the tasks differed so greatly from the level of expertise held by the participants, many of whom may not have engaged in generalising or in exploring growing shape patterns heretofore.

Accepting that co-construction of understanding did not occur between interviewees and myself, as interviewer, it may thus be assumed that the children were not thinking within their Zone of Proximal Development, and that in many cases I was witnessing thinking which they had established prior to the interview and not the thinking which was "in the process of maturation" (Vygotsky, 1978, p. 86). Considering that growing shape patterns are not present in the Irish Primary School Mathematics Curriculum (PSMC), there is a likelihood that many of the children I interviewed would not have encountered such patterns before (Government of Ireland, 1999). It was therefore imperative for this study that children would be supported in their engagement with the tasks, in order to explore how their thinking was developing and concepts they had the potential to engage with, as opposed to robust thinking that has already 'matured' (Vygotsky, 1978, p. 86). I decided therefore to alter the dynamic of the interview by structuring task-based group interviews applying methodology from clinical interviewing.

The interview schedule, and the tasks presented to children

In designing the interview schedule for the first research pilot, I considered varied themes of pattern identification, extension, algebraic thinking within computation and variable use. My aim, however, for the main study was to schedule interview questions which would allow me to extract rich data about the participants' views of the single theme of 'generalisation of shape patterns' in as broad a sense as possible, to allow themes of significance to emerge. In conducting the interviews I aimed to remain open to alternative trains of thought or tangential discussions which might have produced data indicative of participants' views and understandings, even if such tangents strayed outside my initial broad themes.

In my pilot, I included specific tasks which were designed to facilitate children in demonstrating a range of thinking strengths or challenges, e.g. basic repeating patterns, numeric patterns, context based patterns, shape patterns, computation and the use of variables. Difficulties emerged in my aim to explore generalisation from patterning, the application of algebraic thinking to computation and the use of variable, as I found with every participant that I had too many items to cover and I did not allow sufficient time for detailed discussions. The tasks and underlying topics explored in this first pilot study and in the subsequent pilot and main data collection are presented in Table 3.6.

Table 3.6. Tasks and topics explored in 1st and 2nd pilot studies, and during main data collection

1st Pilot	*2nd Pilot and Main data collection*
Repeating patterns: extend, copy, describe 4 patterns of increasing complexity. Create one repeating pattern.	
Growing patterns: Extend, replace missing term, identify rate of change, identify 10th term, identify 100th term for 7 patterns, 4 numeric, 3 with figural representations	Growing patterns: Describe, extend, identify near term, identify far term for 2 patterns with figural representations
Context-based patterns: Extend, identify rate of change, derive rule for 2 patterns	Context-based patterns: Describe, extend, identify near term, identify far term for 1 pattern
Number operations: 3045-356; 2689+357; 7 + 6 = 9 +_; 55 – 34 = 56-_; 40 - _ = 38 – 21; 4 + 8 =_ + 7	
Variables: 5 questions of varying structures where the child is required to solve for an unknown, with or without the use of an expression involving a letter/word to denote the unknown. For example: Mia had €47. After paying for 3kg of fish, she had €20 left. Find the cost of 1kg of fish	

As can be seen from Table 3.6, the interview schedule for the first pilot contained far more items than subsequent data collection cycles. In order to present this vast array of tasks to children in one hour, it was necessary for children to complete each task without delay, and discussion was therefore minimal. A further exploration of this narrowing of focus is included in Appendix J. When redesigning the task-based interview for my main study, I incorporated this finding in the interview schedule, by narrowing the focus to shape patterns and including more probing metacognitive questions which asked children to articulate their thinking (Goldin, 2000), such as "would anyone like to tell me why they placed the tiles in the way that they did?"[7]

Thematic analysis

The question which this research seeks to address is "what strategies do children attending Irish primary schools adopt when seeking to construct general terms for shape patterns? " I adopted a hermeneutical phenomenological approach which I outlined earlier in this chapter, holding the position that any meaning I construed from children's words or actions was at best an estimation of their thinking. Also, I remained conscious during analysis that children's thinking was fluid and changing as they developed their thinking through interaction with each other (Cohen, Manion, & Morrison, 2000). In aiming to explore the strategies adopted by children, I undertook a 'thematic analysis' of the data I had collected, wherein I familiarised myself with the data; coded the data using a grounded approach, and also by applying existing theory; and identified themes for analysis. Braun and Clarke (2006) assert that thematic analysis is a research method for "identifying, analysing and reporting patterns (themes) within data" (p. 79). Thematic analysis not only offers a structure for the organisation and rich description of data, but may also give the researcher insight into the research topic as patterns and themes are identified.

I considered applying grounded theory but there existed a significant body of research in the field of children's constructions from shape patterns which was applicable to my research question. Feedback I received from a Current Report I had submitted to

[7] This question was asked during the interview with the 3rd group of children in the 2nd school. Further details regarding the schools, and the groups of children who participated in the interviews is presented in Chapter 4.

Research in Mathematics Education , and also a paper I had submitted to the Conference of European Research in Mathematics Education advised me that applying existing frameworks would support my analysis of the data I had collected (C. Smith, personal communication, October 16, 2014; H. Stromskag, personal communication, December 7, 2015). I also felt that my role as researcher was very central in that I designed the research instrument, I facilitated the interviews, and I analysed the data. It seemed unlikely that themes could emerge from the data without my interpretation impacting on what emerged (Braun & Clarke, 2006). In the following sections I outline the steps I took in analysing the collected data, and the theoretical underpinnings which guided my decision-making.

Ethics

In all research, it is imperative that researchers abide by strong ethical principles in order to minimise any negative impact on participants or research sites, and also maximise the positive impact of research involvement for all concerned. Creswell (2005) draws attention in particular to matters relating to respecting the rights of participants, respecting the research site and also reporting research fully and honestly. The rights of the participants include voluntary inclusion and freedom to withdraw from the study, information regarding any social consequences the study may have, and information pertaining to the scope and purpose of the study. Respecting the site of the research is of particular relevance when conducting research in a school, and due concern for the work of the school must underpin plans for the involvement of children. In this section I outline the steps I undertook to adhere to strong ethical principles throughout this research study.

A preliminary step in conducting my research was gaining access to the participants. Creswell (2013) advises that researchers should aim to establish a basis of trust and openness with all individuals involved in the research site. Consent may be mandatory from gatekeepers such as parents and personnel involved in management of the school, but consent should also be sought from class teachers with whom a positive relationship of mutual respect will reduce the possibility of barriers to successful data collection. Therefore, I presented school principals, heads of Boards of Management, teachers and parents with the plain language statements and informed consent forms included as Appendices F to K. In distributing plain language statements and consent forms, I met with most of the people whose consent I requested, in order to outline the research to them. I aimed to facilitate discussion regarding the purpose and audience for the research, and I disclosed all information which was requested or which I anticipated individuals should possess. I assured children and gatekeepers of confidentiality and my intentions to protect their anonymity, and I emphasised the ethical underpinnings of my research.

Informed consent for participation in research should adhere to the four principles of "competence, voluntarism, full information and comprehension" (Cohen, Manion & Morrison, 2000, p. 51). In requesting consent from gatekeepers I endeavoured to ensure that I adhered to these four principles, by emphasising that participants could withdraw from research at any time, by withholding no information and by making myself available to answer all questions. When informed consent had been received from the relevant gatekeepers, I met with the children, outlining in detail what I was researching, what their participation would involve, and how I would use the data. I requested the children's assent, adhering again to the four principles outlined above. Samples of the Plain Language Statements and Informed Consent Forms for both gatekeepers and children are included as Appendices A through to F.

In conducting research with children, the theory of informed consent requires close scrutiny, in particular for children with Special Educational Needs. Simple language and icons were used on the literature for distribution to children and I was prepared to make appropriate adjustments, for example large font or audio- tape for children with visual impairment, if such were required (Fraser, Lewis, Ding, Kellett & Robinson, 2004). The needs of all participants were taken into account in the preparation and delivery of the task-based interview. Reasonable adjustments were made for children for whom English is an additional language, for example, I rephrased questions if I suspected a difficulty in comprehension, and I aimed to facilitate sufficient time for children to formulate responses. I reasoned that it would not have been appropriate, however, to engage children whose English language competency was at such a reduced level that it would skew their performance on the interview. Two children were identified by their class teacher as possessing English language competency which would inhibit their engagement in the interview. I was concerned that asking children with such limited English language fluency to participate may place them in a position where they would struggle to understand and engage, thereby feeling isolated and inadequate. I did not believe that it was ethically appropriate to risk placing children in this negative position, and the two children in questions were thus removed from the sample selected.

While it may not be possible to guarantee anonymity, I endeavoured to safeguard the identity of participants and schools. To this end, I considered it imperative to choose schools which were suitably distant from both St. Patrick's College[8] and my home. In this way the schools might be less readily identified by members of the surrounding communities who might encounter the findings of my research. In preparation of video clips for future presentation to others I reduced the possibility of the school being recognised by obscuring any features which might identify the school, for example school crests, statues or buildings. I deliberately selected clips for presentations that minimised the possibility of embarrassment for participants. The focus of any clips

[8] In 2015 when I collected my data, I was a PhD student of St Patrick's College. On September 30th, 2016, St Patrick's College incorporated with Dublin City University, under the umbrella title of Dublin City University.

chosen would be on children's thinking as examples of what was occurring in Irish primary schools in a broad sense. If clips contained children's errors which may be of interest, children's faces would be obscured through the use of video editing software in order to avoid embarrassment. To ensure confidentiality, pseudonyms for children and schools are used within this thesis, and would be used when showing or referring to video clips. All data collection and analysis would utilize pseudonyms selected by the children as opposed to children's names. All children chose gender-preserving pseudonyms. Identifying information, such as school type and location, are not referred to in the analysis of results to ensure that children are not readily identifiable.

There is a further ethical imperative in researching with children that the experience does not cause harm to the participants as would undermining their view of their abilities (Alderson, 2004). The tasks presented to the children were chosen to be sufficiently challenging so as to explore children's thinking as fully as possible without incurring upper limits. There were situations during the interviews when I perceived that a child was uncomfortable with a question I was putting to him/her, or to the group. If there was evidence that a child would prefer not to be drawn on individually, I erred on the side of caution, and did not put the child on the spot, or seek in any way to find fault with their thinking. Also I endeavoured during the interview to provide all participants with praise and recognition of their thinking, which has been seen as a positive benefit to children of participation in research (Dockett, Einarsdottir, & Perry, 2009). When distributing the plain language statements to the children, and in the introduction to the interview, I explained to each participant that I was interested in hearing the ideas they had about mathematics. I emphasised regularly throughout the interview that the ideas they shared were very interesting to me (Dunphy, 2005). The children responded well to this encouragement, and on the video footage it was possible to see children smile warmly when I expressed my interest in their ideas. One child, Alex, commented that he “never knew grown-ups could be interested in children’s thinking”.

Conclusion

To summarise, the research design I adopted was a hermeneutic phenomenological design, underpinned by a sociocultural view of how children develop understanding. In this chapter I have outlined the theoretical stance I took in planning and administering my research study. Groenwald (2004) states that “The aim of the researcher is to describe as accurately as possible the phenomenon, refraining from any pre-given framework, but remaining true to the facts” (p. 44). Within my research I aimed to fulfil this aspiration of describing the phenomenon of children’s strategies in constructing general terms as accurately as possible, while acknowledging the centrality of my role, and the hermeneutic nature of interpretation of children’s communications. The research instrument I utilised was a task-based group interview. A group interview allowed me to explore the personal understandings of the children, with the myriad of approaches and perspectives they brought to their groups. Decisions I made relating to group composition were highly relevant to this exploration, and in Chapter 4 I present such decisions and their theoretical underpinnings. Equally the group interviews afforded the children an opportunity to explore the mathematics, and to consider new concepts, with which they were possibly unfamiliar. Working together, and building upon each other's ideas facilitated the children in working within their ZPDs, and in applying emergent thinking in novel contexts. In preparation for the interviews, I drew from the guidance of experts in the field of research with children, and I was rigorous in ensuring that my research adhered to strong ethical principles. I sought, and gained, consent from adult gatekeepers, and assent from child participants. I maintained confidentiality, aimed at all times to protect the participants’ anonymity, and I endeavoured to facilitate a positive experience for all children.

CHAPTER 4: RESEARCH METHODS

Introduction

Qualitative research methods are inherently interpretive in nature, and it is incumbent upon the researcher to outline in detail the research methods s/he adopted so that they may be scrutinised for rigor (Creswell, 2013). Building upon the presentation in Chapter 3 of my research design and conceptual framework, in this chapter I outline the decisions I made which underpinned my sampling, group composition, the promotion of collaboration during group work, and the tasks I presented to the children.

Also in this chapter, I present the decisions I made in collecting and analysing my data, including the various steps involved in preparing for and implementing the research instrument of a task-based group interview. In Chapter 3 I presented the hermeneutic nature of my phenomenological approach, and in the section of this chapter entitled 'Inferences from children's comments' I outline in greater detail how this was made manifest in the decisions I made in interpreting children's thinking.

In planning the interviews, and in analysing the children's thinking, I drew on the developmental pathway outlined in Chapter 2. The developmental pathway informed my thinking, but I sought responses at the highest level from all groups of children, and I did not limit my expectations of what the children might have been capable of doing. Also described in Chapter 2 were the affordances of task-based group interviews, and I deemed this research instrument appropriate in addressing my research question as I planned to unpick the reasoning behind the children's answers. I provided the children with materials, and I encouraged children to probe their own and each other's thinking, in order to facilitate engagement in the highest cognitive ranges of their respective Zones of Proximal Development. I outline the steps I took to facilitate children in engaging fully with the tasks in the section entitled 'Phase 2 - Conducting the interviews'. I explore theory relating to the interpretation of the children's comments, and my work in approximating the children's underlying thought processes. I present a comprehensive description of the phases of analysis I worked through from when I first interviewed the children to my writing of the analysis chapters. Finally I discuss matters of reliability, validity and replicability.

Phase 1 - Preparation

Children: Sampling frame and sample selected

Participants were selected from two schools, which were quite different in socio-economic terms, and in geographical location. While this research study is small scale and the findings cannot be considered to be generalizable, Eivers et al. (2010) found that achievement in mathematics was "highly correlated" with school socioeconomic status. Significant progress has been made in Ireland in addressing educational disadvantage and more recently Kavanagh, Weir and Moran (2017) found that in many cases DEIS[9] Band 2 Schools were attaining at or above national norms in mathematics. Overall however in the research reported by Kavanagh et al. (2017) children attending 6th Class in DEIS Band 2 schools demonstrated lower mean standard scores than the comparable general population of children, and the mean standard scores for children attending 6th Class in DEIS Band 1 schools was significantly lower than children in Band 2, indicating that achievement in mathematics varied in line with the assessed level of disadvantage. I considered it necessary therefore to avoid selecting the entire sample from one school whose intake may reflect a narrow socioeconomic distribution. Table 4.1 presents some information about the schools from which I selected the sample of children for participation in this research. The school status of School 2 was DEIS Band 1.

[9] Delivering Excellence in Schools (DEIS) is a government programme aimed at addressing educational disadvantage (Government of Ireland, 2005). Schools which are assessed as experiencing socioeconomic disadvantage receive a DEIS designation and numerous associated incentives including reduced class sizes, access to a numeracy support service, and Home School Community Liaison service which develops and promotes numeracy programmes for families. Urban DEIS schools are designated as Band 1 or Band 2, where schools with a higher assessed level of disadvantage are designated as Band 1.

Table 4.1. School Profiles

	School 1	*School 2*
Enrolment (September 2014)	494	755
Location	Small town Population < 5000	Large town Population > 20000
School Status	No DEIS designation	DEIS Band 1

In planning a sampling strategy for my research, I endeavoured to achieve reliability and validity of my research findings while remaining cognisant of ethical considerations. As demonstrated in my research design, ethical considerations drove methodological decisions I made, in line with the highest standard of educational research involving children (Creswell, 2013; Somekh & Lewin, 2011). While there may be research benefits to selecting a stratified sample which is in some way representative of an attainment spread in a classroom, I utilised a random number selector to select a simple random sample from pupils attending Fourth class within each school. If a sample was selected based on attainment, it would have been necessary to inform the parents and children of this to ensure that their consent was informed. It may thus have been detrimental to low-attaining children to learn that their selection was due to their difficulties in mathematics. I did not feel that the benefit of this research to the participants could outweigh any embarrassment suffered by my informing children that they had been selected due to their low attainment.

Dockett et al. (2009) discuss how excluding some children from research can result in a reduced recognition and respect for the diversity of understanding that exist among children. Also Gray and Winter (2011) talk about the rights of all children to participate in research and to have their voices heard, and they highlight the ethical considerations of excluding children on the basis of an additional educational need. Taking the recommendations of Dockett et al. (2009) and Gray and Winter (2011) as guidance, I did not exclude children due to potential difficulties they may have encountered during the interview, whether in terms of the mathematics or their ability to attend. I did however ask the class teachers whether any of the randomly selected children had a level of English language competence which would impact upon their ability to explain their thinking. Two children were identified as having a level of English language competence which would cause them considerable difficulty in explaining their thinking and these children were not included in the sample. As described in the Ethics section of Chapter 3, I aimed at all times to ensure that my research was guided by strong ethical principles, including respect for participants. I intended that, as far as I could facilitate, the overall impact of participation for each child would be positive, adhering to “best outcomes based ethics” whereby benefits are maximised while the potential for harm is reduced or eliminated (Alderson, 2004). I was not convinced that this would be the case for children whose English language capacity was very poor. My emphasis during each interview was for the children to talk and discuss their ideas with each other because I was interested in hearing what they thought. I was concerned that children with reduced English language would be therefore excluded from full discussion with the other group members and have a negative experience overall.

I also asked the school principals to inform me of any children who were vulnerable or who may have found an interview with a stranger to the school an unpleasant experience. Neither principal identified any of the children as at risk of distress from their involvement.

Not only is there an ethical imperative to allow all children access to involvement in research but I also believe that all children have mathematical proficencies which are worthy of exploration. In many Irish primary schools, children are assessed as needing support in mathematics when their percentile ranking derived from a standardised assessment falls below the 12th percentile. As I presented in my Literature Review, there is an inherent difficulty in assessing algebraic thinking and a child's performance on a standardised assessment may not give a strong indication of his/her current level of algebraic thinking. There is scope for doubt about the validity of assigning young children a percentile ranking based on a single standardised assessment, and as such I did not exclude children in this category. Similarly, I did not exclude children who had been assessed as presenting with attention deficit disorders, or other low incidence conditions which might have impaired their ability to learn independently in traditional school settings. Hiebert et al. (1997) state that all children can learn mathematics with understanding, regardless of the challenges they face due to specific learning difficulties. Boaler (2009) concurs, stating that "the ability to do mathematics, at school levels, is not some sort of special gift that is bestowed upon a small number of children" (p. 96).

As outlined in Chapter 3, my research is underpinned by an understanding of learning as socially distributed. As such, it was necessary to facilitate collaboration among the children who participated. Davis and Simmt (2003) discuss the 'emergence of a mathematical community' and conditions which must be met in order for children to learn. One such condition is 'internal diversity' which occurs when the members of a group present with a range of approaches and perspectives to problems presented to them. While Davis and Simmt contend that all school-based groups contain internal diversity, "no matter how homogeneously conceived", I held the position that to in some way exclude children on the grounds of perceived mathematical attainment would not be of benefit in seeking to promote collaboration within groups (p. 149).

For my first pilot research I selected four participants from each of Junior Infants, First class, Fourth class and Sixth class. While it was not feasible for the sample size to be sufficiently large so as to generate generalizable findings, I aimed nonetheless to include four children in each class involved in the research in order to allow comparisons to be drawn and to increase the probability that a spread of mathematical attainment would be represented. I found during this first pilot study that the theoretical base required for an examination of algebraic thinking across eight years of primary school was very broad. In Chapter 2, I discussed the role of shape patterns in developing, and making explicit children's algebraic thinking. In order to investigate children's algebraic thinking deeply, I decided to narrow my focus of attention to generalisation on shape patterns. Rivera and Becker (2011) discuss the role that multiplication and addition play in growing shape patterns. There is a significant requirement on children to think multiplicatively in generalising many shape patterns and the ability to express multiplicative thinking would improve participants' access to communicating their ideas. In the Irish Primary School Mathematics Curriculum (PSMC), children commence multiplication formally in Third Class and I decided therefore to select my sample from Fourth class, where I could have a reasonable expectation that most children would be familiar with multiplicative thinking. Also Frobisher and Threlfall (1999) present the possibilities of under assessment on patterning tasks due to children's abilities in expressive language and in selecting Fourth class, I aimed to interview children who may have been more articulate in terms of their mathematical understandings than children in junior classes. The sample of children who participated in the study is presented in Appendix K.

To conclude, my sample comprised 42 participants, selected from two schools. In selecting schools I aimed to avoid clustering by drawing from different socioeconomic backgrounds and different locations. The participants were selected randomly from a sampling frame which was a subset of all pupils attending Fourth class in each school. Two children were excluded from the sampling frame because they were identified by their teacher as possessing an English language competency that would limit their responses to questions and tasks, and thus their participation may be an overly negative experience for them.

Phase 2 - Conducting the Interviews

Data collection

As outlined later in this chapter, the data for analysis in this research study was drawn from task-based group interviews involving children who were attending Fourth class. In this section I outline the logistical steps I undertook in facilitating the interviews, and in collecting the data. Findings of qualitative research are the interpretations which the researcher brings to the interview transcripts and field notes. For this reason, the research must be systematic and rigorous for the findings to be valid (Denscombe, 1998). While findings which emerge from qualitative research are driven by the data, these findings cannot be generalised to all children or even to a subgroup of children. The task-based group interview employed in my research can only measure how each participant performed at a moment in time. On a different day, in a different environment and with a different interviewer, or different peers, a participant may respond differently to tasks. The findings may however shed light on the ways in which children attending Irish primary schools respond to tasks which required them to identify a pattern, predict terms and consider the general case.

Interviews

For each interview, I collected children from their respective classrooms and brought them to the room which was made available for my use by the school. I collaborated with the classroom teachers and school leadership to choose the most appropriate time for the children to participate in the interviews. I assigned children to seats and activated the video and audio recorders. I introduced the children to the tasks they would be asked to complete using the following introduction:

> What I'm going to ask you to do, I'm going to ask you to do together.

> There are materials on the table for you to use. I'd like you to think out loud and talk to each other, see what ideas other people have and what way they can help you to think about the tasks. Each one of you will have ideas and it's important to talk as much as possible, tell the others your ideas and also what you think about their ideas. Your ideas will help the others to have more ideas.
>
> It's very important while completing these tasks that you talk to each other as much as possible. It's not often that a teacher will ask you to talk a lot, is it?

As I mentioned in Chapter 3, this introduction was carefully scripted to support child agency, and facilitate collaboration between children. The worksheet presented to the children is included in Appendix L.

During the interviews, I aimed to allow children time, and freedom, to construct their personal understandings of the patterns presented. I intervened if I observed that children were straying off task, if I wanted to gain greater insight for the reasons behind their thinking, or to encourage them to compare their thinking to that of others. I read through the questions from the worksheet with some groups, and some children raised their hands, indicating, I interpreted, that they expected me to take responses from them. I regularly asked the children to share their thinking with each other, rather than presenting ideas to me. Examples of the verbal questions I asked and prompts I made are included as Appendix I.

Group composition

When my sample of children for interview had been randomly selected I organised each class cohort into groups of four with input from the class teacher. Webb (1991) collated a meta-reflection of research studies which looked at connections between peer interaction in small groups and educational achievement. Within the studies summarised by Webb, the group size was typically four. Howe et al. (2007) also recommend groups of no more than four or five children, and Mueller et al. (2012) worked with groups of four children in their research.

In planning for the composition of the groups in my study I drew from Webb's overview of an optimal group composition and also her supporting recommendations in establishing and affirming desirable peer interactions within groups, which I discuss below. In considering the gender composition of groups, Webb (1991) recommend avoiding groups with unequal numbers of boys and girls as they may "disadvantage at least some members of the group" (p. 382). Webb explains that due to issues of mathematical identity and status among boys and girls, studies have shown that when the gender distribution of a group favours one gender there is a reduced tendency among group members to ask and answer each other's queries, and requests for clarification. Of the eleven groups interviewed for this study, nine had exactly 2 boys and 2 girls, one group had all boys and one group had 2 girls and 1 boy. Of the 11 groups I selected 4 for in-depth analysis of their responses to the patterns. A comprehensive description of my reasoning behind the selection follows in a subsequent section of this chapter. Of the four groups I chose for closer analysis, all included exactly two boys and two girls.

Davis and Simmt (2003) highlight aspects of groups which support children in collaborating successfully in striving to solve shared tasks. They contend that groups should uphold conditions of "internal diversity", and "redundancy" (p. 147), but that such conditions do not guarantee genuine collaboration. Redundancy refers to sameness in characteristics of group members, which Davis and Simmt contend is essential "in triggering a transition from a collection of *me's* to a collective of *us*" (p. 150, the authors' emphasis). Within my research redundancy was upheld, as all children in each group attended the same school and encountered the Irish curriculum interpreted through the same series of books. While they may not have all received instruction from the same classroom teachers, I reasoned that the children within each group shared many common school norms in terms of how mathematics was perceived as a subject to be taught and learned. As mentioned earlier, I aimed for diversity of gender, and mathematical attainment, within groups to support the condition of internal diversity.

Promoting collaboration

Mueller et al. (2012) point to teacher moves which facilitate collaboration between peers in group settings. They suggest that "contrasting opinions are encouraged" (p. 370), and during the group interviews I regularly encouraged children to compare their work with each other, and to explore the differences. I aimed to support an understanding of the value of different approaches, and how somebody else's thinking, while different to my own, may support me in thinking more broadly or in adopting a different perspective. As discussed in the Literature Review Mercer and Littleton (2007) identify rich group work discussion as 'exploratory talk', wherein children build upon each other's thinking through questioning, discussion and justification. Mercer and Littleton advise that teachers support children in engaging in high quality discussions by modelling desired behaviours, for example, seeking multiple opinions, or answers to a question; asking 'why' when relevant; drawing out reasons for answers proffered. During the interviews, I sought to implement this advice by asking for many children's ideas to questions, rather than taking one answer; regularly asking children to describe their reasoning behind an answer; and also by asking children to explain their thinking to their group.

Webb (1991) emphasised the value of a group reward in promoting meaningful interactions. It would not be appropriate in this setting for me to reward the children in any concrete way for their progress in tasks, but I foregrounded the identification of activities as belonging to the group in order to promote a sense of working as a group. Webb also highlighted the importance of children within the group sharing their thinking and responding to each other's ideas, and stated that there are strong connections between such successful peer interactions within groups and achievement in group tasks. Ginsburg (1997) suggests adopting a manner which would create a sense of difference between the interview setting and the classroom setting, which I aimed to do in explicitly comparing my requirement for the children to talk with an assumption of the norm within the children's classrooms. For example in opening the interview with School 1, Group 1, I stated "I'm going to ask you to talk a lot to each other. It might not be too often that you hear a teacher asking you to do that, to just talk, and talk and talk."

Child agency

The establishment of child agency in how the children engaged with the patterning tasks presented was central to my facilitation of collaboration. Mueller et al. (2012) describe agency as

> a key player in any mathematical learning, and following its footprint as it is imprinted on mathematical argumentation and reasoning will assist us in tracing the origin of mathematical ideas as well as the way that discourse, and the participants in the discourse, influence the ultimate mathematics that is constructed (p. 373).

In seeking to establish child agency during the group interviews, it was necessary for me to be mindful of the 'expert authority' which children often perceive teachers as possessing, as the presence of expert authority can work to inhibit agency in classrooms (ibid.). I sought to construct a setting wherein the children would express what they thought, rather than what they believed I wanted to hear. I also intended that the children would be motivated to take risks, and to make tentative suggestions without fear of being incorrect.

Cobb, Gresalfi and Hodge (2009) discuss the interplay between authority and agency in the mathematics classroom. They state that authority is "about "who's in charge" in terms of making mathematical contributions", and contrast classrooms where the teacher sits as judge over all comments, with classrooms where the validity of comments is agreed by children, and teacher, based on a justification of the reasoning used. If the children who participated in the interviews did not have previous experience of shared authority over mathematical thinking, it would not be possible for me to dissipate their sense of a teacher as possessing all, or superior, mathematical knowledge. In my introduction and throughout the interviews I stressed that children should present their ideas to each other, rather than to me. I encouraged the children to ask questions of each other, and to seek explanations if somebody else's thinking was not clear to them. In this way, I sought as far as was possible, to facilitate a sense of child agency, wherein the children decided whether suggestions made by group members were valid, and the children took the lead in deciding which strategy to adopt. For example, one child Emily in her group's discussion of the near generalisation for Pattern 2 adopted a strategy which required attention in order to produce a correct term construction. I drew Emily's attention to the original terms presented and to an earlier term she had constructed, but I did not point out her error. Rather, I allowed her to figure out the required modification to her approach through further manipulation of the concrete materials, and through her need to explicate and justify her thinking with her peers. Further investigation of Emily's thinking is presented in Chapter 5, in a section entitled *Emily's thinking*.

Selection of groups for in-depth analysis

The aim of this research is to explore the strategies employed by children in constructing general terms from shape patterns, and also what factors contribute to their thinking about the patterns. I aimed to produce 'thick' rather than 'thin' description of the different facets of the approaches adopted by children (Miles & Huberman, 1994). Within this thick description, I intended to draw attention to relationships that children observed or overlooked within pattern structure, the aspects of the pattern upon which they commented, and also the multiple influences on their thinking. Schwartz-Shea and Yanow (2012) highlight that researchers engage in "analytic sense-making" during their engagement in field work, in addition to analysis during subsequent stages of research studies (p. 90). During the task-based group interviews I conducted for this research study I began to perceive that the volume of data I was producing was possibly greater than necessary for me to produce thick descriptions in addressing my research questions. For this reason, I reconsidered the number of group interviews it would be appropriate to analyse, and I felt that four interviews would give me the optimal balance of variety in the strategies adopted without excessive repetition. In this section I expand upon this decision-making, and outline the process I undertook in selecting the four groups for interview.

When reviewing the footage of the children's group interviews, I found similar themes emerging from multiple interviews. As discussed above, qualitative research affords the researcher opportunities to reconsider decisions made, as one learns though the ongoing research. Powell, Francisco and Maher (2003) highlight that video data presents a large quantity of information, and that this enormous dataset creates a challenge both in developing the necessary familiarity for analysis of the data, and also in dealing with the fine detail. As mentioned above, when conducting the interviews, I realised how potentially rich was the data generated by the interviews, and I was motivated to ensure that I could analyse these data in sufficient depth. My initial observations were that during each interview, the children were adopting a broad and varied range of strategies. I reasoned that more would be gained in terms of addressing my research question by in-depth analysis of some groups than broad analysis of all, both in terms of the range of strategies adopted, analysing the language children used to unpick their approaches, and also analysing the interactions which supported the children's thinking. During the interviews interactions occurred between children; between children and the resources; between children and me, as facilitator; and between children and the tasks. These many and varied interactions had a significant impact on the strategies adopted by children, and as such, were relevant to the research question. In order to facilitate in-depth, rather than surface level analysis, I chose, therefore, to conduct a focused analysis on four interviews. As described in the next paragraph, I used a systematic approach to select four groups, by exploring the attainment of the children on the patterns, and also the interactions between the children in each group. During analysis, I found that the four selected groups demonstrated a sufficiently large range of approaches, to facilitate me in addressing my research question.

After each interview, I examined the notes from the growth-point informal assessment spreadsheets I had aimed to complete during each interview (presented in Appendix M), and I also watched the video footage or listened to the audio footage. As presented in Chapter 2, the growth-point framework is not envisaged as a linear pathway, along which children progress, and as such, placing children along the pathway as a form of assessment, is at best an approximation of their potential to solve the patterning tasks presented. For the purpose of selecting four groups, I accepted that such an approximation could afford me a method of considering the spread of attainment within groups. I placed each child on the growth-point framework, focusing on the pattern for which s/he made the most progress. A sub-section of the growth-point framework showing the relevant growth-points is presented as Table 4.2.

Table 4.2. A sub-section of the growth-point framework

Growth-point	*Description*
GP2	Can offer some description of a growing pattern verbally, and extend maintaining number and shape
GP3_1	Demonstrates some elements of GP3 for some patterns. Hazy thinking, inconsistent, maybe inhibited by Whole Object thinking, or other inflexible and unhelpful ideas.
GP3_2	Demonstrates some elements of GP3 for some patterns. Inconsistent across patterns. Some evidence of explicit thinking, some recursive, whole object or counting.
GP3	Can correctly identify a near term. Can describe a pattern explicitly. Can offer a possible far term with reasoning.

In addition to looking at the attainment on the mathematical tasks, I sought to compare the balance of dispositions within groups. Mercer (1995) discusses the impact of learner's "social identities" on how they interact with peers during group-work. He outlines the tendency of boys to take up dominant roles, and of how it is possible for girls to feel compelled to submit to the "verbal bossiness" of their male peers in order to complete a task. In preparing my research design, I was conscious of the need to maximise group collaboration in order for the children to support each other in working within their ZPDs. In selecting groups for in-depth analysis, I decided to attend to the progress children had made on the patterning tasks, and also the social identities of the children as key indicators of the peer interactions. In doing so I aimed for diversity, both in attainment, and in social identity, seeking to fulfil the condition of 'internal diversity' as described above.

Mercer (1995) draws attention to the relevance of children's social identity within classrooms and within groups when children are asked to work collaboratively. Mercer emphasises that the social identities of children play an important role in how children interact with the mathematics tasks at play, and are often erroneously overlooked by researchers. Mercer points to the impact of children dominating discussions, and steering peers along a choice of path of inquiry or action that may inhibit some children from contributing or exploring their own thinking. He advocates supporting children in engaging in 'exploratory talk' as highlighted in Chapter 3 in the 'Assessing Algebraic Thinking' section. In selecting groups for in-depth analysis, I aimed to examine the dynamics within groups, by analysing the social identities of the children.

When taking field-notes, I included observations of whether children demonstrated confidence in their approach to the mathematics, and to working within the group. In many groups, I needed to draw out some children more than others, and among these children some made no suggestions to the group without my prompting them individually. I deemed children who presented in this way to be taking up a reactive role within the group. In contrast, some children presented very confidently, and made continuous suggestions to their groups, without any need for prompting from me. I deemed such children to be adopting 'dominant' roles. It is pertinent to mention that not all children behaving in a dominant manner were highly attaining, but no highly attaining children behaved reactively. Children who presented neither as particularly dominant, nor reactive, I categorised as undeterred. In Tables 4.3 and 4.4, for Schools 1 and 2 respectively, I present a categorisation of each child by the roles they adopted during their interviews, and their attainment on the patterns.

Table 4.3. Children in School 1, categorised by growth-point attained, and how they presented in group interactions

Group-point attained	*Dominant*	*Undeterred*	*Reactive*
GP3	Arina	Grace	
	Claire	Billy	
		Willis	
		Jamie	
		Alice	
		Kay	
		Wyatt	
		Paul	
Some, but not all, elements of GP3	Ciaran	Stephen	BillyBob
	Adam	Mark	Amber
		Lucy	Alex
		Emily	
GP2	Daniel	Jay	Orla
	Luigi	Shane	Cherry
			James
			Fiona

Table 4.4. Children in School 2, categorised by growth-point attained, and by how they presented socially in group interactions

Group-point attained	*Dominant*	*Undeterred*	*Reactive*
GP3		Precious	
		Melody	
		Angelina	
		Christopher	
		Jane	
		LilyRose	
Some, but not all,	Simon	Sarah	
elements of GP3	Desmond	Courtney	
		Peace	
		Danny	
GP2		Trip	Robert
			Lloyd

When I had categorised all children I looked at the composition of each group. In Table 4.5 I present the eleven groups interviewed, and I show the category which is relevant to each group-member, both in terms of their attainment on the patterns, and their interactions within their group. Group identifiers take the form School.Group, where School 2, Group 4 would be identified as Group 2.4.

Table 4.5. The interview groups from both schools, with all children categorised by attainment on the patterns, and social presentation during the interviews

Group	*Growth Point 3*		*Parts of Growth Point 3*			*Growth Point 2*		
	Dominant	*Undeterred*	*Dominant*	*Undettered*	*Reactive*	*Dominant*	*Undettered*	*Reactive*
1.1		Grace	Ciaran			Daniel		Fiona
1.2	Arina			Jay	Alex			Cherry
1.3		Colin, Jack, Jodie			Clodagh			
1.4	Mary Ann	Lauryn			Adam			William
1.5		Wyatt		Emily		Luigi		Orla
1.6		Ella, Ross	Mark	Sarah				
1.7				Enda, Sava			Elie	
2.1			Simon	Courtney Sarah				Robert
2.2		Angelina Melody	Desmond					Lloyd
2.3		Christopher Jane LilyRose					Danny	
2.4		Precious		Peace			Trip	

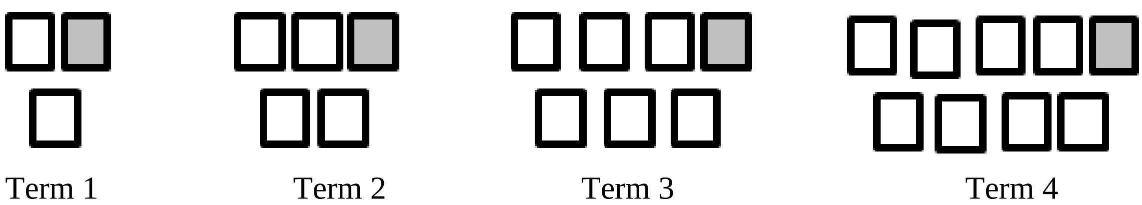
From School 1, I selected groups 1, 2 and 5. Each of these three groups included four children; the children were attaining across all three levels, and also presented with differing approaches to the group-work scenario. From School 2, I selected Group 3, as this group included four children who participated throughout the interviews, and engaged well with the patterns. While there were no children who attained at the middle level included in this group, I selected this group rather than group 2 because of the minimal contribution of Lloyd in Group 2. Lloyd struggled to extend patterns, and made no comments during the hour-long group interview that gave any insight into his mathematical thinking. He also expressed frustration with the manipulatives, and his own attempts at drawing the terms. Consonant with my research design, ethical implications drove methodological decisions I made. As I mentioned in the Ethics section of Chapter 3, I did not intend to put children under pressure to speak when they seemed unwilling to do so.

Tasks

Stromskag (2015) defines a shape pattern as a sequence of terms, composed of ‘constituent parts’, where some or all elements of such parts may be increasing, or decreasing, in quantity in systematic ways. While a limited number of terms of a shape pattern may be presented for consideration, the pattern is perceivable as extending until infinity. As discussed in Chapter 2, in order to construct a general term for a shape pattern, children must “grasp a regularity” in the structure of the terms presented, and generalize this regularity to terms beyond their perceptual field (Radford, 2010, p.6). In this section I outline aspects of the patterns I used in my research, including a presentation of their previous use in the research of others. All three patterns are presented in this research as ‘growing patterns’ in that they are perceived as progressing beyond the terms shown in a sequential manner, wherein elements of constituent parts increase in number. It would be possible to imagine the presented terms as a unit of repeat and to perceive the pattern’s progression as further repetitions of this unit. Such an interpretation would not be appropriate within this research study. If a child had demonstrated thinking which suggested this interpretation, I would have prompted him/her to review the pattern, and to imagine the pattern as a sequence that was progressing sequentially rather than repeating.

Pattern 1

Figure 4.1. Pattern 1



As presented in Figure 4.1, Pattern 1 is a Beams pattern used in previous research by Radford (2010; 2011). The terms in this pattern include two rows of tiles, where the bottom row contains *n* tiles, and the top row contains *n*+1 tiles. Many children perceived terms of this pattern as consisting of a top row and a bottom row, as presented in Figure 4.2, and some children considered each term to consist of n pairs of tiles presented diagonally, with one additional shaded tile, as presented in Figure 4.3.

Figure 4.2. Pattern 1, presented as two rows of tiles, where the bottom row contains n tiles, and the top row contains n+1 tiles

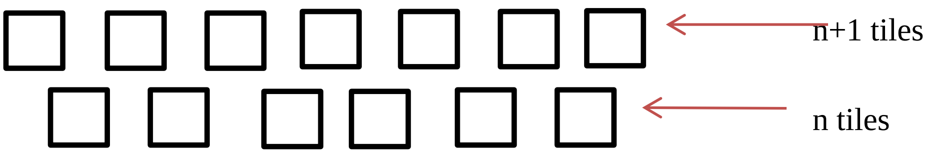
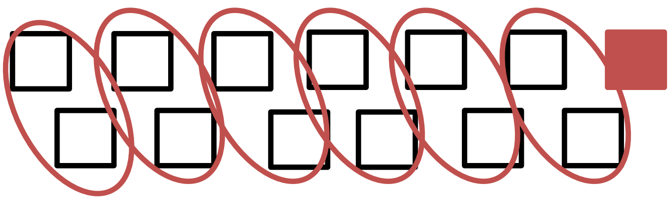
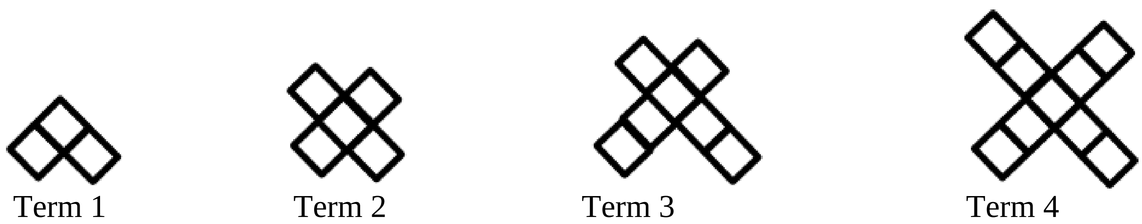


Figure 4.3. Pattern 1, presented as n pairs of tiles presented diagonally, with one additional shaded tile



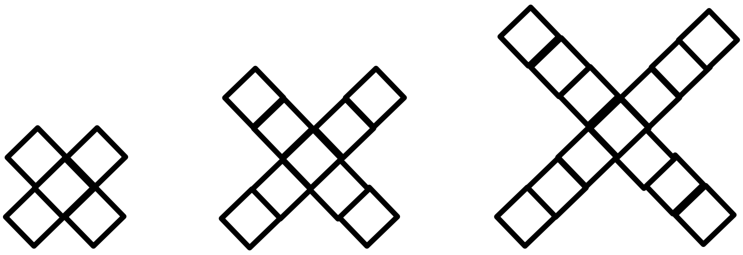
Pattern 2

Figure 4.4. Pattern 2



Pattern 2 is presented in Figure 4.4, and was based upon a pattern taken from research conducted by Warren (2005), where she conducted a teaching experiment and utilised pre and post-tests. Warren's research was focused on children's ability to generalise a pattern rule from a growing pattern and the participants had an average age of nine years and six months. Question 1 of the pre and post-test was designed to assess children's understanding of growing patterns. The pattern presented to the children who participated in Warren's research is presented in Figure 4.5.

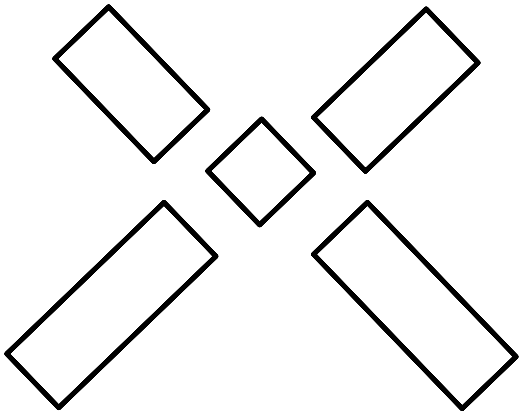
Figure 4.5. The diamonds pattern referred to in Warren (2005)



Rivera and Becker (2011) also presented children with the same pattern in their research into generalisation strategies of children in 3rd grade in California. For my pilot research, I was interested to see how the children would respond to the non-linear representation, but felt that this pattern was too similar to Pattern 1, as both were representations of the numeric pattern 2n+1. I decided to present a variation of this pattern to the children participating in the pilot, to see how they would respond.

The variation includes an asymmetrical growth pattern, where additional tiles are added to the *top* of term 1 to construct term 2, and to the *bottom* of term 2 to construct term 3. The even numbered terms have rotational symmetry, whereas the odd-numbered terms do not. If children isolate the central tile, and seek to explore the number of tiles on the legs of the diamond, as presented in Figure 4.6, they must incorporate whether the term number is odd or even into their thinking. If applying recursive thinking, children are required therefore to consider the rate of change, the element of the pattern which is growing, and also the direction of growth. If applying explicit thinking, whether the term number is odd or even must be included for consideration when constructing far terms, and an abstract generalisation which describes both the quantity of tiles, and their position in the shape must describe even-numbered and odd-numbered terms separately.

Figure 4.6. A general term for Pattern 3 presented as a central tile, and four legs



Alternatively, children could isolate the central diamond, and see the pattern as two symmetric sets of n-tiles arranged on either side of the central diamond, as presented in Figure 4.7, or see the terms as two strips of tiles overlapping at the central tile as presented in Figure 4.8. In either of these two scenarios, odd-numbered terms will contain an additional tile below the central tile, on each side, or on each strip.

Figure 4.7. A general term for Pattern 3 presented as two symmetric sets of n-tiles arranged on either side of the central diamond

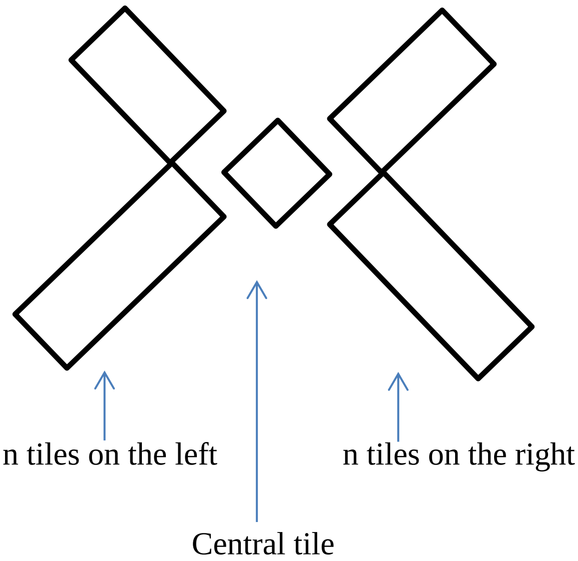
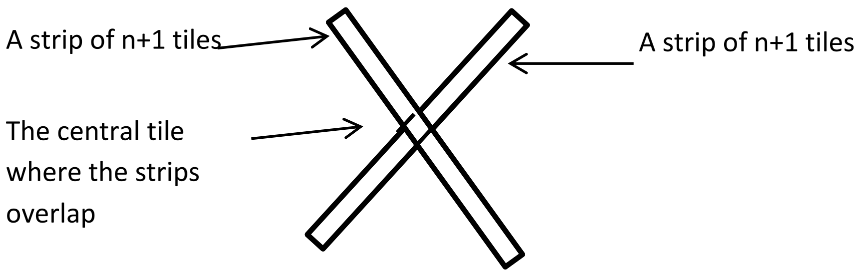
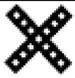
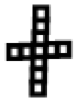


Figure 4.8. A general term for Pattern 3 presented as two strips of tiles overlapping at the central tile



During my first pilot, children were interviewed individually, and the four participants from Fourth class were girls. I asked each child to describe the pattern, and to extend the sequence by drawing the 5th term. Without indicating whether her response was correct, I then asked each girl to describe the 10th and 100th terms (as near and far generalisations of the pattern). All four girls found this task challenging and as the interviews were individual they did not have peers with whom to discuss the pattern, and could not therefore build upon the thinking of others, or broaden their individual perspectives. In order to support their progress, I asked "suggestive questions" (Ginsburg & Pappas, 2004, p.182). I asked Bella, Nikki and Tara whether they could find a connection between each term and its position in the sequence, and I asked Natasha whether she could identify the changes between successive terms. Table 4.6 presents the girls' drawings of the 5th terms and their verbal constructions of the 100th term. Each girl used the same strategy for constructing the 10th and 100th terms. All of the girls who responded pointed to the legs of the 4th term when identifying how many diamonds would comprise the 10th and 100th terms.

Table 4.6. Drawings of the 5th term and descriptions of far terms

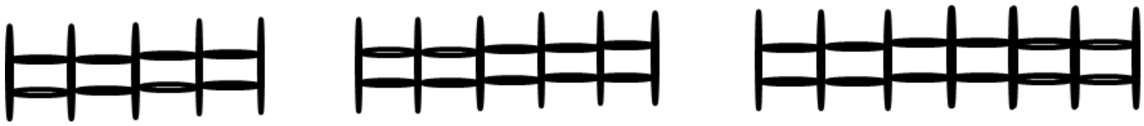
Participant	*5th term*	*100th term*	*99th/101st term*
Natasha	 Extended all legs	100 there, 100 there, 100 there and 100 there.	
Bella	 Correct 5th term	50 there, 50 there, 50 there and 50 there	99th: 4 there, 5 there [indicating legs on the left of the x-shape], 4 there, 5 there [legs on the right]
Tara	 Correct 5th term	Did not respond	
Nikki	 Rotated and extended opposite legs	There's 50 going across there, and 50 there, and 50 there and 50 there.	101st: 49 there and there and 2 extra on the bottom.

In order to understand girls' intended descriptions of patterning terms, it was appropriate to attend to their use of 'deictics', or words which pointed towards some element of the terms (Radford, 2010). The girls made no verbal reference to the central diamond, but it may be said that they pointed to its presence, through gesture and use of the deictic 'there', as they indicated the distribution of diamonds on legs of the x-shape. Having described the 100th term, Bella and Nikki were asked to consider the 99th and 101st terms respectively. Bella seems to have constructed the term using a commonality in the pattern, by distributing the term number (using 9 in place of 99) between the top and bottom legs on each side of the x-shape. Nikki may have used a recursive approach, constructing the 101st term by working from the 100th. The girls' engagement with the pattern was such that interesting conversations were facilitated which revealed aspects of the girls' strengths in isolating elements of the terms, and identifying rates of change. I decided therefore to include this pattern during data collection.

During the group interviews I asked the children to verbally construct the 10^{th}, and 75^{th} terms for this pattern, and to quantify the tiles required. In this way I aimed to explore how the children thought about any constant or variable elements of the pattern, both numerically and spatially, and how they would deal with the additional challenge of the asymmetric nature.

Pattern 3

Figure 4.9. Pattern 3



Term 4 Term 5 Term 6

The third pattern presented in Figure 4.9 is a modified version of a pattern mentioned as a typical growing pattern by Cooper and Warren (2011). Within the task-based group interview I had included two patterns which were purely abstract and I considered it appropriate to include a pattern which could be considered to have a context and could be discussed in relation to a real world phenomenon with which the children may be familiar. This pattern I presented as a sequence of fences. The fences are constructed with posts and are comprised of an increasing number of panels, that is, the fence in position 4 is comprised of four panels, term five has five panels, etc.

On presentation of each pattern, I asked the children what they could tell me about it and also to describe the pattern, asking them to share their descriptions with each other. Through asking open-ended questions, I aimed to allow the participants freedom in their responses and thus elicit “the expression of personal ways of thinking” (Ginsburg, 1997, p. 126). I then proceeded through warm up exercises, asking participants to extend the pattern to the 5^{th} and the 6^{th} terms of Patterns 1 and 2; and to the 3^{rd} and 7^{th} terms of Pattern 3, by construction using the manipulatives provided and by drawing. When the children had discussed extensions to the pattern, I asked them to describe near and far terms. Children could construct the near term by counting on from the extensions they had constructed, but solution of the far generalisation would require the application of a general rule. In this context, I was of the opinion that the position numbers of the far terms I asked the children to construct (86, 75 and 56), were sufficiently large to play the part of a general number (Stacey, 1989).

To conclude work with each pattern the children were asked to verbally describe a general term. Drawing from the literature discussed in the 'Generic example' section of Chapter 2, I did not ask children to express their generalisation using symbols, but rather I took the position that abstract symbolism is not a necessary component of algebraic thinking (Radford, 2010). As discussed in Chapter 2, the use of variables in generating rules from patterns is semantically challenging, as children must see a single variable as simultaneously fulfilling the roles of "dynamic general descriptor" of terms in relation to their position, and also a generic number in an expression (Radford, 2000). Variable use is not present in the Primary School Mathematics Curriculum in Ireland before Fourth class, and would be entirely novel to most, if not all, of the children who participated in this research. It may not have been appropriate therefore to expect children participating in this study to work with the position number of terms in a pattern in both ordinal, and cardinal roles (ibid.). Equally children's verbal comments which referred to 'any term', or 'a term' were considered to be generalisations, as were expressions using a numbered term as a generic example.

Concrete materials

Manipulation of concrete materials plays a key role in supporting children in constructing understanding, particularly in a context where they are encouraged to interact with their peers, learning from, and questioning each other (Bruner, 1966; Dunphy, Dooley & Shiel, 2014). In planning the interview schedule I expected that the physical construction of terms from concrete materials would facilitate children in engaging with an alternative form of representation to the iconic presented on the worksheet, or the abstract which they may need to visualise in order to construct a generalisation. Radford (2010) describes the construction of a generalisation as resting on “highly evolved mechanisms of perception and a sophisticated rhythmic coordination of gestures, words and symbols” (p. 7). Also, Pape and Tchoshanov (2001) emphasise that representations may be used as tools for thinking, and discuss how producing, and then using, representations for mathematical questions may reduce the cognitive load for children, and facilitate them in justifying or explaining their thinking.

Consonant with the assertions of Radford, and Pape and Tchoshanov, Pouw, Van Gog, and Paas (2014) assert that the physical manipulation of concrete objects supports children’s developing understanding in two non-trivial ways, by alleviating cognitive load, and supporting internalisation of processes. In order to differentiate and further illuminate details of both aspects of the use of manipulatives, Pouw et al. (2014) describe embedded and embodied cognition as follows:

> 1. Under certain conditions, perceptual and interactive richness can alleviate cognitive load imposed on working memory by effectively embedding the learner’s cognitive activity in the environment (Embedded Cognition claim).
>
> 2. Transfer of learning from manipulatives does not necessarily involve a change in representation from concrete to symbolic. Rather, learning from manipulatives often involves internalizing sensorimotor routines that draw on the perceptual and interactive richness of manipulatives (Embodied Cognition claim) (Pouw et al., 2014, p. 52).

Pouw et al. emphasise that they do not present a case for an extreme view of ‘embedded embodied cognition’ wherein children’s thinking develops, and is applied, entirely in interacting with their environment. Rather they suggest that embedded embodied cognition plays a key role in education, along with other cognitive structures, such as visualisation, and mental arithmetic.

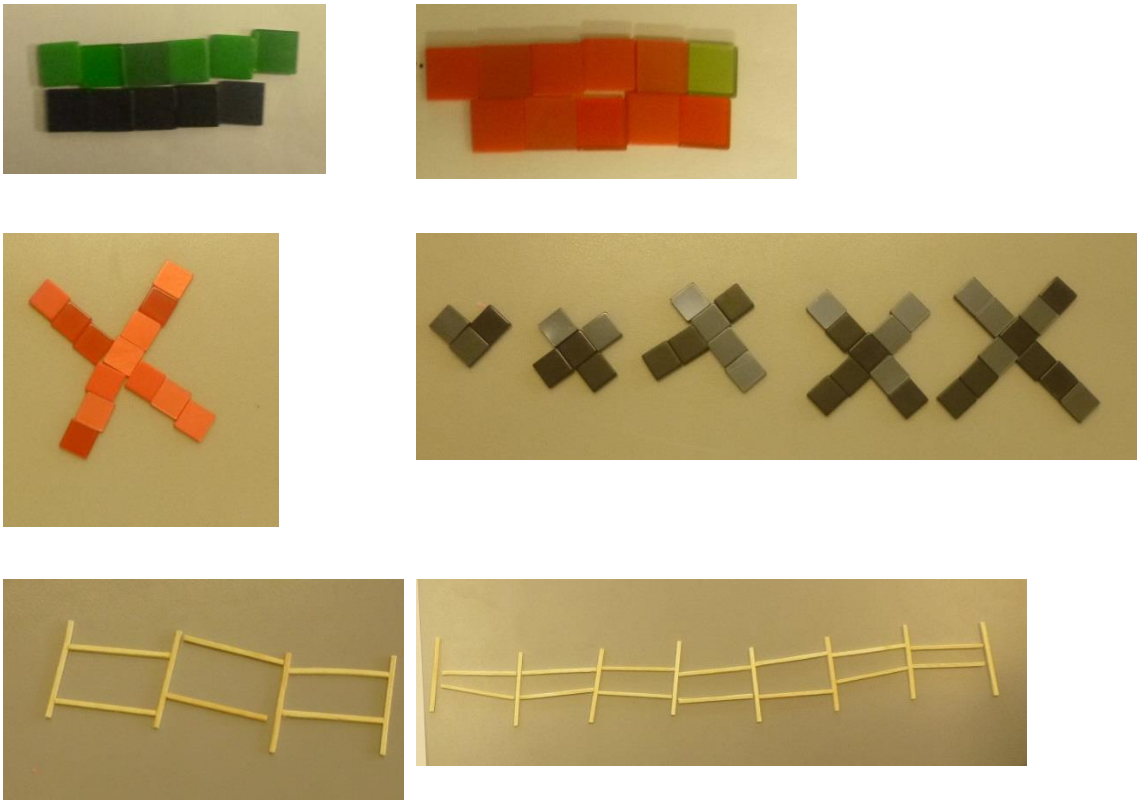
In line with theories of embedded cognition, during the task-based interviews, I anticipated that some of the children would coordinate their developing perception of the structure of the patterns with their construction of pattern terms. In this way their external environment may support their thinking about the patterning tasks by serving as "external working memory". Kyttala and Lehto (2008) state that poor visual-spatial working memory may be a key factor in children's success with mathematics, and I considered the inclusion of such supports for children highly appropriate, as they engaged in novel and abstract mathematical tasks.

Complementing theories of embedded cognition, theories of embodied cognition explore how children transfer learning from one context into another, or from the environment into a decontextualised abstract sphere (Pouw et al., 2014). While much learning with the use of manipulatives facilitates the internalisation of sensorimotor routines which children can transfer to multiple contexts, the transfer of learning from children's lived experiences to a mathematical context, may in some situations increase the cognitive challenge as the child must dis-embed the cognitive activity from the context. Within this study, children were asked to transfer their algebraic 'powers' of "specialising and generalising", which they have developed outside of the study of mathematics, to the construction of generalities for shape patterns (Mason, 2008). I believed that it was appropriate to support the children in performing this transfer of understanding from one context to another through the use of manipulatives.

Of specific relevance to the research presented in this thesis, Warren and Cooper (2008) conducted research which explored children's algebraic thinking, and found that concrete representation of patterning terms supported the children in their research in broadening their thinking to include an explicit approach. Equally, in assessing children's mathematical thinking through task-based interviews Goldin (2000) recommends that children should be afforded "various planned, external representational capabilities", to provide a problem-solving environment that is rich in diversity (p. 542). This will allow the researcher to observe a broader variety of responses from which s/he may make inferences about the children's internal representations.

I decided to provide coloured tiles, and match-sticks, to the children, and to ask them to construct pattern terms using them. I also planned to encourage the children to refer to their constructions of earlier terms, as they sought to consider subsequent terms. Figure 4.10 presents a selection of the pattern terms constructed by the children using the concrete manipulatives provided.

Figure 4.10. Pattern terms constructed by children using the concrete manipulatives provided



Data sources

Each interview was recorded. Most interviews were video-recorded, but two interviews were audio recorded, as some children did not assent to video-recording. During the interviews, I took photographs of the children's physical constructions of the patterning terms, and I collected all of their written drawings and jottings.

Before each interview I printed a two-sided assessment framework for each child, drawing from the framework of growth-points outlined in Chapter 2. A copy of this assessment framework is presented in Appendix M. I sought to prepare a practical assessment framework, which would allow me to make notes on children's progress, while not impinging on my focused attention to the children's interviews. During the interviews, I endeavoured to take notes about the children's progress, but I found that on many occasions, it was challenging to oversee the children's discussions, remain vigilant about the recording, take photographs of the children's work, and also take notes. As a result the notes I took during the interviews were less comprehensive than I had anticipated they would be.

After each day of interviewing, I replayed the footage, and made field-notes. When all interviews had been completed, I transcribed sections of the interviews where the children were engaged in discussions about the patterns. As discussed earlier in this chapter, I had become aware during the interviews that each interview was very rich in terms of the depth of the children's discussions, and also that there was some overlap between the approaches children were taking. I undertook a process of inspection of the groups, and selected four groups for closer analysis. I transcribed the entire footage from the four groups, and imported the transcriptions into NVivo.

I also documented my reflections on the research as it progressed in a reflective journal. The data collected therefore comprised recordings of the interviews, transcripts of the recordings, field notes, a reflective journal, children's written responses to the patterning tasks, and photographs of their constructions.

Inferences from children's comments

In line with my hermeneutic phenomenological approach described in Chapter 3, a significant amount of my time was spent seeking the underlying meaning from children's comments. In this section I outline decisions I made in this regard, but I would like to foreground the discussion by asserting that I believe it is not possible to ever be completely sure about what somebody else is thinking. Steffe, Thompson, and von Glasersfeld (2000) draw attention to the reality that researchers “may not, and possibly cannot” fully interpret the mathematical understanding reflected in children’s comments, as this understanding may be distinctly different to the understanding of the adult researcher (p. 268). That said, it was necessary to make every effort to explore what a child’s utterances indicate, to include preceding, and subsequent comments, and also gestures to elucidate the meaning of the words used.

Rowland (2007) outlines the various types of vague language used by children in articulating their opinion, or taking a stance, on a mathematics problem. He discusses ‘hedges’, which may blur statements that would otherwise be precise, or which may moderate the level of commitment of the speaker to the statement uttered. In seeking to explore children’s meanings I included an analysis of hedges that children used. I anticipated that in doing so, I could explore the possible intended meanings of children who used very vague language. In referring to Rowland’s categorisations of hedges, I sought to unpick the possible underlying intentions of the child, and thus to explore whether children were struggling to make sense of the mathematics, or being vague for some other reason, for example a personal tendency to be tentative or guarded in proffering an opinion. I sought to build upon Rowland’s analysis of the generalisations constructed by children in his research where, for example, he used an analysis of hedges to consider whether a child, Ishka, was generalising, or estimating when she used the rounder ‘about’ in a proposition she made.

An additional aspect of vague language lies in the children’s use of pronouns. Rowland (2000) discusses children’s use of the pronoun ‘it’, and he attests that ‘it’ is used as a ‘linguistic pointer’ to concepts on which the speaker is focused. The range of concepts to which ‘it’ may point is broad, and in different contexts, ‘it’ could refer to a shared idea, or a mathematical referent. During this study the mathematical referent could be a term in general, a specific term, an element of a term, a tile, a construction of a term, or another object within the perception of the speaker. During my analysis of the children's thinking, it was necessary to remain alert to the range of possible meanings underlying a child's use of a pronoun, rather than making assumptions based upon my personal mathematical constructions.

In aiming to take a comprehensive account of children's meanings, I also attended to their use of gesture. In particular, I highlighted in my analysis when children pointed to elements of shapes, and used deictics, such as 'there'. As I explored in Chapter 2, there are many ways in which algebraic thinking may be communicated, along with mathematical statements containing the abstract symbolism most readily identified with formal algebra (Radford, 2010). Radford highlights the "sophisticated rhythmic coordination of gestures, words and symbols" that underpins children's algebraic thinking. He refers to children's use of "pointing gestures, and linguistic deictics" as children describe general terms for sequences, and use deictics to describe an object and where it is positioned within a term. For example, in referring to Pattern 2 used during my research, children pointed to the legs of the x-shape stating the number of tiles which would sit 'there'. In this context 'there' fulfils the role of the deictic, not only referring to the place, but also the object, in this case the leg of the x-shape. When reading and coding transcripts, it was frequently necessary that I reviewed clips from the video or audio footage to observe any gestures used by the children, to ensure that I was approximating as closely as possible their thinking about the patterning tasks.

Assessing children's responses

In Chapter 2 I drew attention to the many challenges inherent in assessing children's algebraic thinking. The interviews I used as my research instrument were designed to be exploratory tools, investigating the range of children's responses, and factors influencing their constructions. As part of this investigation, I considered that it was appropriate to assess each child's progress on each pattern. Such an assessment could afford me a starting point for comparison of patterns, and also an indication of the extent to which some children's thinking was consistent across all patterns, or inconsistent. A more complete exploration of this analysis is presented in Chapter 5: An investigation of children's progress across the three patterns presented. In this section I outline the methodological decisions I made regarding the scoring of children's progress.

In seeking a scoring scale which could accurately differentiate between the levels of progress achieved by children on the patterns presented, I sought a scale which contained a range of scores, but was not so cumbersome as to render the scoring process redundant. I avoided a simple correct or incorrect scoring system, as many children could be deemed to be incorrect in their answers, even though their comments demonstrated progress in their thinking with respect to the tasks. I considered applying an intricate scoring system with an array of scores, but suspected that it would blur the comparison of progress achieved across patterns for each child, and that the level of differentiation achieved was too minor to justify this loss of clarity. I chose to base my scoring system on the 5-point scoring scale applied by the Subject Knowledge in Mathematics Audit (SKIMA) (Rowland, Martyn, Barber, & Heal, 2001).

The SKIMA research team initially used a 3-point scoring scale, where students' responses to questions were marked as achieving a 0, 1 or 2, indicating responses which were 'not secure', 'possibly secure', and 'secure' respectively. The research team later expanded the scale to become a 5-point scoring scale (Huckstep, Rowland & Thwaites, 2003), which I have presented as Appendix N. This expanded 5-point scale gave me greater scope for an accurate reflection of the children's progress, while remaining sufficiently concise for practical purposes. While this 5-point scale affords greater nuance than an incorrect/correct assessment, it remains challenging to capture the breadth of children's responses within a hierarchical scale of discreet scores. In Table 4.7, I present the scores I applied to the children's responses.

Table 4.7. Scoring protocol for children's responses to patterning tasks

Response	*Score*
Perfect generalisation, articulated for all terms	4
Perfect far generalisation for term number given, or generalisation where arithmetical difficulty impeded completion, or generalisation with single numerical error, e.g. 86+85 in place of 87+86 for Radford 86th.	3
Progress towards a generalisation, e.g. correct near generalisation, described some aspects of general terms	2
Extended correctly	1
No progress	0

As may be seen from Appendix N, a score of 0 on the SKIMA Scoring scale indicated no progress, and I included this score within my scoring scale, awarding 0 to children who did not extend the pattern correctly, or make any progress in constructing a general term. A score of 1 was achieved on the SKIMA scoring scale by responses with a 'partial and incorrect solution'. I mapped this partial solution onto situations where children extended the pattern correctly, but could not construct a general term. 'Correct in parts, incorrect in parts' was allocated a score of 2 under the SKIMA scoring scale, and I attributed a score of 2 to the progress of a child who made some progress towards constructing a generalisation, but who did not succeed in constructing a far generalisation. Under the SKIMA scoring scale, a score of 3 was achieved by responses which were 'correct with small errors' or where justification was incomplete. The equivalent score of 3 within my scoring scale was achieved by responses where a correct far generalisation was constructed for a specific term number, or when minor errors were present. The maximum score of 4 was awarded to full solutions with 'rigorous explanations' for participants undertaking the SKIMA, and I mapped this maximum score to responses wherein a general term was constructed without reference to a specific term number. As I have highlighted in Chapter 2, there is disagreement within the research field of algebraic thinking regarding the distinction between general terms which refer to a generic example, and general terms which refer to abstract generalities. In scoring the progress of children who participated in my interviews, I held the position that there is a greater cognitive challenge in articulating a general term without referring to a specific term number, and that while not necessary for children at this stage in their algebraic thinking to do so, it is important that a scoring scale captures instances when children do attain this level of abstraction.

To summarise, I took the decision to score each child's progress on each pattern, for the reasons outlined above. In order to do so, I developed a scoring range for the children's responses, modelled on that used by the SKIMA (Rowland, 2007). In using the scale presented in Table 4.7, I watched the video footage, or listened to the audio footage, while referring to transcripts, and artefacts from the interviews. The validity of this process depended upon my objective and consistent application of the scoring scale to each child's progress on each pattern. As explored above, my interpretation of the children's thinking could not be considered to be an exact description of the thoughts of each child, but rather an "adequate elucidation" of their response to the patterning tasks. Similarly, it is appropriate to consider my scoring of the children's responses as the current 'best match' of an on-going process.

Analysis

Step 1: Familiarisation with the data

Due to my role as facilitator of the interviews which comprised my data collection instrument, I was somewhat familiar with my data before I began this phase of my research. During the interviews, I made minimal field-notes as my focus was on facilitating discussion and engagement among the children. For most groups the notes I made were limited to comments on the progress of individual children on the template I had prepared for this purpose (Appendix M). I interviewed two groups on each day of interviewing, and following the interviews, I watched the video footage, or listened to the audio footage. I made field notes at this point, where I made an initial assessment of the progress of individual children, and also of the levels of engagement. I began at this stage to consider possible themes, and I observed that many children seemed to have demonstrated a higher level of competence in their engagement with the patterns than I had expected. Braun and Clarke (2006) caution against complacency at this point as immersion in the data through repeated reading remains vital. Such reading may be enrichened by the early familiarity with the data which arose from the interview process. Having gained some familiarity with the data, I transcribed sections from all interviews where the children were articulating mathematical thinking.

During the process of observation and transcription some tentative questions began to emerge, which I have presented in Table 4.8.

Table 4.8. Tentative research questions emerging from initial observations of children's engagement with the patterning tasks

Is there a connection between initial observation and subsequent approach taken?
Do suggestive questions play a role in broadening children's approaches?
Do children who adopt numerical approaches succeed less well?
Do children who adopt recursive approaches experience limitations resulting from this approach?
Do individual children adopt both numerical and figural modes of generalising – do tasks have an impact on the approach the children take and/or input from peers/questions from researcher?
Do children lead themselves/each other along successful/unsuccessful routes – does justification play a role in supporting them in distinguishing which route is successful?

All questions at this point were quite closed in nature, but reflected some observations I made of children who, for example, seemed to be focusing on numerical aspects of patterns, and whose attempts at generalisation seemed to be somewhat constrained as a result.

Step 2: Coding

In using thematic analysis, Braun and Clarke (2006) state that themes or patterns within data can be explored in one of two ways: in an inductive way, wherein the themes emerge from the data, or in a deductive way, wherein themes from existing research are explored within the data. In this section I outline how I applied both inductive and deductive approaches to the coding of the data collected.

Deductive

When I had imported the transcriptions into NVivo, I inspected each comment, and I considered what each comment could indicate in relation to existing frameworks of the variety of responses children may make to generalisation tasks from shape patterns. In Chapter 2 I presented the frameworks developed by Lannin (2005), Barbosa (2011), and Rivera and Becker (2011), and at this stage I utilised their frameworks in coding the children's comments. The specific codes I assigned are presented in Table 4.9.

Table 4.9. Codes assigned based on existing research on algebraic thinking

Code	*Framework Source*
Distinguishing between elements	Rivera and Becker (2011)
Figural	Rivera and Becker (2011)
Numerical	Rivera and Becker (2011)
Explicit	Lannin (2005)
Recursive	Lannin (2005)
Whole Object	Lannin (2005)
Final Adjustment	Barbosa (2011)
Counting	Barbosa (2011)

Adopting the categorisations presented in Table 4.9, I considered whether each comment indicated that the child was thinking about the pattern at hand in a way that was consistent with one, or more, element(s) of the categorisation. I coded a comment as ‘figural’ if it attended in any way to figural aspects of the term. Some comments coded as ‘figural’ could be said to be wholly figural, such as the following comments from Lily Rose and Christopher when discussing Pattern 2:

Lily Rose Ehm, I think you’re just making a bigger times sign every time you just keep trying to make a big ehm kind of x thing bigger and bigger.

Christopher First it grows that side then it grows that side.

Other comments coded as figural, incorporated both numerical and figural aspects of the pattern structure, such as the following comments from Lily Rose and Jane, when discussing Pattern 3:

Lily Rose Each time there is going to be five and then four panels in the middle. It is going to be five fences, and then it is five panels in the middle. It is going to be 6 fences posts

Jane And the poles are the same amount of sections and one extra.

Comments were deemed to indicate numerical thinking, when they referred to numerical aspects of the pattern, *without any reference to figural aspects*. Examples are the following comments by Daniel and Alex, when talking about Patterns 1 and 2 respectively:

Daniel 1, 2, 3, then, 1, 2, 3, 4, 5, that’s going up

Alex If you add two 5s you add half 2 on to that. That would make 11.

I coded a comment as ‘distinguishing between elements’ if a specific reference was made to one element of a term, e.g. a leg on the x-shape in Pattern 2, or the top row in Pattern 1. I deemed a comment to indicate ‘explicit’ thinking if the quantity of components of an element of the term was seen as a function of the position number of the term, or if an explicit link was made between the two quantities (Carraher et al., 2008). For example when talking about Pattern 1, Wyatt said “Because on term 4 it’s number 4 and it has 5 on the top”. In comparison, when a comment was made that involved constructing a term based on the quantity of components of a preceding, or subsequent term, I deemed such a comment to indicate 'recursive' thinking. As an example, when thinking about Pattern 2, Ciaran said "it's going up in two's each time, look cos these grow, then these grow, then these grow, so it'd probably be like 16, at the other end". When a comment involved some sort of adjustment to a construction in order to satisfy the structure of previous terms, I coded the comment as 'final adjustment'. For example, Emily said "6 plus 6 equals twelve, so I doubled the term 6, ehm, 12, so I doubled the top, 12 on the bottom, then put the one 13" about her construction of the 12th term for Pattern 1. In this case, Emily seemed to be treating the number of tiles of the 6th term as a 'Whole Object', doubling the quantity on each row to construct the 12th term, and making the final adjustment of adding an additional tile to the top row. I coded a comment as 'Counting' if a child seemed to be counting the components of an element in order to find the quantity, for example when discussing the structure of Pattern 1, Alex counted all tiles presented, as he said "So, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20. And so 21, 22, 23, 24".

I coded many comments in several categories, for example, many comments appeared to indicate an explicit approach, and also a figural mode of generalising. In coding, I aimed to conduct a 'latent thematic analysis', aimed at incorporating gesture and patterns in children's expressions of their ideas to interrogate underlying assumptions, rather than operating at purely semantic level and taking every utterance as an exact representation of children's thinking (Braun & Clarke, 2006). If I was unsure about a child's thinking in a comment, I referred to preceding and subsequent comments, to the video or audio footage, and also to the child's written work.

Inductive

Using NVivo, I examined exchanges between children during the four hour-long interviews, and sought to identify features of the children's comments, situated as they were within the group discussions (Braun & Clarke, 2006). Table 4.10 presents the themes which emerged from the data during this stage of analysis.

Table 4.10. Codes which emerged from inspection of children's comments using NVivo

Code	*Description*
Contradicting Self	A child expressed a contradictory idea to one expressed previously
Estimation	A child used vague language to give an approximate answer
Justifying thinking	A child explained why or how he/she thought about a task
Learning from peers	A child built upon, or drew from, the thinking of another child.
Odd or even	A child described the quantity of components of a term as odd or even.
Striving to make sense	A child questioned his/her own thinking, or that of the group, seeking clarity, or noticing that an idea might not fit the structure of the pattern.

The first code presented in Table 4.10, 'contradicting self', was the code I assigned to a comment which was a contradiction of a previous comment by the same child. Some such comments may indicate that a child had changed his/her mind about some aspect of a pattern, or that a child held conflicting views about a task. Such comments may reflect a level of confusion or uncertainty on the part of the child, or alternatively a trial and error approach to the solution of a task. I used the code 'estimation' to capture incidents when children contributed approximate solutions to questions, for example in discussing Pattern 3, Daniel stated that "I'd say it would be more than 54" and shortly thereafter "around 71, 71, 72". At times children spontaneously justified an idea they were sharing with their group, and at other times they were prompted to do so by a peer, or by me as facilitator. When a child did explain their thinking, I coded their justification as 'justifying thinking', for example in discussing Pattern 3, Emily explained the need for one more post than the number of panels by stating "because if you had, if it was, if you had six vertical ones, it would be, the other three would just fall off".

The code 'learning from peers' I assigned to a comment which reflected a previous contribution of another child in the group, or where a child asked a question of another child, and adjusted his/her thinking, or construction, accordingly. For example in extending Pattern 1, Luigi counted the tiles in the extension he had constructed, and stated "Yeah, there's 12. What do you get? 11?" Luigi then removed one tile from the shape he had constructed. The code 'odd or even' I assigned to comments relating to whether the quantity of components of an element of a term, or of the term in totality, was odd or even. This code only applied to Group 1 in School 1. I assigned the code 'striving to make sense' to a comment which indicated that a child was in some way seeking to make sense of the pattern by asking questions, trying out hypotheses, or trying to figure out where two ideas seemed contradictory. For example, Arina appeared to be a little unclear about the structure of the 75th term of Pattern 2 when she said "It would have, like, 75 on each side I think and… [trails off]", but when I probed her for clarification, she replied "Like 75 here and 75 here. Like 1 middle square here and 75 here and here and here and here."

Accepting that the coding process involves my interpretation of the children's comments, I aimed at all times to be consistent and systematic in how I coded. I devoted significant time and attention to this process, and read the transcript of each interview three times, referring to other artefacts from the interviews at all times. In Appendix O, I present an excerpt from the interview of Group 2 in School 1 with the codes I assigned to each child's comment.

Step 3: Searching for themes

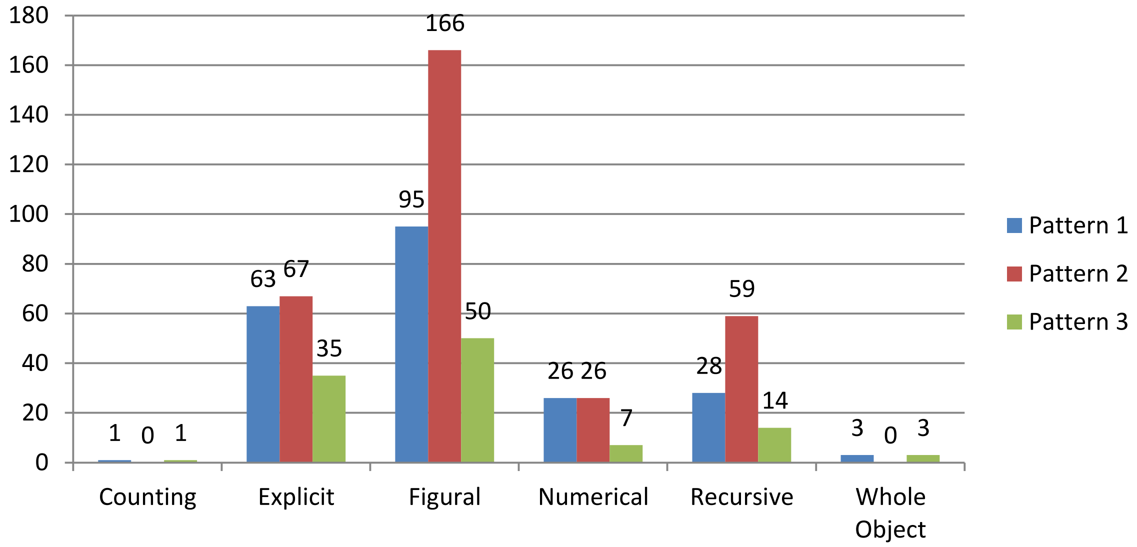
At this stage of the research, my analysis of the data incorporated three aspects, memo-writing, categorising strategies and contextualising strategies, to reflect my understanding that a research question exploring themes which may be interlinked, cannot be answered by one analytic strategy (Maxwell, 1999).

Memo-writing

Memo-writing involved my writing of any thought that occurred as a response to my reading of the transcripts, or thinking through the data, or relevant writings. A complete list of memos generated during this phase of my research is appended to this thesis as Appendix P. I would like to highlight one theme which emerged through the memo-writing process, that is the inter-pattern variation in the strategies employed by children. As I familiarised myself with the data, I sensed that there was variation in how some children responded to each pattern. I wondered whether the structure of specific patterns had an impact on how children thought, and how they worked through the tasks relating to each pattern. I reviewed the comments which I had coded Explicit, Recursive, Numerical, Figural, Whole Object and Counting, to see whether there were children who demonstrated a range of approaches, and I found that many children made comments across many of the six codes. When I examined the underlying figures, I saw that 12 of the 16 children made comments which I had coded under four codes or more. There were only three children whose corpus of comments were limited to three codes or fewer.

I carried out a similar study of the inter-pattern variation across these codes. Figure 4.11 presents the number of comments coded as Explicit, Recursive, Numerical, Figural, Whole Object and Counting, and grouped by pattern.

Figure 4.11. The quantities of comments coded as Explicit, Recursive, Numerical, Figural, Whole Object and Counting, and grouped by pattern



As can be seen from this chart, I coded comments under most of the six codes from the children's discussions of each pattern. No strong pattern emerged from this early stage analysis, other than an indication that Pattern 2 generated more discussion from the children. While I coded more comments as figural from children's discussions of Pattern 2, than of Pattern 1 or 3, there was no corresponding lower number of numerical comments relating to this pattern. In the analysis chapters I present an exploration of the inter-pattern variation in the children's responses.

Categorising strategies

Maxwell (1999) recommends categorising codes, by identifying patterns, and drawing themes together. Miles and Huberman (1994) highlight the necessity of seeking clusters, in order to "understand a phenomenon better by grouping and then conceptualising objects that have similar patterns or characteristics" (p. 249). At this stage in my research I examined the codes presented in Table 4.10 above, which had emerged from my collected data. Seeking similarities among the codes, I deemed the codes 'contradicting self', 'learning from peers', and 'striving to make sense' all related to children's urge to make progress in the tasks, and to figure out the structure of each pattern. In relation to my research question, and my exploration of the strategies that children adopted in constructing general terms, I reasoned that an understanding of *how* children came to favour one strategy was very relevant, and worthy of further analysis. In my analysis, I sought to explore the influences of children's interactions with their peers, with the environment, and the materials provided, and with the tasks themselves. Included in this analysis is discussion about situations when a child experienced a perturbation, and needed to refine an idea that didn't fit with other evidence about the structure of the pattern. Explorations of why children adopted particular strategies in their construction of general terms are contained in Chapters 5, 6 and 7.

In this study, contextualising strategies involved investigating the thinking of specific children, or the examination of a single pattern across all groups. Along with coding data in NVivo, I generated tables in Word from the transcripts, and I structured the tables so that I could sort them by child, and by pattern. In this way I could isolate the thinking of one child across all patterns, or isolate the thinking of all children on one pattern (examples are presented in Appendix Q). While I found both NVivo, and the Word tables useful in focusing in on the thinking of specific children, I regularly referred to the video or audio footage, as well as the photographs, and copies of children's work to support my understanding of comments children made.

Step 4: Reviewing themes

Having familiarised myself with the data, coded the data under existing frameworks, and coded the data to allow additional features to emerge, I had identified the following themes:

Under the existing frameworks of approaches to generalisation presented by Rivera and Becker (2011), Lannin (2005), and Barbosa (2011), what strategies did the children adopt? Within this, in what ways did strategies overlap with each other, and support children's thinking?

Why did children adopt particular strategies? What role was played by their peers, the environment, and the individual structures of the patterns?

Were children consistent across patterns in the strategies they adopted? In situations where they were not consistent, what role was played by their peers, the environment, and the individual structures of the patterns?

These sub questions guided my interrogation of the children's thinking about the patterns presented, and in Chapters 5, 6 and 7 I present my exploration of these questions as I strove to answer the research question of this study. In so doing I drew on my interpretations of the children's verbal comments along with their constructions, drawings and written calculations.

Validity and Reliability

Inter-observer reliability

In planning and conducting research where the research instrument is a task-based interview, there is a necessity to consider specific issues relating to reliability which arise from the nature of the data collection. Firstly, in conducting an interview, the researcher has the flexibility to vary the questions depending on his/her interpretation of a participant's response, and therefore one must consider whether two researchers would ask the same schedule of questions and elicit the same responses. Secondly, in coding responses, there is an inherent complexity in the responses participants may give to 'why?' questions which creates difficulty in establishing reliability. In task-based interviewing, when similar 'why?' questions are prevalent, Ginsburg (1997) states that it is possible to obtain acceptable levels of inter-observer reliability but that due care and attention needs to be paid to consistency of experience for participants and consistency in coding responses. To that end, I shared video footage from my data collection with my supervisor, Dr Thérèse Dooley, and requested her opinion on the codes I assigned. In transcriptions referred to in the analysis presented in Chapters 5, 6 and 7, I included my inputs to the children's discussions, in the form of supporting questions and prompts, aiming to remain explicit about my role as facilitator of the children's engagement with the mathematics.

Validity

Also in any research with children there are issues regarding the validity of the research in terms of the reliability of the child's responses and whether a participant's responses give a true reflection of his/her thinking. Decisions I made regarding interpretation of children's comments and gestures are presented above in the section entitle 'Inferences from children's responses'. While my understanding of children's thinking is at best an approximation, I aimed to be systematic and rigorous in how I explored their possible intentions.

Alongside the difficulty of interpreting children's thinking, extraneous variables may impact on how a child responds to a task. Such factors include the Hawthorne effect, whereby some children may give what they perceive to be the required answer; the experience of being interviewed by a stranger; the novelty of the tasks involved and the context of the tasks. In planning this research I used group interviews to minimise the pressure on children to seek a required answer, rather than taking risks, and articulating tentative ideas, as outlined in Chapter 3. It is also probable that the use of group interviews would reduce the intimidating impact of being interviewed by a stranger (Cohen, Mannion & Morrison, 2000).

Breen and O'Shea (2010) attest that novelty within a task creates a need for participants to engage in "non-routine thinking and reasoning" rather than reverting to imitative reasoning (p. 287). Shape pattern solving is not an element of the PSMC and I expected the tasks of the interview to be quite novel for many participants. In choosing tasks for use during the interviews, which are included in Appendix R, my aim was to find patterns which were context-free or where the context was potentially universal in nature, in order to maximize the inclusivity for children from all sociocultural and economic backgrounds (Shannon, 2007). While endeavouring to minimise the impact of extraneous variables in children's responses, I accept that there may remain an element of variance in responses of some participants. Ginsburg (1997) highlights inconsistency in children's responses to questions and suggests that they may reflect "lack of stability" in a child's thinking which would be very interesting in terms of my research. One possible reason for such a lack of stability may be that a participant is in a transitional phase between developmental stages of algebraic thinking (Burgher & Shaughnessy, 1986).

In determining the validity of interview based research one of the most significant issues for exploration is that of suggestibility or leading questions (Ginsburg, 1997). In particular when the researcher occupies a role analogous to that of teacher and the participant that of pupil, there is a tendency for the participant to seek cues from the researcher in order to get the right answer rather than thinking deeply about the task. During a task-based group interview, when it is not possible to prepare for every possible contingency, there is a greater risk that the researcher may purposely or inadvertently lead the participant towards a solution. In planning for anticipated situations, and in conducting a task-based interview, there is an onus on the researcher to foreground his/her sense of the participant as an individual (ibid.). It was necessary, when facilitating children's engagement with the tasks, that I be prepared to adapt my approach depending on the group's demeanour and responses. As an example, in some cases, it was necessary to adapt a care free and playful style to remove pressure and create a sense of the interview as removed from school, to encourage more taciturn children. In the introduction to each interview, I encouraged children to talk as much as possible, and drew a comparison to their classrooms where they might not regularly receive similar encouragement. In this way, I aimed to lessen any pressure upon children to give responses which they felt they should ‘know’ from content covered in school, rather than thinking independently in response to the task presented (Ginsburg, 1997). My use of group, rather than individual, interviews aimed to create a collegial atmosphere which may have further reduced any sense of intimidation or timidity, as mentioned earlier in this chapter.

While qualitative research provides instruments which lead to rich data collection and analysis, the researcher must be rigorous both in data collection and analysis. One must remain cognisant that “knowledge residing a priori within our minds shapes the images we receive” (Rossman & Rallis, 2003, p. 6). Balanced with this challenge is the richness of understanding which the researcher’s background and perspective contribute to the research (ibid.). In my research the qualitative findings are the understandings drawn from analysis of the children’s responses to the interview questions. In planning for my research and developing the interview schedule, I was aware of the possibility that if I expect to observe certain responses, this expectation would be present in the tasks I choose, and the accompanying questions that I asked the children. In this way my expectations may influence any findings from the research both in foregrounding those responses which I had sought and possibly obscuring other prevalent responses. For this reason, it was necessary for the research to be systematic and rigorous for the findings to be valid and reliable. While findings which emerge from this qualitative research are driven by the data, these findings cannot be generalised to all children in Irish primary schools. The findings may however shed light on some strategies that children attending Irish primary schools may tend to adopt when seeking to construct general terms for shape patterns, and factors which influence their thinking in this regard.

Rather than deciding to research a particular response, for example a misconception or a barrier to learning, the tasks underpinning the group interview, and the manner of their presentation, are designed to allow themes to emerge which are significant to participants' views of generalisation in mathematics. While the deductive element of my analysis drew from existing framework in the field of algebraic thinking, the inductive analysis should generate theory, as suggested in the grounded theory approach (Glaser & Strauss, 1967).

Replicability of findings from a task-based interview

In discussing the use of a group interview in exploring children's thinking, it is pertinent to mention concerns regarding the level of finesse required in administering an interview as an assessment instrument. Piaget (1929) suggests that only after a year of daily use of the method may practitioners move beyond "the inevitable fumbling stage of the beginner" (p. 20), which for teachers may involve excessive talk and suggesting answers. Piaget cautions that there is a challenge involved in striking the balance between preconceived ideas which may skew the direction of the interview and approaching the interview with too little knowledge about the subject matter to form a reasonable hypothesis for research. Equally, it is imperative to adopt a balanced approach to the utterances of the child, which may express factors extraneous to algebraic reasoning, such as a willingness to impress, an inclusion of spontaneous imagination, or fatigue. Piaget (1929) cautions that the greatest risk to the interview is the researcher who considers every utterance of the child as either 'gold' or 'dross' as thereby the findings are rendered relatively meaningless. There is sophistication required in the approach taken to consider and analyse the findings, incorporating information regarding the child's temperament and mind-set at the time of interview.

Verbal mediation by the interviewer is central in the design of a task-based interview. As presented in Chapters 1and 2, our interaction with our external world is mediated by a vast array of cultural artefacts, including language, tools and tasks (Vygotsky, 1978; Bakhurst, 1996; Daniels, 1996). Children's engagement with and understanding of the patterns and the tasks presented to them were mediated by my verbalisations, along with their own and those of their peers. There are implications, therefore, for the reliability and validity of research when the role of the researcher is so embedded in the research design, in design of tasks, mediation during assessment and analysis of results. The social background of the researcher and his/her understanding of how children think will play a powerful role in the results obtained. Smagorinsky (1995) explores the implications of mediation and cautions that “higher mental processes are culturally shaped rather than universal in nature” (p. 203). He suggests therefore that assessments of children’s performance on a task will in effect measure the extent to which the participants’ high mental processes mirror that of the researcher. Some highly developed cognitive strengths may be undetected such as those identified by Moll and Greenburg (1992) in their study of Latino students who displayed great prowess in many skills but were deemed to be of low cognitive ability in school settings.

In order to maximise the replicability of results, Goldin (2000) suggests ten methodological principles for designing quality interviews. Among his suggestions are the development of explicitly described interviews and established criteria for major contingencies; deciding in advance what will be recorded and recording as much of it as possible; a robust pilot-test of the interview; and leaving a margin for compromise for when deemed appropriate. There is a need for researcher mediation to be scripted precisely preparing for all predictable contingencies and for the mediation to be recorded as much as possible in order to identify when mediation may have been inappropriate or excessive. There may be a need for the researcher to compromise and she should therefore be prepared to record such interactions and discuss their implications in the analysis (ibid.).

In response to Goldin’s methodological principles, I developed group worksheets which contained the questions I required the children to consider in relation to each pattern. I also piloted the group interviews, and altered aspects of my approach based upon my findings from this pilot. In particular, during the pilot group interviews, I provided the children with both individual and group worksheets. For some children however, this seemed to give them freedom to disengage from the group, and to work privately without responding to my encouragement to contribute. When I removed the individual worksheets, there seemed to be a notable increase in participation by many children. For my primary data collection, therefore, I did not furnish the children with individual worksheets, but made blank sheets of paper available for rough work, or for individual representations of their constructions of terms. Also during the pilot group interviews, I had one group of two children, and I did not succeed in engaging this group in discussing their thinking with each other. While I only had experience with one very small group, the research evidence of Mueller et al. (2012) and Webb (1991) would concur that groups of four or more are more supportive of collaboration and exploratory talk.

With regards to my mediation as facilitator of the children's engagement with the mathematics presented, my aim was to minimise my involvement. The perceived 'expert authority' of teachers may limit children's inclination to collaborate, or to consider in detail their own thinking and that of their peers (Mueller et al., 2011). I did plan to ask probing questions, to encourage children to articulate, and reflect upon their thinking. I also planned to ask children to share their thinking with each other, in particular when they were in disagreement. I had planned to be vigilant for occasions when it may be appropriate for me to intervene because the children were expending considerable time and attention on an aspect of the pattern without seeming to move forward in any constructive way. An example was the tendency in School 1, Group 1 to focus on whether the total number of tiles in a term, or in an element of a term was odd or even. I did not suggest that this attribute of the quantity was irrelevant because I was open to the possibility that their focus may support them in observing other aspects of the structure of the respective patterns. Also, they may have used their observations to apply recursive thinking to establish a generalisation wherein multiples of two would be added to a base quantity. More generally, I was concerned that if I suggested to the group that the path they were pursuing might not support their thinking, the children might infer that there was one specific strategy that was required, and a willingness to find this one strategy would hinder their thinking. With hindsight, however, I feel that the children's engagement with the patterns were not supported by this focus on whether quantities were odd or even, and their thinking about the patterns may have benefitted from my suggesting to them that this aspect was not particularly relevant.

Lastly in keeping with Goldin's principles, I recorded all interactions I had with the children during the interviews, and I transcribed my comments meticulously. During analysis, I included my inputs in all relevant transcripts, and I also referred to preceding inputs if there was any possibility that they directed the children's thinking in any way.

Conclusion

In this chapter I have attended to the theoretical base for decisions I made during the practical elements of data collection and analysis. I presented my reasoning behind the selection of my sample of 42 children and my in-depth thematic analysis on the thinking of sixteen of children, to facilitate analysis which was rich and deep.

Due to the novelty of the mathematical thinking I was asking the children to engage with, it was incumbent upon me to facilitate children's engagement beyond a mastery level, but within their Zone of Proximal Development. During the interviews I aimed to apply theory from research findings into my practice in mediating the children's engagement with tasks through peer interactions, concrete materials and supporting prompts and questions. I sought to encourage, and support collaboration between peers as they worked together to solve tasks. I aimed to facilitate child agency by encouraging the children to report to, and seek justification from, each other, rather than reporting to me. I organised the groupings to adhere to research findings regarding the ideal group-structure when collaboration is valued as an important aspect of group-work.

CHAPTER 5: AN INVESTIGATION OF CHILDREN'S PROGRESS ACROSS THE THREE PATTERNS PRESENTED

Introduction

In this preliminary analysis chapter I present an examination of how the children responded to the patterning tasks. I aim to describe children's responses, and also through a hermeneutic phenomenological lens I aim to analyse the underlying thinking of the children and the factors that contributed to how they thought about the patterns. Within this examination I compare the three patterns to each other in terms of how the children interacted with the questions asked of them, and the progress they made in constructing terms. I focus on specificities of the patterns, drawing upon examples of children who succeeded well with one pattern, but not with another.

In seeking to assess children's proficiency in any area of mathematics, well designed tasks are key to the validity of the assessment (Burkhardt, 2007). Krainer (1993) emphasises the dual role of powerful tasks as both supporting children in engaging with material, and also allowing them to construct their own pathway through the mathematics. In selecting tasks for this research it was necessary to find tasks which would fulfil both purposes. This duality may at times be difficult to achieve, in that some tasks may require mathematical thinking which is beyond the reach of the child, while other tasks may involve very little algebraic thinking. In this section, I discuss my choice of the tasks, building upon the discussion in Chapter 2.

The tasks chosen were similar in nature, but differed through the variations implicit in the three patterns used. Rivera and Becker (2011) assert that there are variations in how different children may perceive a pattern presented to them. In this chapter, I explore how the variations which occurred between different children's understanding of each pattern supported their progress. I present, therefore, a statistical comparison of the three patterns to explore the relative challenge they presented to this cohort of children, and I explore why some tasks may have been more challenging than others to some children.

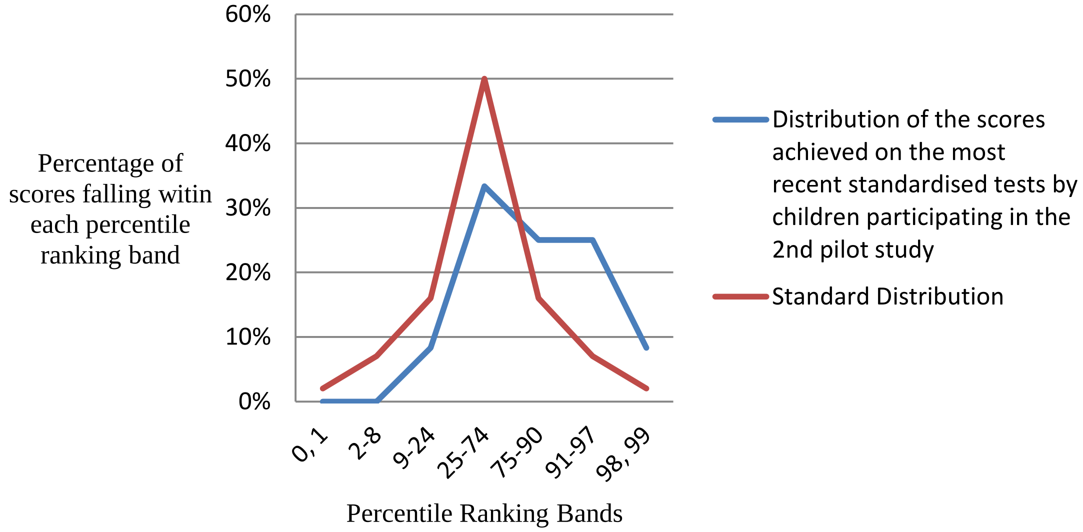
Additionally, in exploring children's progress on specific patterns, I discuss other factors in the children's environment which may have supported or hindered the children's progress in constructing generalisations for the patterns presented to them. I discuss situations where specific children tended to commence with a limited or flawed approach, and whether their focus shifted as they engaged with each other, and with the patterns. In analysing any shift in focus I aimed to consider any identifiable external catalyst for change in the peer interactions relating to the patterning tasks at hand. I also aimed to identify and analyse situations where children self-corrected when they discovered that their approach would not be successful, or when they experienced a perturbation when their expressed view was contradicted by a new finding as they progressed through the task. I look at group interactions which in some cases may have broadened children's thinking and supported their progress; while in another case may have hindered one child's thinking.

Drawing from pilot studies

In my first pilot, I aimed to present a broad range of tasks to participants, and to select the tasks which afforded the greatest insight into the children's thinking. Of the tasks presented, the repeating patterns were achieved by all children, including those in Junior Infants, and there was no evidence of mathematical challenge in how the children responded to them. The number patterns presented resulted in a greater range of responses, but I felt that these tasks were dependent upon children's multiplicative understanding, and that children's varying mastery of number may skew the assessment of their algebraic thinking. Equally Cai and Knuth's equations and variable tasks were onerously challenging for many children, and their design did not support children in making progress. The tasks which were accessible to all participants, while still challenging at some level to all children, were the tasks involving shape patterns.

As discussed earlier, the findings of my first pilot strongly suggested that using diagnostic interviews involving groups of children would be a more advantageous research method. It was necessary for me to conduct a second pilot to ensure that the tasks I was considering would lend themselves to supporting collaborative sense-making. During my second pilot the tasks I presented facilitated rich discussions among the children. Children had been chosen at random, and while the sample was not large enough to be representative in any way, a broad range of attainment was represented. The distribution of the mathematics attainment percentile rankings from the most recent standardised tests for the participants is compared to the standard distribution in Figure 5.1. As can be seen from Figure 5.1, the scores of the participants are similar to the normal distribution for the standardised test, but skewed to the right.

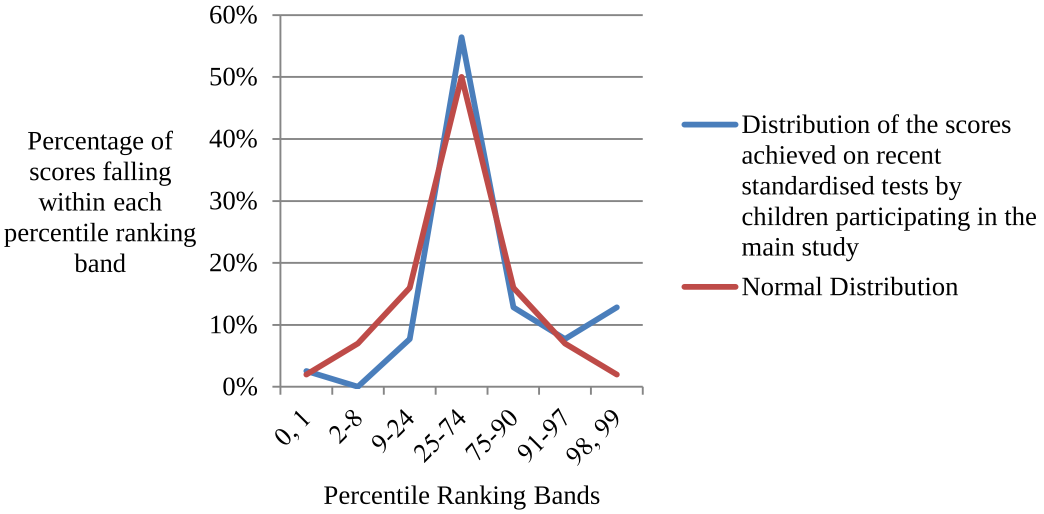
Figure 5.1. A comparison of the distribution of the scores received by children participating in the second pilot study on a recent standardised test, and the associated normal distribution for the test population



In engaging with the tasks, all children contributed by extending at least one pattern, and all children experienced challenge at some level, in that no child correctly described a general term for all three patterns.

In my main data collection, the recent standardised scores of the children were more normally distributed, as may be seen from Figure 5.2.

Figure 5.2. A comparison of the distribution of the scores received by children participating in the main study on a recent standardised test, and the associated normal distribution for the test population

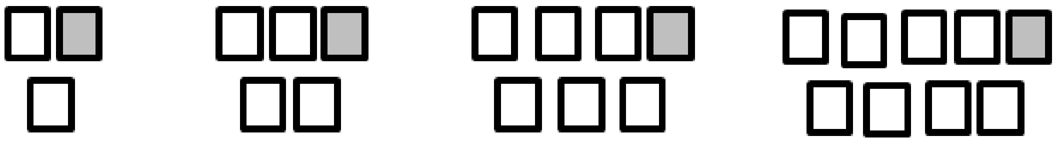


Patterns Used and Questions Asked

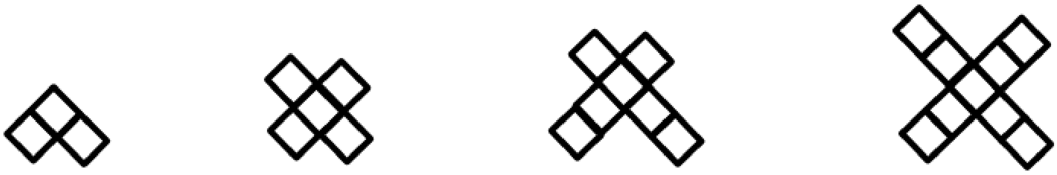
During the diagnostic interviews three patterns were presented to the children, as shown in Figure 5.3. The structure, and source of the three patterns is discussed in Chapter 3.

Figure 5.3. The patterns as they were presented to the children

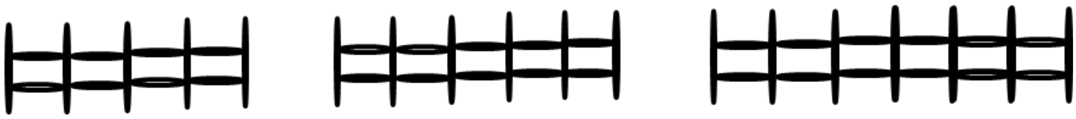
Pattern 1



Pattern 2



Pattern 3



Additionally in Chapter 4 I drew attention to pertinent details of each pattern presented to the children, for example the asymmetric rate of growth of Pattern 2. Among the tasks relating to Pattern 2, I asked children to construct the 75th term, as a far generalisation. To do so explicitly would require children to halve 75, and round down the answer achieved in order to find the number of tiles on each of the upper two legs of the x-shape. Lily Rose demonstrated that the challenge presented in dealing with this number hindered her progress in thinking about the task, as demonstrated in the following transcripts.

Lily Rose	I think it might be 35 either.
AT	And why 35, Lily Rose?
Lily Rose	Because half of 60 is 30 and then half of 70 is 35.
AT	Great and for the 75th would you need to adjust it?
Lily Rose	Ehm, ehm, I think it might be, ehm
AT	I can help you with that now if you like?
Lily Rose	Can you?
AT	I can. You could say half of 74 is 37. So, this is the 75th term?
Lily Rose	37 and a half.

AT	Would you have 37 and a half tiles do you think? Does that fit into the pattern, do we have any half tiles? Okay Lily Rose, I am going to ask you to take a look back at the terms there and see what happens, what could you do when you can't half 75 and Jane can you tell me why Lily Rose halved 75, why did she want to half 75?
Jane	'cause the same way in term 10, it was like half of 10 was the amount of tiles on the legs. So probably half of 75 would go on the amount of the legs.
AT	Okay lovely right, now all four of you need to look at the terms that you have there in front of you, find out what happens when you can't halve the number. What terms do you have where you couldn't halve the number?
Lily Rose	You will have to give one side, one extra once. Like down at the bottom two they might have one extra than the top two.
AT	Okay and so 75 how would that work do you think, what do you think Danny, does that make sense?
Danny	Yeah.
AT	That if the number can't be halved the bottom one is one extra tile?
Danny	Yeah.
AT	OK. And then if we know the two 37s, that two 37s is 74 could anybody describe what the 'X' shape would look like?
Jane	I think there would be an extra one at the top on each line at the top… [trails off]
AT	What do you think Christopher?
Christopher	I think that there is going to be 32 at each side and 22 then at the ..[inaudible]
AT	And Lily Rose any ideas what – about how the shape would look exactly?
Lily Rose	I think it would be like a big cross, and would be like – I don't know...

During this section of the interview, none of the four children in this group seem to have succeeded in describing the 75th term, even though I aimed to scaffold them by taking responsibility for some of the computational load. Jane and Lily Rose described elements of the structure of the pattern, and both seemed cognisant of the need to halve the term number, but struggled to manipulate half of 75 in order to distribute the tiles on the figure. Later in this chapter I compare the thinking of Grace across all three patterns, and there seems to exist a parallel between challenge she encountered with manipulating half of 75, and the challenge experienced by Lily Rose and Jane during their interview. In hindsight, given the relative challenge of the asymmetrical rate of growth of this pattern, it would have been advisable to ask the children to find a term with a position number that was easier to manipulate, for example a multiple of 10.

Multiple Approaches to All Patterns Taken By Children

When presenting children with a pattern, it is pertinent to remain open to the possible variations in how the pattern is perceived (Rivera and Becker, 2011). Such variations were evident in my research, as a number of approaches were taken by children to the patterning problems presented. Table 5.1 contains descriptions of the approaches used by children during this research project, along with examples taken from the group interviews.

Table 5.1. The possible approaches to construction of general terms for each pattern, with examples from the interviews of approaches adopted during this research project

Description of approach	Example from interviews
<u>Pattern 1</u>	
Term number on each of the top and bottom, with the additional shaded tile on the top.	Alex: "Maybe twice this. Twice as much as the term? Well the shaded one would be one added on.
Term number on bottom, term plus one on top	Emily: "The bottom number is the proper number and the top is just one more than it"
Diagonal pairs of tiles plus extra one	Ciaran: "you have six [pairs] going diagonally, and then you need to put a red one in the corner which is the weird one"

<u>Pattern 2</u>	
Central diamond plus a number of tiles on each leg, equal to half of the term number.	Jane: "I think, I think there would be 5 on each leg, […],Well some middle square, [….], 21?"
A number of tiles on each leg, equal to half of the term number, without a central tile.	Orla: "20: 5, 10, 15, 20".
Two strips each containing a number of tiles equal to the term number, intersecting at the "middle", of the first strip if containing an odd number of tiles, and at the (n/2)th tiles from the top if containing an even number of tiles.	Wyatt, "I think it would have 11 going down slanted and then 11 in the opposite way because you add on one more."
<u>Pattern 3</u>	
n horizontal posts, and (n+1) vertical posts	Arina: "Like 56 twos, […], I'm going to add… I'll add 57 to know what the total".
n*3 + one additional post added at the end 4+ (n-1)*3	All children in this research project separated the posts into vertical and horizontal posts, rather than adding successive groups of 3.

Children who struggled to engage with the patterning tasks presented

I had one participant, Dylan, for whom the tasks were not accessible. On his most recent standardised test for mathematics, Dylan achieved a score which placed him on the 1st percentile ranking for that assessment. I felt during his group and individual interviews, that there may be an emotional, as well as cognitive, reason for Dylan's complete lack of progress through tasks. He receives support in school for both academic challenges he faces, and also emotional and behavioural difficulties. One other participant, Daniel, scored a 0 on all three patterns, because he did not extend any pattern independently, but received support and guidance from his peers. Daniel, did however engage well with the patterns, and made many descriptive comments when presented with each pattern. A discussion of the possible reasons contributing to Daniel's difficulties is presented within the section of this chapter devoted to interactions with peers.

A Comparison of The Patterns Presented

In developing a rich thick description of the children's progress in constructing generalisations, I firstly summarised the children's responses on each pattern, and compiled a table which is included as Appendix S. In compiling the table I referred to transcripts, video footage, and artefacts from the children's group interviews, such as photographs, drawings and calculations, as discussed in Chapter 4. I developed a scoring range for the children's responses, modelled on that used by the Subject Knowledge in Mathematics Audit (Rowland, 2007). Children's responses were scored within the range from 0 to 4, as indicated in Table 5.2.

Table 5.2. Scoring protocol for children's responses to patterning tasks

Response	*Score*
Perfect generalisation, articulated for all terms	4
Perfect far generalisation for term number given, or generalisation where arithmetical difficulty impeded completion, or generalisation with single numerical error, e.g. 86+85 in place of 87+86 for Radford 86th.	3
Progress towards a generalisation, e.g. correct near generalisation, described some aspects of general terms	2
Extended correctly	1
No progress	0

If a child made no progress with a pattern, and did not succeed to extend to the next term, I awarded a score of 0. I aimed to plot children's progress between the extremes with the scores 1, 2, and 3. I allocated a score of 1 to responses which demonstrated some progress, including when a child extended to the next term, but made a reasonable error, such as including two additional diamonds on the top legs of the 5th term. A score of 2 was awarded to responses which demonstrated a correct extension and some indications of considering a general term, and a score of 3 to responses wherein children generalised but made an arithmetical error, were hindered by previous errors in their workings, or encountered challenge with computation, or articulation of ideas.

A score of 4 was awarded when a child correctly described a general term for the pattern in question, without needing to rely on specific numeric examples. Radford (2006) defines generalisation as “grasping a commonality noticed on some particulars (in a sequence), extending or generalizing this commonality to all subsequent terms, and being able to use the commonality to provide a direct expression of any term of the sequence.” Rivera and Becker (2011) extend this definition to include justification. In this study, children were asked to describe any term for the pattern, to facilitate them in seeking to fulfil the definition by Radford (2006), and in some cases, they justified their definitions. As discussed in Chapter 2 it would also be possible to have asked children to derive an algebraic expression using variables for the general task they have described. During their longitudinal investigation of children’s algebraic thinking, Rivera and Becker (2011, 2008, among others) asked children to “find a direct formula” where children were required to derive an expression using abstract symbols to represent the structure which they had observed within the pattern. In this research project I took the position that abstract symbolism is not a necessary component of algebraic thinking (Radford, 2012). Also, as outlined in Chapter 4, the use of variables in generating rules from patterns is semantically challenging, and variable use is not present in the Primary School Mathematics Curriculum in Ireland before 4th Class. I considered therefore that it would not be appropriate or necessary to expect children participating in this study to express their thinking using symbols such as ‘x’ or ‘n’ but rather to encourage them to express their mathematical thinking using words, constructions and sketches. In Table 5.3 I describe examples of responses which merited each score on the scoring protocol.

Table 5.3. Examples of children's thinking and the scores assigned

Score	*Child*	*Pattern*	*Response*
0	Daniel	1	Expressed confusion regarding the existence of terms beyond those presented, and did not construct independently the next term in the pattern
0	Fiona	3	Referred to aspects of the pattern such as vertical posts, parallel posts, etc, but did not succeed in constructing a fence independently, and said that the 9-panel fence would have 2 more posts than the 7-panel fence.
1	Danny	2	Constructed the 5th term, and drew the 6th term, but remained silent during discussion of the near generalisation, and offered "23 on each side" as a description of the far generalisation
1	Alex	3	Extended correctly, offered an incorrect answer of 27 for the near generalisation, and did not construct a far generalisation
2	Grace	2	Struggled to halve 75, and could not therefore describe the 75th term – "Because ehm, you could just find eh, the nearest number to, eh, but like eh, you could choose the doubles that equal 74, and then you just add one more… I think , like all the sides would be like, at least 40 something cos like if you add 30 that would be around like 60"
2	Emily	1	While Emily seemed to describe a general term correctly as "The bottom number is the proper number and the top is just one more. than it" she consistently applied flawed whole object thinking, which hindered her from describing correctly the far generalisation
3	Wyatt	2	Correctly calculates 151 tiles, and describes 4 legs with one central tile. Struggles to articulate thinking clearly, and does not describe a general term.
3	Arina	3	"Like 56 twos…I'm going to add… I'll add 57 to know what the total."
4	Jane	3	So the panels, we double them and then the posts would be

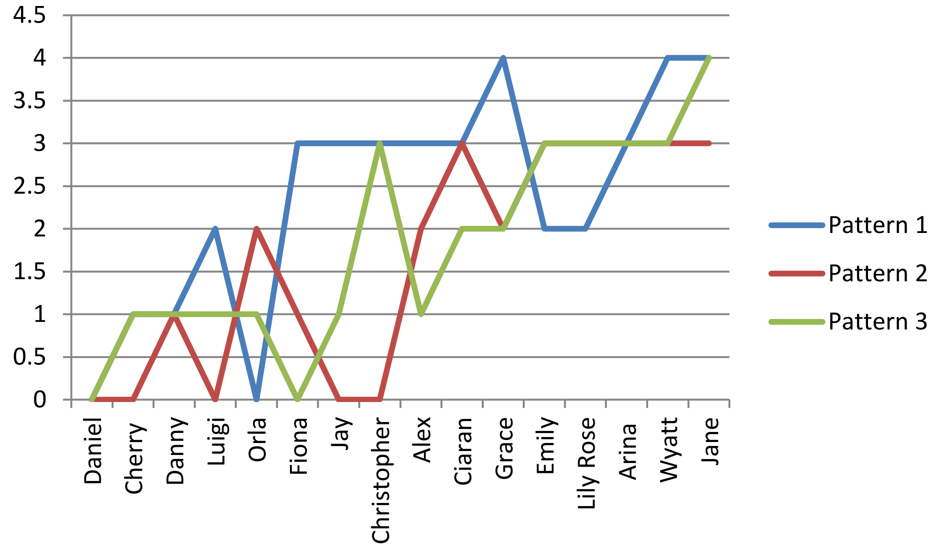
			the number of the panels and one extra.
4	Wyatt	1	Describes the bottom row as containing the term number of tiles, and the top row as one more.

Referring to the summary overview I scored each child on each pattern. I found the need to also refer regularly to the transcripts, video footage and artefacts to ensure that scores fully reflected the children’s responses. The scores for each child are presented in Table 5.4, and Figure 5.4 presents a comparison of scores obtained on individual patterns. The group code given includes a reference both to the children’s school and to their group within the school, for example group 2.3 was the third group interviewed in the second school.

Table 5.4. Scores achieved by each child on each pattern

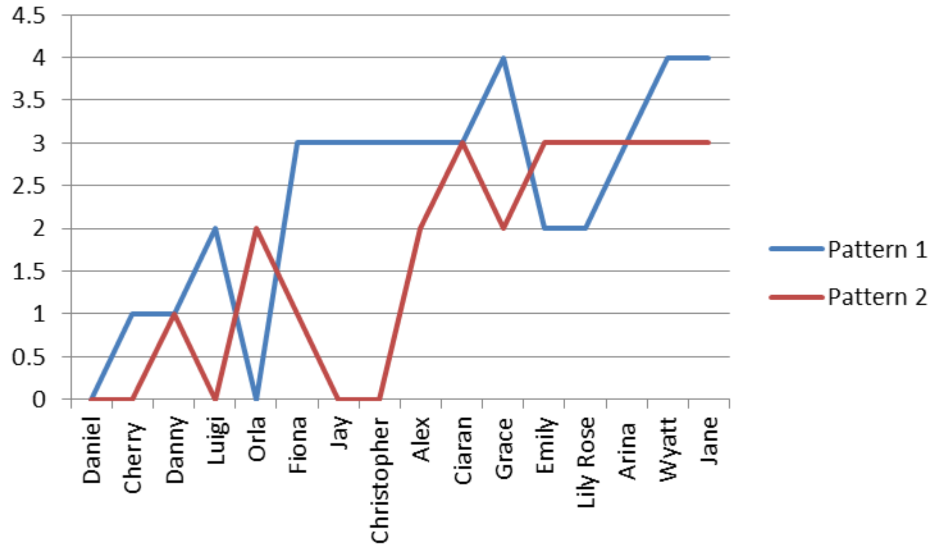
	Sch.Group	*Pattern 1*	*Pattern 2*	*Pattern 3*	*Total*
Daniel	1.1	0	0	0	0
Cherry	1.2	1	0	1	2
Danny	2.3	1	1	1	3
Luigi	1.5	2	0	1	3
Orla	1.5	0	2	1	3
Fiona	1.1	3	1	0	4
Jay	1.2	3	0	1	4
Christopher	2.3	3	0	3	6
Alex	1.2	3	2	2	7
Ciaran	1.1	3	3	2	8
Grace	1.1	4	2	2	8
Emily	1.5	2	3	3	8
Lily Rose	2.3	2	3	3	8
Arina	1.2	3	3	3	9
Wyatt	1.5	4	3	3	10
Jane	2.3	4	3	4	11
Average Score		2.4	1.6	1.9	

Figure 5.4. A comparison of scores achieved on the three patterns



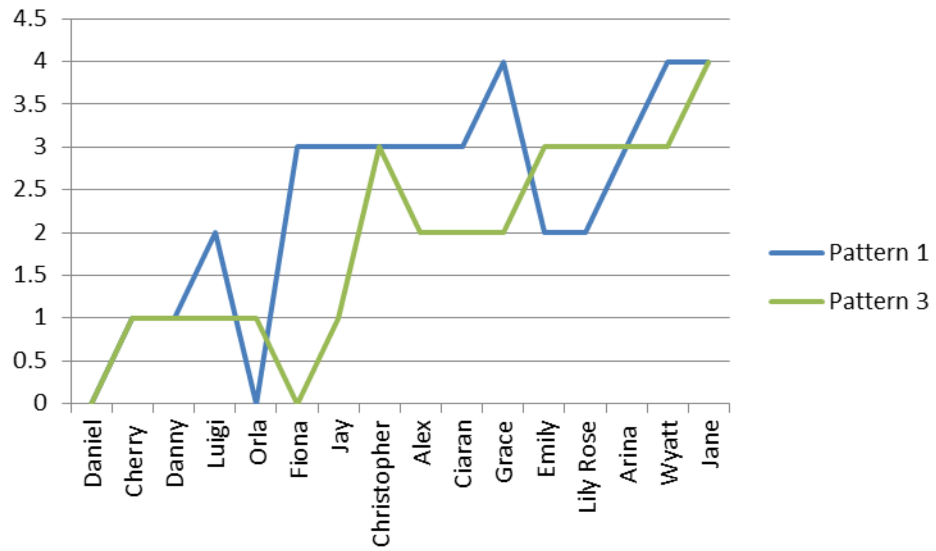
It is evident from Figure 5.4, and Table 5.4 that Pattern 1 resulted in higher scores for many children, but it is difficult to compare Patterns 2 and 3 as they seem to be reasonably similar and overlap in parts of Figure 5.4. For clarity, in Figures 5.5, 5.6 and 5.7. I juxtapose each pair of patterns.

Figure 5.5. A comparison of scores achieved on Patterns 1 and 2



Reading from Table 5.4, and Figure 5.5 it becomes clear that for most children Pattern 1 resulted in higher scores than Pattern 2, indicating that children may have experienced more challenge in engaging with Patterns 2. As discussed in Chapter 2, and alluded to earlier in this chapter, Pattern 2 contained a challenging asymmetric design. This pattern was included to explore how well children would reason with generalisations, in making the final adjustment for the odd numbered far generalisation (75th term). In this chapter, and in Chapters 6 and 7 I explore children's approaches to the construction of general terms for this pattern, and examples from children's work demonstrate the relative challenge experienced by children who adopted recursive approaches or focused excessively on numerical aspects of Pattern 2.

Figure 5.6. A comparison of scores achieved on Patterns 1 and 3



Given the linear nature of Pattern 3, I had anticipated that children may have similar levels of success in constructing general terms for this pattern as they had for Pattern 1. In fact, I had expected that they would apply thinking they had engaged with in working on Patterns 1 and 2, and thus possibly encounter less challenge with Pattern 3. However, fatigue and time constraints may have contributed to the poorer results evidenced for Pattern 3 for many children. In two of the interviews, children received announcements relating to lunch or to an upcoming activity in their classroom. Also, some children were noticeably quieter when engaging with Pattern 3, as shown in the total number of comments for each child on each pattern presented in Table 5.5.

Table 5.5. The total number of comments made by each child on each pattern, and averaged across all children

	Pattern 1	*Pattern 2*	*Pattern 3*	*Total*
Alex	57	62	20	139
Arina	71	49	9	129
Cherry	17	11	11	39
Christopher	15	18	23	56
Ciaran	88	77	40	205
Daniel	88	96	42	226
Danny	12	11	28	51
Emily	26	66	23	115
Fiona	19	20	7	46
Grace	47	52	26	125
Jane	10	22	17	49
Jay	44	34	9	87
Lily Rose	9	35	9	53
Luigi	36	87	52	175
Orla	15	16	10	41
Wyatt	20	68	28	116
Total	574	724	354	1652
Average per child	35.875	45.25	22.125	103.25

Table 5.5 evidences that on average children contributed twice as many comments in their engagement with Pattern 2 as they did with Pattern 3. There may be many contributing factors to this disparity, and it could not be said that the number of comments alone would give any indication of children's level of challenge or uncertainty, as Pattern 1 which generated higher scores also generated fewer comments than Pattern 3. Also, the type of discussion the children engaged in may have contributed to the number of comments made, and some of the additional comments generated during children's engagement with Pattern 2 could have been due to uncertainty with children arguing, disagreeing, or questioning each other. The reduced number of comments relating to Pattern 3 could however point to fatigue, when coupled with some indications from children, for example, Jay leaning his head on the table, and

Daniel asking “what time do we close at?” Considering that the children had engaged in an hour of novel, cognitively challenging tasks some element of fatigue might have been expected during their engagement with Pattern 3 in comparison to Pattern 1 in particular.

Figure 5.7. A comparison of scores achieved on Patterns 2 and 3

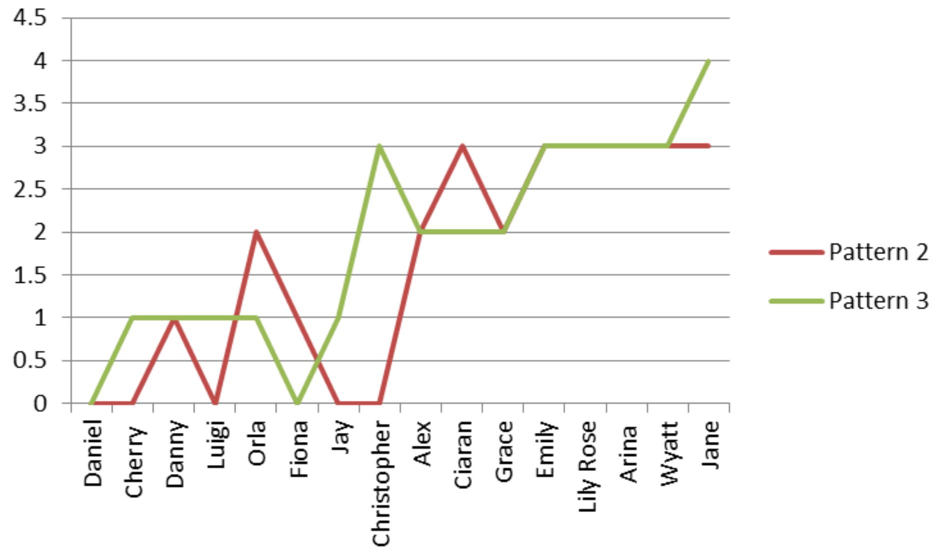


Figure 5.7. clarifies that Pattern 2 was consistently equally or more challenging than Pattern 3, but the difference in the scores achieved on the two patterns was not greater than 1 for any child other than Christopher. This difference may be attributable to the more challenging mathematics of the asymmetric nature of Pattern 2, and may indeed have been more pronounced if the order in which the patterns were presented was reversed.

Consistency across patterns

In order to examine the level of consistency children presented across patterns, I compared the highest score achieved to the lowest score achieved for each child. I deemed a child to be consistent across patterns if his/her lowest score was 2 less than his/her highest score, and highly consistent if his/her scores were equal across patterns or if his/her lowest score was 1 less than his/her greatest score. As presented in Table 5.6, of the 16 children, eight demonstrated a high level of consistency across patterns by this benchmark, five demonstrated some level of consistency, and three demonstrated inconsistency between responses.

Table 5.6. The level of consistency demonstrated by children in the scores they achieved across all patterns. The children are presented in this table in the order of their total score across all patterns

	Lowest Score	*Highest Score*	*Highly Consistent*	*Consistent*	*Inconsistent*
Daniel	0 All Patterns		√		
Cherry	0 (Pn 2)	1 (Pn 1 & 3)	√		
Danny	1 All Patterns		√		
Luigi	0 (Pn 1)	2 (Pn 2)		√	
Orla	0 (Pn 1)	2 (Pn 2)		√	
Fiona	0 (Pn 3)	3 (Pn 1)			√
Jay	0 (Pn 2)	3 (Pn 1)			√
Christopher	0 (Pn 2)	3 (Pn 1 & 3)			√
Alex	2 (Pn 3)	3 (Pn 1)	√		
Ciaran	2 (Pn 3)	3 (Pn 1)	√		
Grace	2 (Pn 2 & 3)	4 (Pn 1)		√	
Emily	2 (Pn 1)	3 (Pn 2 & 3)	√		
Lily Rose	2 (Pn 1)	3 (Pn 2)	√		
Arina	3 All Patterns		√		
Wyatt	3 (Pn 2 & 3)	4 (Pn 1)	√		
Jane	3 (Pn 2)	4 (Pn 1 & 3)	√		

Children with consistent scores across all patterns

As evident from Table 5.6, of the sample of 16 analysed, three children encountered significant challenge across all patterns, scoring a 0 or 1 on each, and three children scored a 3 or 4 on each pattern. Ciaran and Emily scored a 2 or 3 on each pattern, indicating that they could make some progress towards constructing near and far generalisations for all patterns, but did not describe an abstract generalisation for any pattern. In exploring differences between patterns, it may be more constructive to analyse more closely the work of children who were less consistent.

Children presenting as inconsistent across patterns

Christopher's thinking

Christopher scored a 0 on Pattern 2, and a 3 on both Patterns 1 and 3. While most of the thinking exhibited by the children in Christopher's group connected terms to their positions in the pattern, recursive reasoning was also present. In their initial discussion when presented with Pattern 2, Christopher's group focused on the rate of change between terms, and for example Christopher described the pattern's growth as:

> Christopher It's like going up, so first it goes that, and then the two blocks is, and term two it goes there, and then for term three you add another one, the same again on the other four posts, on this side and then I think that that term 5 that it's going to add onto the right side.

While all three patterns were presented as linear sequences of four consecutive terms, the structure of this pattern may have obscured the relationship between terms and their position more than was the case with the other patterns, and encouraged children to focus on the 'incremental change' between terms. Christopher seemed to struggle to broaden his thinking about this pattern beyond the comparison of consecutive terms. Referring to the categorisations of Barbosa (2011) presented in Chapter 2 and also discussed in Chapter 6, Christopher does not identify a unit or use guess and check, but refers persistently to the "extra" 2 tiles, or to the terms growing, without specifying exact dimensions of near or far terms. All his comments when describing and extending the pattern related to "adding" and "extra" tiles, and at no stage does he mention the term number when aiming to construct a term.

Alternatively, Christopher may have reverted to recursive thinking when he encountered this pattern, due to the level of challenge he experienced, while with other patterns he applied connections between the terms and their position numbers to tackle the patterning tasks confidently. As discussed earlier in this chapter, Pattern 2 involved a challenging requirement to generalize differently for odd and even numbered terms.

While it may not be helpful to consider recursive and explicit thinking as hierarchical, it is very likely that Christopher was more familiar with recursive thinking, and that he was less secure in using an explicit approach. Christopher did not succeed in generalising from this pattern, and confident assertions which are present during discussion of Pattern 1 and Pattern 3 such as “this is easy” or “I know what [this term] is” are absent from his contribution to the discussion relating to Pattern 2.

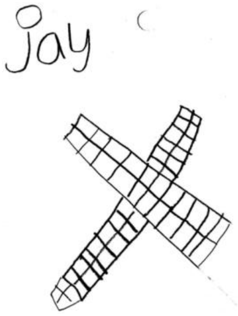
Jay’s thinking

Jay scored a 3 on Pattern 1, indicating that he constructed a far generalisation, without articulating an abstract definition of a general term, and he scored a 0 on Pattern 2 and 1 on Pattern 3. Examining the language he used in explicating his thinking on Pattern 1, it could be said that Jay showed some degree of uncertainty in constructing the far generalisation. His initial description of the pattern shows evidence of a strong link between the quantity of tiles and the term number when he says “Don’t count the odd one out, 1 four and 1 four, cos 3, 3, odd one out; 2, 2, odd one out”. When articulating his thinking in constructing the far generalisation requested for this pattern (the 86th term), Jay suggests “How about 88 and then 86 at the bottom”, and then suggests 171 as a total, before rethinking in light of other children’s ideas, and adding 87 to 86. While it could be said that Jay benefitted from the thinking of the others in his group on this pattern, he seems to construct his final answer independently. Jay’s initial descriptions of Patterns 2 and 3 are less clearly articulated than his initial observation of Pattern 1. His extension of Patterns 1 and 2 are presented as Figure 5.8.

Figure 5.8. Jay's extensions of Patterns 1 and 2

Pattern 1

Pattern 2



It is evident from Figure 5.8 that Jay grasped the structure of Pattern 1, but he seems to have only noticed the outline of the terms in Pattern 2. Not only does his extension have an incorrect number of tiles for the next term in this pattern, but he drew two columns of tiles running in each direction, which was completely unlike the terms presented to him. Jay's non-completion of a correct drawing may also have made his further engagement with the pattern more challenging, both in terms of his observations of elements of the pattern, and also in understanding the comments and perspectives of his peers whose drawings were accurate. Jay suggested "75+75" as the possible number of tiles on the 75th term, but when I questioned why, he replied "to get a bigger answer", raising his voice at the end of the sentence as if querying his own thinking.

When engaging with Pattern 3, Jay proffered no opinion during the children's initial description of the pattern, and while cooperating well with Alex when constructing the 3rd and 7th terms, he remained very quiet. While the video footage shows him working as part of the group, and seeming to consider the pattern, he proffers few thoughts, making only nine comments, in comparison to 33 during the group discussion on Pattern 1. It may be that Jay didn't build his understanding of Patterns 2 and 3 through observing, and then extending in the way that he did on Pattern 1.

Fiona's thinking

Fiona scored a 3 on Pattern 1, a 1 on Pattern 2 and a 0 on Pattern 3, where she failed to extend the respective pattern to the next term correctly. Fiona demonstrated some signs of fatigue during her discussion of the 3rd pattern with Grace. She rested her head on her hands, and contributed less to the discussion than she had on other patterns. She worked with Grace for the early element of the discussion, and Grace was very lively and obviously enjoying her engagement with the tasks. Grace's exuberance, and Fiona's waning energy may have worked together to reduce Fiona's engagement, and she only spoke 7 times during the entire discussion on Pattern 3, in comparison to 19 utterances on Pattern 1, and 20 on Pattern 2. In comparison, when engaging with Pattern 1, Fiona spoke more, and independently constructed a near generalisation (the 12th term). For the far generalisation, Fiona correctly stated that 87 should be added to 86, but did not justify her thinking. She proffered this opinion after a very unfocused discussion led by Ciaran into whether the 86th term would be odd or even. While Fiona received a lot of support, and guidance from Grace during their work on this pattern, I believe that she understood the structure of the pattern, and that the suggestion of 87+86 was her own thinking, rather than her expression of something she had heard another child say.

While there is a possibility that Christopher, Fiona and Jay would have found the patterns challenging even if their groups' discussions had supported them, all three seem to have missed out on an opportunity to engage fully with the perspectives of the other children. Mueller, Yankelewitz and Maher (2012) explore the varying ways that children contribute to each other's thinking and emphasise that reasoning emerges from shared discourse in incidents of co-construction. The experience of Fiona, Christopher and Jay highlighted here may be examples of children who did not engage in collaboration, and the resulting progress in reasoning about the patterns, due to their reticence or the large variation between their perspective and that of their peers.

Children demonstrating some inconsistency

Three children, Luigi, Orla and Grace, demonstrated some inconsistency in the scores they achieved on different patterns. Their scores on all patterns are presented in Table 5.7.

Table 5.7. The scores of children who demonstrated consistency but were not highly consistent

	Pattern 1	*Pattern 2*	*Pattern 3*
Luigi	2	0	1
Orla	0	2	1
Grace	4	2	2

Of these, I will focus my analysis on the thinking of Grace and Orla. Luigi seemed to revoice the thinking of his peers very often during his group's discussions. It would not have been ethical for me to put him on the spot repeatedly to ascertain whether he was constructing his own thinking, but he did not succeed in giving me either a valid answer on the occasions when I asked him for a suggestion (before he heard those of the children around him), or a justification for an answer. As outlined in Chapter 4, in the section entitled 'Inferences from children's comments', it is not possible to be certain about the thinking children seek to express in the comments they utter during a task-based group interview, or indeed in any setting. For example, I sought to unpick Luigi's thinking in order to assign him the most appropriate score for his progress on Pattern 1, and in the paragraph below I refer to comments made during the conversation presented in the following transcript:

Wyatt	It's going to have 87 on top and 86 on the bottom.
Orla	87 on the bottom.
Luigi	So we're going to have to make all of ours together.
AT	And Wyatt, tell me why you think that?
Wyatt	Because on term 4 it's number 4 and it has 5 on the top.

Luigi	Yeah, 87 on top and 86 on the bottom.
Wyatt	I think the 86th will have one more than 86 that's 87 on top and then take away and it's 86 on the bottom.
AT	Ok, very interesting.
Emily	I think it could be the ehm it could be the number, like, for all of them the number that you're on, sort of like, 86 on the top and 85 on the bottom because since ehm, since here the term for…
Luigi	Yeah it's one lesser than the… so that's supposed to be the proper number then that's supposed to be the one…

Luigi described the far generalisation (86th term) as "Yeah, 87 on top and 86 on the bottom" with some confidence, which might indicate that his progress on Pattern 1 deserved a score of 3. However, this comment followed immediately after Wyatt commenting that the 86th term would consist of "87 on top and 86 on the bottom", and justifying his answer with reference to earlier terms. Luigi's subsequent remark of "Yeah it's one lesser than the… so that's supposed to be the proper number then that's supposed to be the one…" which he expressed in his own words, (as no other child had referred to the term number as "the proper number") seemed far more hesitant, and taking into account Luigi's previous comments, plus his drawings and constructions I was not convinced that he had constructed this 86th term, but felt there was a strong possibility that he had revoiced the thinking of Wyatt. Therefore, I judged 2 to be the most appropriate score to assign Luigi for Question 1 as I was confident that he had made progress towards a generalisation, and could understand the constructions of others. I was less convinced that he had succeeded in independently constructing a perfect far generalisation for the given term number. I therefore chose to focus more in-depth analysis on the thinking of Orla and Grace who had not engaged in revoicing of the thinking of others in this manner.

Orla's thinking

Orla was a participant in Group 5 of School 1. She participated with Luigi, whom I referred to in the preceding paragraph; Emily, whose thinking is examined closely later in this chapter; and Wyatt, who scored a 3 or 4 on each pattern presented to him. Orla scored a 0 on Pattern 1, and seemed very uncertain about how to extend, for example, the 5th term she constructed is identical to the 6th term which she drew. She also answered "I've no idea" when I asked her what she thought about the 12th term, aiming to draw her into the group discussion. After a discussion which highlighted that her peers had very different ideas about the structure of the 12th term (near generalisation), Orla remained resolute in presenting her incorrect 12th term, as can be seen from the following transcript:

Emily	Mine's the same cos I took the 6th term and then I kept adding two, so I'd one on the bottom and one on the top, and that was term 7, and I kept doing that til it reaches 6, and I got the number that I said it was gonna be. [trails off]
Orla	I just did 6 on the top and 6 on the bottom
Luigi	I had 1, 2, 3, … 13 on the top and 1, 2, 3, … 11; 1, 2, 3, ...11, I put 11 on the bottom
Wyatt	I did 13 on the top and 12 on the bottom
Luigi	I put 11 on the bottom and 13 on the top
AT	Ok, and is everybody happy with what they did themselves or would they like to think about it, and ask somebody else? [Emily], how many do you have on the top row?
Emily	What?
AT	How many do you have on the top row?
Emily	12
AT	And on the bottom row?
Emily	11
AT	And, Wyatt has something different, and [Orla] has something different. So could you talk to each other for a few minutes and see why they're different, or...

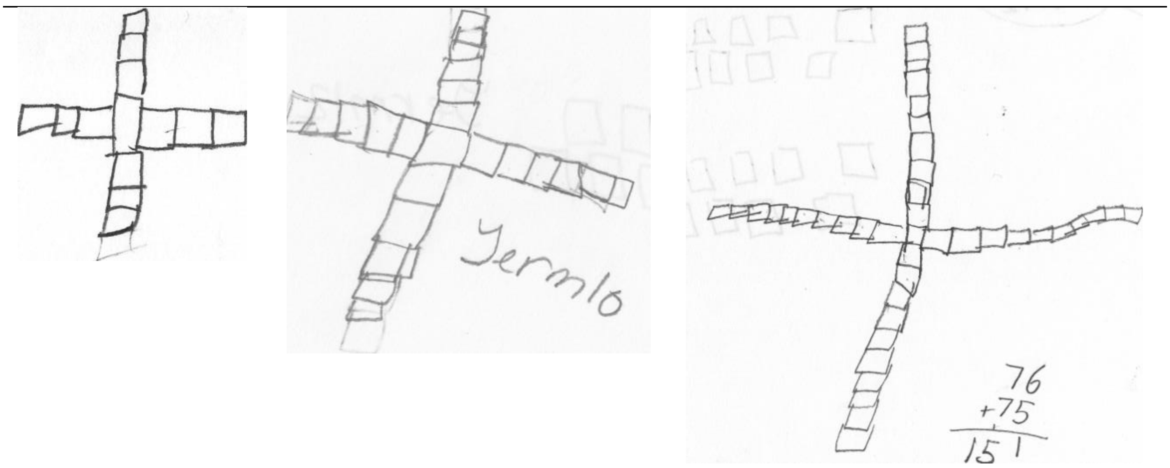
Orla I just done them 6 on the bottom and 6 on the top, and that's fine.

Ideally, in this situation, Orla would examine the contrast between her thinking and that of her peers, in a way which could facilitate a cognitive restructuring leading to improved understanding (Mercer, 1995). Unfortunately, she doesn't demonstrate any evidence of doing so, but she does progress onto more constructive thinking when working with Pattern 2. It may be relevant to this discussion to note that English is an additional language for Orla, and not the language spoken in her home. This may have had implications for her engagement with the discussions, and also her disinclination to seek to establish intersubjectivity with her peers.

Orla seemed more comfortable and assertive when working on Pattern 2, in comparison to Pattern 1. She asked in quite a determined manner "can I join in?" when she felt that the other children were not including her in the discussion. Her drawings of terms 6 and 10, and her workings in constructing term 75 are presented in Figure 5.9.

Figure 5.9. Orla's workings on Pattern 2

Term 6 *Term 10* *Constructing Term 75*



During the group discussions about Pattern 2, Orla seems quite disengaged from her peers, as she looks towards the camera. She spends some time constructing the early terms of the pattern, by placing tiles over the diagrams on the task sheet presented. She also seems to refer to a construction on the table when drawing the 6^{th} term. In extending and constructing a near generalisation, Orla seems to ignore the central tile. She stated that she used “5 in each row”, and when I asked her how many tiles in total she drew, she replied “20: 5, 10, 15, 20”. This could be seen to be a valid response for this pattern, as the pattern was presented to the children as a diagram drawn with black lines on white paper, and there were no spaces between the tiles (Figure 5.3 above). The central tile may just be the space between the strips of tiles, or may be an additional tile. Most groups of children believed there to be a central tile, but some children believed there to be a space in the centre, and Orla may have understood this to be the case. It is interesting again that she did not engage with the thinking of her peers, as Wyatt described the 10th term as “I think it would have 11 going down slanted and then 11 in the opposite way because you add on one more”. The group discussed the number of tiles needed for this tenth term and beginning with 22, they adjusted their thinking to incorporate the overlapping of the central tile, resulting in a total of 21 tiles. Orla did not take part in this conversation, and when I asked her whether she agreed with the others’ thinking, she replied “OK”. In constructing the near generalisation, Orla again worked alone, and did not contribute to the conversation. She drew a diagram of the 20th term, and later, after her peer’s discussed how they would calculate the number of tiles she wrote 76+75. Orla seems to have grasped the structure of this pattern, and if she had been part of a group where others saw the pattern in the same way she did, I feel that she could have moved on to construct a far generalisation. The similarities between the approaches taken by the others in her group, and her disinclination to present her thinking to her peers, caused her to be isolated, and possibly to feel less confident in her thinking. This is regrettable both in terms of Orla’s thinking, and also for the others in the group whose thinking could have been broadened to consider an alternative perspective on this pattern.

Grace's thinking

Grace commenced her work with the patterns very confidently. Having engaged in a lively, but unfocused, discussion with her peers on Pattern 1, she stated "and like number 86 is kind of obvious cos all the time when you do it, whichever number it is at the top it will just be one more than it, and at the bottom". When I asked the group to consider a general term, independent of a term number, the conversation focused on whether terms were odd or even, and Grace did not offer an appropriate response to the question. In working through the tasks on Pattern 2, Grace seemed to start well, but showed some signs of uncertainty. For example, she correctly explains how to calculate the number of tiles required for the 6th term, and used a recursive approach to calculate the number of tiles required for the 10th term, as follows:

> Because like em you see the 6 that had em 3 on all each sides and then one diamond in the middle. So, em two threes would be six and then two sixes would be twelve and then eh twelve add one that would be like 13, and then the 10, it would be, like each time whenever you add something like 7, that would be just 2 more, so it would be eh 15, and then 8 that would be 17, and then the 9 that would be the 19, so the 10 has to be the 21.

From this point there followed a conversation where many uncertain and hazy comments are made by all the children in this group, for example, Grace suggested "Well it could be, it could kinda be like 20 something because like ehm, that one's 3, that one's 5, that one's 7, that one's 9, it's very near to the teens though". In this statement, Grace's use of the hedges 'kinda' and 'like' attach vagueness to her response (Rowland, 1995). During the exchange Grace and her peers also used many hedges, referring to "20 something" or "in the teens". Rowland suggests that such hedges may possibly shield children "against possible error in the cognitive basis" for their suggestions. Grace did refer to aspects of the pattern such as "well, what's staying the same, that's the middle one because it stays" and "Because ehm, you could just find eh, the nearest number to, eh, but like eh, you could choose the doubles that equal 74, and then you just add one more." While her speech indicates possible uncertainty, her thinking was valid, and she seemed to be on the cusp of constructing a valid far generalisation, particularly as during her initial descriptions she identified the differences between terms with odd and even term numbers. Grace did not progress from this point, however, and it is difficult to see whether the group discussion encouraged her to consider different approaches which didn't support her thinking, or whether connecting together all her observations, and quantifying the number of tiles for the 75th term posed too great a challenge for her at the time of the interview.

When working on Pattern 3, Grace again extended to the 3rd and 7th terms without much difficulty. She referred to figural aspects of the pattern in her initial description, and referred to the number of posts when talking about the 3rd and 7th terms. When I asked the group to consider the 9th term as a near generalisation, Fiona suggested 24 posts would be needed, and Ciaran suggested 28 posts. Grace remained silent at this time, and looked straight ahead, as if she was puzzled, or concentrating on figuring it out. Ciaran and Fiona argued about the validity of their answers, but were unfortunately interrupted by Daniel, who wanted to return to counting the number of posts in the sixth term. When the group progressed on to considering the 56th term, Grace suggested "59, […], Because like everytime you add 1 on, you add… you're just adding 3 on". Independently of the group discussion she then multiplied 54 by 4, which she justified by saying "Yeah, each panel it has at least 4 sticks. So ehm if you multiply it by 54, then that could be the answer". Grace may have adopted a Whole-Object approach without final adjustment to this pattern by drawing upon her observation of one panel, rather than looking at each fence as a collection of panels. Equally as mentioned above, this comment of Grace's was made at the end of an hour long session of engagement with novel tasks in an unfamiliar setting, and Grace's suggestion may reflect fatigue.

Summary

In this section I have compared the three patterns in terms of the children's engagement with them, and the relative ease or difficulty each presented. I have compared the progress of all 16 children on each pattern, and I have looked at groups of children whose progress was more or less dependent on the pattern at hand. While overall Pattern 2 seemed to present more challenge than Patterns 1 and 3, I have presented examples of children who made good progress with Pattern 2 and also children for whom Patterns 1 and 3 were challenging. Some aspects of Pattern 2 which may have contributed to the varying levels of challenge were the asymmetric nature, along with the requirement to halve 75. For some children issues of fatigue may have impacted on their engagement with Pattern 3, and in this section I drew attention to missed opportunities for Orla to contribute to, and learn from the ideas of her peers. In the next section of this chapter, I explore, with greater detail, the role of peer interactions in the children's engagement with the patterns.

Interactions Among Peers

As discussed in Chapter 1 and further developed in Chapter 2, an understanding of mathematics learning as situated and distributed underpins the research outlined in this thesis. Also, on the basis of my initial pilot, and supported by the findings of other researchers, I felt that it was not possible to support children in engaging with novel material at the highest cognitive ranges of their ZPD in an individual interview with a stranger researcher. Within the research field of algebraic thinking, Rivera and Becker 2011 attest that

> While we acknowledge the constructivist nature of pattern generalization among individual students (every individual sees what s/he finds meaningful to see that influences how and what s/he constructs), collective action - that is, shared ways of seeing - makes the above characterisations even more meaningful than when performed in isolation (p. 329).

This, therefore, presented an imperative that diagnostic interviews would be conducted with groups of children. In this section I discuss the implication for some children of engaging with new and challenging tasks in tandem with their peers, by drawing on examples of children who benefitted from peer interaction in varying ways and to differing degrees. I analyse Daniel's thinking, which may not have been supported by the inputs of other participants in his group, the thinking of Emily who seemed to learn from the inputs of Wyatt in her group, and progress to apply her new learning in subsequent tasks, and the thinking of Alex who may have missed an opportunity to build upon the thinking of his peers by acceding to the thinking of Arina. In order for peer interactions to best support collaborative sense-making, it is supportive if children engage in exploratory talk, where they feel comfortable disagreeing with each other, and calling upon each other to justify their thinking (Mercer and Littleton, 2007). As outlined in Chapter 2 Nic Mhuirí (2014) and Kavanagh et al. (2015) draw attention to low levels of group discussion during mathematics lessons in Irish primary school classrooms. It is probable that the children engaged in this research may not have much experience of group work where exploratory talk was facilitated and encouraged. Boaler (2002) states that the skills inherent in group work, and the construction of an exploratory talk scenario requires a skill level that must be honed over time. She also discusses the disadvantage experienced by children from working class backgrounds, or some ethnic groups, due to the norms of discussion which would be prevalent in their home, and differ from the disputational approach required to support a robust mathematical discussion.

Daniel's thinking

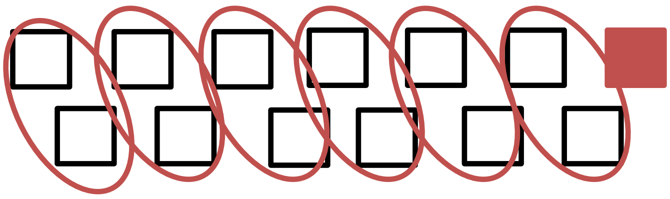
Daniel as mentioned earlier in this chapter, scored a 0 on each of the 3 patterns presented. In this section I will discuss possible reasons why this may have occurred, other than difficulties he may have experienced in processing aspects of the patterning activities. Daniel presented as a very confident, articulate child, and spoke more than anyone else in his group, uttering 226 comments in comparison to 205 from Ciaran, 125 from Grace and 46 from Fiona. However, Daniel seemed to be easily distracted, both by non-mathematical topics, and also by the thinking of children around him, as evidenced by the following comments made during his interview.

Daniel	I'm more of a chatterbox… I don't really like being a chatterbox… Because if people call me a chatterbox, that makes me feel like I chat every single time in school and I like only have one dojo point.
Daniel	Is it still recording?
Daniel	That was so close, look how close that is. This red was mixed up with the green? Oh maybe that was supposed to be the dark one.
Daniel	Now it feels like I'm a different person, like, when you're calling me Daniel, I feel like I'm a different person… I feel like I'm not popular, I feel like I'm not even in this school
Daniel	And you, nobody actually was, we're not famous on tv. Once I was on tv, but that was like, when I was 3

Not only was Daniel distracted by ideas which occurred to him during the interview and were unrelated to the mathematics, he also seemed to be distracted by the mathematical thinking of other children. While other children's thinking which is relevant to solving a task may often support and broaden a child's thinking, Daniel did not progress along his own train of thought, nor did he experience the 'cognitive restructuring' which could support him in deepening his understanding as a result of engaging with the ideas of other children (Mercer, 1995).

One example of this occurred during his attempt to extend the first pattern. He was the first child in his group to offer an observation on the pattern, noticing that each subsequent term had 2 more tiles than the previous term, and he should therefore have had little difficulty in extending. However, Ciaran, who sat next to him, was very animated about his observation that all the terms were "even", an observation which he quickly recanted to observe just as exuberantly that they were all "odd". Daniel agreed with both observations. Wells and Arauz (2006) explain that "for dialogue to proceed satisfactorily, participants have to make a persistent attempt to understand each other's perspectives —to achieve a state of intersubjectivity," (p. 382) and it is possible that Daniel's urge to achieve intersubjectivity with Ciaran caused confusion as he aimed to incorporate the relevance of Ciaran's observations with his own. Daniel did not seem to understand at the beginning of this task that the pattern would continue beyond the terms presented to the children, as he asserted "there is no 6th one", and later "ummm, 12th term, there is no 12th term". When Daniel commenced construction of the 5th term, he mentioned 3, and it is difficult to tell whether he intended to collaboratively construct the 5th term with Ciaran, or whether he felt he needed to construct the 3rd term, incorrectly considering that to be a term where the quantity of tiles was 6. Ciaran corrected Daniel, and advised him to make the 6th term saying "you have six of them going diagonally, and then you need to put a red one in the corner which is the weird one". Ciaran was describing a possible decomposition of the 6th term as presented in Figure 5.10, which seems to incorporate explicit thinking.

Figure 5.10. A diagram based upon Ciaran's description of the 6th term of Pattern 1

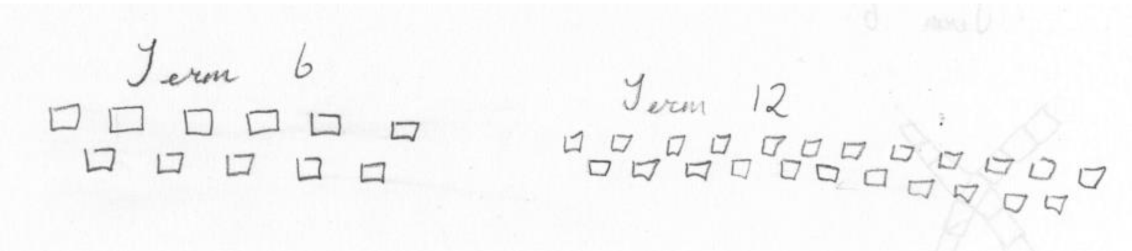


Daniel's observations up to this point were recursive, and again he did not succeed in solving the conflict between his thinking and that of Ciaran's in a way that supported his understanding (Mercer, 1995). It is difficult to be certain whether Daniel's thinking was constrained by the conflict caused by Ciaran's thinking, or whether his approach was hindered by his misunderstanding about the continuation of the pattern beyond the terms presented to the children.

Emily's thinking

Emily was an interesting participant in her group. In working with Patterns 2 and 3, she appeared confident and competent. She was attaining at a high level in mathematics in her class, and in her most recently completed Standardised Test had achieved a score which placed her on a percentile ranking of 99. Her initial descriptions of Pattern 1 referred to recursive and figural aspects, for example "It's not going up in one, that's a three and that's a five, so the top goes up in one and the bottom goes up in one." Emily extended correctly and then commenced to construct the 12th term in the pattern, which was the near generalisation requested in this case. In her first attempt to construct the 12th term, Emily doubled the quantity of tiles on the top row of the 6th term, and deducted one to calculate the number of tiles on the bottom row, articulating clearly her thinking: "I think it could be since this one is 6, you double the top and that's 12, it could be 12 on the top row, then since it's 5 on the bottom, it could be 11 on the bottom. 12 on the top and 11 on the bottom." In this way she seemed to be using a Whole Object approach with final adjustment, as described by Barbosa (2011). Although she was treating the six tiles on the top row as a whole object, she did not double the bottom row in the same way, but adhered to her observation of the bottom row as containing one fewer tile than the top. Emily's drawing of terms 6 and 12 is presented in Figure 5.11.

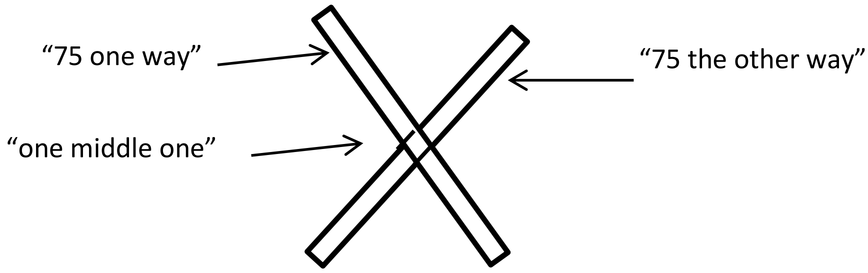
Figure 5.11. Emily's drawing of Terms 6 and 12 of Pattern 1



Emily seemed quite confident in how she justified her decision to describe the pattern as a top row of 12, and bottom row of 11, until Wyatt, a fellow participant in her group offered an equally confident assertion of a shape with 13 tiles on the top row and 12 on the bottom, without justification. While Emily defended her positon, she wavered in responding to the request to describe the 86th term. When the group began to discuss the 86th term she remained silent. Wyatt confidently stated that "It's going to have 87 on top and 86 on the bottom" because "on term 4 it's number 4 and it has 5 on the top". Emily responded with "I think it could be the, em, it could be the number, like, for all of them the number that you're on, sort of like, 86 on the top and 85 on the bottom because since ehm, since here the term for…" where her reticence may indicate a lack of certainty, or that she was working to reconcile her previous construction of the 12th term, and the seemingly logical thinking of Wyatt. Rowland (1995) discusses the use of "I think" as a 'Plausibility Shield' and suggests that its use may indicate doubt that the associated statement would stand up to scrutiny. As facilitator I deliberately did not affirm or reject any of the children's suggestions during this exchange. Shortly after this comment Emily suggested "the bottom number is the proper number and the top is just one more than it." By the 'proper number' in this instance I understand her to mean the term number, indicating that she has adapted her thinking to incorporate the explicit approach demonstrated by Wyatt. Later in the conversation Emily seemed to construct an element of the general term for this pattern when she suggested "because the bottom number is equal to the term [number]", but she proceeded afterwards to again double the 6th term in constructing the 12th. Referring to the research discussed in Chapters 2 and 3 on developmental levels in algebraic thinking, it is possible that Emily is in transition between levels, and that her thinking is tending to waver, and was not robust when she sought to construct a general pattern for this term (Burgher and Shaughnessy, 1986).

In seeking to construct the 75th term of Pattern 2, Emily described the term as "I'd have thirty… like, yeah, 75 one way and 75 the other way but you would only, then you'd have 76… in the middle, you would have another one in the middle because you're only using one middle one." From her description, and her pointing to the x-shape I understood her statement to mean that she was separating the x-shape into two perpendicular strips which crossed at the centre, as demonstrated in Figure 5.12. She thus described the 75th term correctly, and scored a 3 for this pattern.

Figure 5.12. Emily's description of the 75th term for Pattern 2

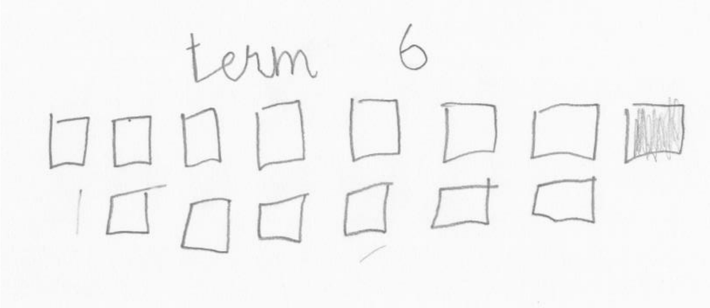


Similarly, when working with Pattern 3 Emily successfully constructed a pseudo-general term when she described the 100th term as "if you have 100, say if you had a hundred, you wanted to make a hundred panelled fence, you'd have a hundred across, a hundred at the bottom. Add those 2 together but you'd have 101 going vertical". Pattern 1 was the only pattern where Emily did not adopt an explicit approach, and thereby succeed in constructing a general term. It is possible that Emily's discussions with her group in relation to Pattern 1, supported her in considering how an explicit approach could be used in order to construct far terms, and a general term. She then applied her new understanding successfully when thinking about Patterns 2 and 3.

Alex's thinking

Alex struggled to extend Pattern 1, and drew a sixth term which contained one superfluous tile in the top row, as shown in Figure 5.13.

Figure 5.13. Alex's 6th term for Pattern 1



He seemed to be unclear about the rate of change, and when I asked him to tell me why he had drawn what he had, he suggested that "Because I thought because it was a higher and new term it should have higher and new numbers". As he thought more about the term, and engaged in the discussion with his group, Alex appeared to begin to draw connections between the term number and the number of tiles. When asked to construct the 86th term, he firstly suggested "So term 86 yeah would be 86 at the top and 85 at the bottom, but the 86th square on the top would be the shaded one", before adjusting after discussion with Arina to conclude that the term would have "86 plus 87 then".

In considering Pattern 2, Alex quickly noticed that each subsequent term contained two additional tiles. From here, he seemed to progress very quickly to connect the number of tiles in Term 5 and the position number in the pattern, as he stated "Wait, I know, term five would have eleven. If the fifth term [inaudible], […], If you add two 5s you add half 2 on to that. That would make 11." Alex correctly constructed the fifth and sixth terms from tiles, and seemed quite confident in voicing his opinions about the number of tiles required, for example "it would have to be 13 [tiles]". When I asked the group to consider the 10th term, Alex appeared to begin counting in twos from thirteen, but was interrupted by a very confident "21" from Arina. When Arina was justifying her answer Alex continued to count, and when I asked for alternative ideas to Arina's he suggested "I think there might be 23, [..], I agree with Arina, [..], But I'm not sure so I agree with Arina". This may have been an unfortunate interjection from Arina, in terms of Alex's construction of understanding in relation to Pattern 2. As he did not have the opportunity to complete his construction of the 10th by counting on in two's from the 6^{th} term, he missed an opportunity to interrogate his approach for accuracy. Ideally, if Alex had questioned Arina at this point, and if they had compared their approaches to each other, the exchange may have contributed to both of their understanding of the structure of this pattern.

It seems unclear to me during this exchange whether Alex is seeking to figure out how the term would be constructed, as he firstly repeats Jay's idea and secondly states that he agrees with Arina. This may be an example of 'cumulative talk' wherein the children are coming to common understandings, but the ideas are not justified (Mercer and Littleton, 2007), and Alex, for example, does not seem to critique Arina's thinking, but accede to it. Later in the transcript, Alex did demonstrate an independent perspective on the pattern that had not been expressed by the others when he used recursive thinking to describe the 10^{th} term.

Alex had no difficulty in constructing the third term for Pattern 3. He stated that "To make the 3 panel fence you would have to take away 3 posts from the 4 panel" with certainty, and also correctly added six onto the number of posts in the seventh term to quantify the number of posts in the ninth term. He struggled however to progress beyond this point, and made no real attempt to construct a far generalisation, other than to suggest "I'm trying to do, like, how much would I need to add if I'm trying to make 57" and "I think maybe multiply or divide". Alex presented as a very pleasant, placid character, and seemed very often to accede to the opinions of others, in particular Arina. While he was both more confident, and more successful with Pattern 1, I would suggest that he may have been capable of greater progress if he had worked through his own thinking rather than agreeing so often with Arina. That said, he did not move beyond a recursive approach in thinking about Pattern 3, even though he had considered the construction of terms in Pattern 2 in relation to their position in the pattern.

Conclusion

Van Manen (1990) attests that hermeneutical phenomenology is both description of a phenomenon on the one hand, and "description of meaning *of the expressions* of lived experience" on the other (p. 25, Van Manen's emphasis). In this preliminary analysis chapter I have set out to present the varying levels of success children achieved across the three patterns. I have outlined the multiple approaches children adopted, drawn attention to the relative challenge of patterns for the cohort as a whole, and zoned in on particular groups of children and individual children, in order to unpick the meaning of their expressions as they discussed the patterns, and their attempts to construct terms.

Many aspects of the patterns and of the children's environment acted upon the progress they made in the tasks presented to them. In relation to the patterns, I have described and analysed when the structure of each pattern, or the specific term requested in the questions I asked the children impacted on children's constructions. In addition, I have explored a variety of situations where peer interactions played a role in children's explorations of the patterns presented. I have described the thinking of Emily when her peers supported her in considering alternative approaches, and also Daniel and Alex's situations where peer interactions may have hindered them from pursuing their efforts to make sense of the pattern structure. What aspects of the patterns children attended to seemed to play a central role in the children's interactions with each pattern. In this chapter there are the examples of Grace and Emily who attempted to apply a whole object approach to the construction of terms, and the examples of Christopher and Alex who struggled to construct terms as they focused solely on the relationships between consecutive terms. In Chapters 6 and 7, I explore in more depth the consequences for children's thinking of the relationships they perceived within patterns, and the aspects of the patterns they attended to.

CHAPTER 6: CHILDREN'S EXPLORATION OF RELATIONSHIPS WITHIN THE PATTERN STRUCTURE

Introduction

> Children naturally detect relationships between details, and they shift their attention from specific elements to relationships between elements, and from relationships between specific elements to relationships as properties that objects may or may not have (Mason, 2008, p. 61).

In seeking to investigate the strategies children adopt in their construction of shape patterns, the relationships that children detect between details are worthy of close attention. In Chapter 2, I presented a review of the literature concerning the relationships children detect within the structure of patterns, and in this chapter I will draw upon such literature in order to present an analysis of the relationships detected by children during this study.

One relevant framework is that presented by Lannin, Barker and Townsend (2006), which I included in the section of Chapter 2 entitled 'How a term may be situated within the structure of a pattern'. Within the framework of Lannin, Barker and Townsend a 'recursive' strategy involves an examination of the mathematical relationship between consecutive terms in a sequence, and if using an 'explicit' strategy, a child identifies a rule for the relationship between a term and its position in the pattern. While children may tend to adopt recursive approaches more intuitively when first encountering patterns, it is pertinent to not view explicit thinking as of higher order (Lannin, 2004; Watson, et al., 2013). Both approaches are appropriate in certain situations with certain patterns, and children's thinking is best supported by familiarity with both. Along with purely explicit and recursive approaches, children may be observed adopting ways of thinking about general terms for patterns, which include aspects of either explicit or recursive thinking, such as a whole-object approach. 'Whole-object' describes an approach wherein a child uses a given term as an object, and multiplies the quantity of elements in constituent parts by some number in order to generate subsequent terms. Barbosa (2011) observed that when using multiples of a term in order to construct subsequent terms, some children made a final adjustment based on either numerical or figural properties of the pattern. In Table 2.1 I presented the findings of research conducted by Barbosa (2011) when she built upon previous work by Healy and Hoyles (1999), Stacey (1989), Swafford and Langrall (2000), Lannin (2001) and Lannin, Barker and Townsend (2006).

Remaining cognisant at all times that my observations are an approximation of children's underlying thinking, in this chapter I present the pattern term constructions of the children who participated in my research project, focusing on the relationships they identified within the structure of the pattern. As I am adopting a phenomenological approach, I will analyse not only how children thought, but also what may have supported or inhibited their thinking.

To begin I present an overview of the children's work, identifying the strategies children used to construct general terms for the three patterns presented. The overview is followed by an exploration of when children came to use explicit thinking, a whole-object approach, recursive thinking, and counting. Very few children reasoned explicitly from the outset, but many did apply explicit thinking when seeking to construct general terms, and I include an analysis of how some children came to reason explicitly. In so doing, I aim to analyse elements of the interview which may have supported them in broadening their approach from recursive or whole-object thinking. In presenting an analysis of the whole-object approach adopted by children in this study, I will draw attention to Emily's justification of her thinking, and the difficulty she experienced in leaving her flawed approach behind even after she described an explicit solution to the pattern. I will present examples of children who used recursive thinking and counting in their solution of the patterns, and the challenges presented by their chosen approaches as they sought to describe general terms.

Overview of Children's Approaches

As described in the Research Methodology, the children were presented with each pattern in groups of four, and strongly encouraged to discuss their ideas with each other. The children had concrete materials available to them for the construction of terms, and also blank paper and pencils for sketchings, jottings, or any rough work they felt inclined to undertake.

As outlined in the Coding section of the Research Methods Chapter, I deemed a comment to indicate an explicit approach if it identified, described, queried or drew upon a relationship between terms and their position in the pattern. Comments which spoke of relationships between consecutive terms were deemed to be indicative of a recursive approach. Table 6.1 shows the number of comments coded as recursive or explicit broken down by pattern. For reference the three patterns are presented in Appendix R.

Table 6.1. Comments made by children, coded as either recursive or explicit

Total	*Counting*	*Explicit*	*Final Adjustment*	*Recursive*	*Whole-Object*
Pattern 1	18	64	2	28	4
Pattern 2	15	77	0	63	1
Pattern 3	16	39	1	14	4
	49	180	3	105	9

It is important to emphasise while presenting these figures that they are an approximation of the proportions of time and focus given by children to the four approaches. Many factors impacted upon my production of these figures which could alter the true proportion of focus given by the children, particularly with such a small sample size. For example, it was not possible for me to hear every comment uttered, some children may have repeated themselves at different times of interviews and I may have included this as two comments, and also all coding depended upon my interpretation of the children's comments. These figures must be seen as the best information available at the moment, but not an exact representation of the situation.

Taking the numbers of coded comments as indicative of a pattern, it may be said that children expressed more ideas of an explicit nature, than of any other, when describing, extending and generalising from the patterns presented to them (180 comments in total). It may be considered surprising that an explicit approach seemed to dominate the groups' thinking about the patterns presented to them, considering previous research in the area, and also the approach to patterning within the PSMC, as discussed in Chapter 2, (Lannin, 2004; Government of Ireland, 1999). Twelve of the sixteen children demonstrated explicit thinking at some stage during their engagement with the patterns, while one participant, Danny, remained resolute in seeking to only identify and apply relationships between consecutive terms. Daniel experienced difficulty in progressing beyond description of the terms presented to the children. In seeking to extend Pattern 1, he indicated that he did not perceive any terms in the pattern beyond those presented on the page: "there is no 6th one", and later "umm, 12th term, there is no 12th term". When working with Pattern 2, he failed to conserve the external shape of the terms. Frobisher and Threlfall (1999) highlight the possibility of children not realising that patterns may extend beyond that presented on the page, and Daniel seemed to encounter that challenge to his thinking about the pattern presented.

Also it is pertinent to highlight that the high figure for counting relates predominantly to children's comments during the initial description and extension elements of their engagement with the patterns. As identified by Barbosa (2011), counting is a natural and appropriate approach to take to one's exploration of the structure of a pattern. Later in this chapter I will draw attention to the thinking of children who relied on counting as an approach to the construction of generalisations.

In Appendix T, I include a breakdown of comments coded for each child, accompanied by my field notes from the interviews, and from the transcription process. The coded comments, and observations, indicate that few of the children used a single approach consistently across all patterns. In the next section I will explore how those children who did reason explicitly came to do so, and whether their thinking was supported by my prompts and questions, peer group interactions, the use of concrete materials, or aspects of the patterns presented.

Strategies Involving an Explicit Approach

As may be seen from the overview of children's strategies presented above, many children demonstrated explicit thinking in their construction of near and far terms for the patterns. As there seem to be indications from research that recursive thinking is likely to be the intuitive approach of many children, I found the inclination of children to demonstrate explicit thinking to be of particular interest. Adopting the hermeneutic phenomenological stance outlined in my Research Methodology, I analysed contributing factors to children's inclination to think explicitly. In this section I outline the impact on the children's thinking of a) my contributions as facilitator; b) the concrete materials; and c) the children's interactions with each other within the interview groups.

Facilitator prompts and questions

During two of the interviews, I felt that children's discussions indicated that their perspective was focusing on the relationship between consecutive terms, or how the pattern was growing, to the exclusion of any other aspect of the structure. I considered it necessary, therefore, to encourage the children to consider a relationship between terms and their position in the patterns. For example during their discussion of Pattern 2, Group 1 was making slow progress, and tending to use very vague language. I felt at that point that a prompt would be appropriate in focusing their attention, and supporting them in engaging with the task. I asked three times whether the children could see any connection between the term number and the quantity of tiles on the legs of the x-shape. The children at this time seemed to be struggling to perceive the rates of change of constituent parts of the pattern terms, and my prompts were not supportive of their thinking, as can be seen from the following transcript:

AT	Can you see any connection between the term number and the length of the legs? Or between the term number and the term? If we look at term 1, the number is 1 look at the term, term 2 the no is 2, look at the term, term 3, term 4. Ok Grace can you explain to us where… why you decided 21?
Grace	Because like, em, you see the 6 that had em 3 on all each sides and then one diamond in the middle. So, em two threes would be six, and then two sixes would be twelve, and then eh twelve add one that would be like 13, and then the 10, it would be… Like each time whenever you add something like 7, that would be just 2 more, so it would be eh 15, and then 8 that would be 17, and then the 9 that would be the 19, so the 10 has to be the 21
Daniel	Oh yeah
AT	Ok, now em, I'm going to ask again for you to look back and see can you see any connection between the term and the term no.
Daniel	That, they're longer

Daniel's response to my question may indicate a persistent focus on the relationship between consecutive terms, but Grace's response was more complex in nature. She began by considering the 3 tiles on each side of the 6th term, whereby she combined four groups of three in order to find the total quantity of tiles on the legs, to which she added the one central tile for the total number of tiles. From this 6th term however, she seemed to progress recursively in order to find a total number of tiles for the 10th term. The children's recursive approach did not seem at this point to be supporting them in thinking about bigger terms, but neither did my prompts to consider an explicit approach. Most of the children continued to struggle with this pattern, with only Grace succeeding in constructing the near generalisation outlined above. Later in the conversation I also drew the children's attention to figural aspects of the patterning terms, as I felt that their focus was limited to numerical aspects. After receiving this prompt, Ciaran proceeded to construct a far generalisation. A more in-depth analysis of the perspectives children took regarding their attention to numerical and figural aspects is presented in Chapter 7, Children's Observations of Figural and Numerical Pattern Structure.

Group 2's discussion of Pattern 2 also seemed to be dominated by recursive reasoning, to the exclusion of any other perspective. I prompted the children by asking whether anyone could see a connection between terms and their corresponding term numbers in Pattern 1, but none of the children replied to my question and they continued to discuss the terms recursively.

Aside from prompting the children, each of the worksheets on which the patterns were presented included a final question "Can you see a connection between the term number and the term?" When designing the interview schedule I was aware that it was possible that this question at the end of children's discussions about Pattern 1 could have prompted some children to broaden their thinking to include an explicit approach to Patterns 2 and 3, but I felt that it was improbable that they would exclude recursive reasoning as a result. I deemed the question to be appropriate therefore as a means of exploring the children's observations of the structure of the pattern. As will be seen from the analysis in this chapter, not all children adopted an explicit approach in seeking to solve Pattern 2 or Pattern 3, and many did not respond to this question when it was presented to them. As an example, in interviewing the second group in the first school (School 1, Group 2, or S1.G2, hereafter), I pressed them to consider whether there was a connection between a term number and the number of elements within the associated term, and our discussion is presented in the following transcript:

Arina [Reading] So, can you describe a connection between the term number and each term?

Arina It's just… the connection… term one, sorry.

Alex Well I can't see any connection, I can't see any connection between the term but I think I can see a connection between the numbers

Arina Yeah, the numbers, term one.

Alex Oh yeah, like all of them, when term one is going on to term two it skips, it skips four and then it just goes on to this.

AT What do you think, Jay? Do you see a connection between the term and the term number?

Jay Erm… no.

AT Cherry?

Cherry I think… because like term one is like, there's like one on the bottom.

[A child visits the interview room, and interrupts the interview with a message about leaving to visit a junior class to teach them maths games.]

AT Now Cherry, you were saying what connection you saw between the term and the term number?

Cherry Like the term number has one and then the bottom one is one.

AT Great and is that true for every term?

Cherry Yeah.

In this excerpt from the children's discussion, Arina and Jay proffer no opinion on the question, and Alex demonstrates recursive thinking in describing the sequence of the total quantities of tiles in consecutive terms. Cherry does make a link between the term number and the quantity of elements in one constituent part of the terms, the bottom row. It is surprising that none of the other children built upon this observation, as from my viewpoint as facilitator I expected it to be enlightening for others. Unfortunately, I feel that the message delivered by the child who visited the interview room drew children's attention away from the mathematics, as they were obviously looking forward to their visit to the junior class. Also, Cherry did not indicate an inclination to think explicitly when she was presented with Patterns 2 and 3. In this case, the suggestive questions I asked seemed to fall on deaf ears, and the children seemed unmotivated to broaden or alter their perspectives to accommodate prompts from me. My observation of this phenomenon encouraged me to explore what other factors may support children in broadening their perspectives, and in the next section I present an exploration of the role played by concrete manipulatives as the children engaged with the patterns.

Concrete manipulatives and the physical construction of terms

Six of the sixteen children could be said to have spontaneously explored connections between terms and their positions, before observing their peers making this connection. For example, in response to Pattern 1, when asked to construct the 12th term, Ciaran and Grace (who were both participants in Group 1) demonstrated spontaneous explicit thinking independently of each other, in saying:

Grace	it's obviously going to have 13, because term 5 had 6 squares on the top and 5 on the bottom, so term 12, so it's gonna be really easy
Ciaran	how many tiles are needed for term 12? 1, 2, 3, 4…..12; 12 multiplied 2 is 24, plus one is 25

Analysing her verbal comments and her physical construction of term 12, as presented in Figure 6.1, I deduced that Grace had identified each term as consisting of a top row and a bottom row, containing t+1, and t tiles, respectively. From Ciaran's construction and comments, I ascertained that he had identified terms as containing t diagonal pairs, with an additional tile on the top right corner, as presented in Figure 6.2.

Figure 6.1. Grace's interpretation of Pattern 1, presented as two rows of tiles, where the bottom row contains n tiles, and the top row contains n+1 tiles

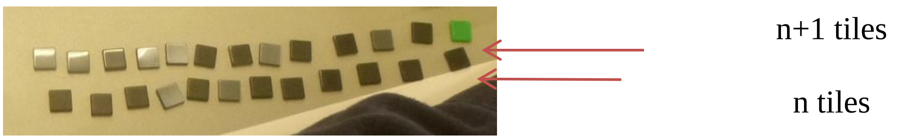
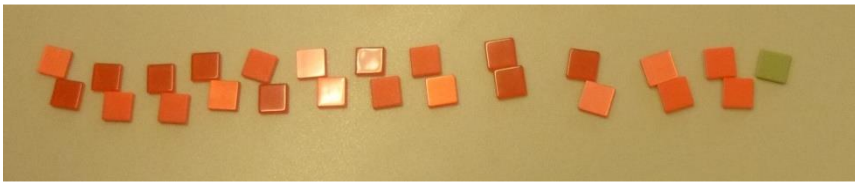


Figure 6.2. Ciaran's interpretation of Pattern 1, presented as n pairs of tiles presented diagonally, with one additional shaded tile



Grace and Ciaran made these comments following their physical construction of pattern terms using tiles. Preceding their construction of the terms, both children had tended strongly towards recursive thinking where comments such as "each time you're adding 2" were typical along with references to the total number of tiles as always being an odd number. A focus on 'numerical' aspects of the terms also seemed to dominate their group's interactions before construction of the terms, but following their use of the manipulatives, both children succeeded in drawing on 'figural' aspects in constructing general terms (Rivera and Becker, 2011). I thought it pertinent therefore to analyse in more depth the pathway followed by Grace and Ciaran in their thinking about this pattern.

Warren and Cooper (2008) found that the use of manipulatives combined with number card identifiers for terms, supported children in shifting their focus from relationships between subsequent terms, to relationships between terms and their position. In exploring the strategies Ciaran and Grace adopted, I sought to consider whether the use of manipulatives played a role in broadening the children's focus in this way. I examined, therefore, excerpts of the children's conversation preceding and during the period of the interview when their perspectives appeared to shift. The following transcript is the initial reaction of the children when they are presented with four terms of Pattern 1 and asked to describe what they see.

Daniel	[reads question] Oh look, it's going up in twos.
Grace	I already said that.

Ciaran	Yeah, but you see, [points], 3.
Daniel	3, and then 2, and then another 2.
Ciaran	Yeah, but look, each time it gets bigger.

Within this transcript, Grace's first comment of "I already said that", referred to a previous comment which was inaudible, and immediately preceded Daniel's reading of the question. While Grace and Daniel immediately refer to the difference between subsequent terms, Ciaran seems to prefer to construct his thinking independently, as he says "but you see" and points to the three tiles comprising the 1st term in the pattern. Having counted the tiles in term 1, he then observes that each subsequent term is "bigger" without specifying the quantity of tiles. Interpreting this comment, I would suggest that Ciaran was not listening closely to his peers at this point, as he seems to be trying to convince them that the terms are growing in size, when both Daniel and Grace have both already acknowledged this aspect. In Table 6.2 I present the children's discussion following on from the transcript above, and I focus on Ciaran's comments, as he refines his thinking, and begins to identify aspects of the structure of the pattern, such as the difference between consecutive terms.

Table 6.2. Discussion between Grace, Ciaran and Daniel where the progress of Ciaran's thinking is isolated for examination

	Comment made	Description
Daniel	1, 2, 3, then, 1, 2, 3, 4, 5, that's going up	
Ciaran	[Counts, pointing with finger.]	
Ciaran	They're all even	Observes that the total quantity of tiles in each term is an even number.
Fiona and Grace	[talking inaudibly]	
Daniel	Yeah, they're always going up in twos.	
Ciaran	Yeah, but, Daniel, look, can't half a 3, can't half a 5, can't half 7, can't half 9. And then they're all odd.	An observation that the quantity of tiles within every term is an odd number.

Daniel	They're all odd, and these are evens	
Grace	They're all odd, 3s are odd, 5s are odd, 7s are odd, so they're all odd	
Ciaran	No these are odd. See look can't half a 3, can't half a 5, can't half a 7, can't half a 9, can you?	An observation that the quantity of tiles within every term is an odd number.
Daniel	No, and also 2s, get it? 3 and then another 2, and then another 2	
Ciaran	Yeah so, they're all going up, each time there's more	A reference to "more" rather than "bigger"
Daniel	No 1 they're all odd and no 2 there's adding 2	
Ciaran	Yeah each time you're adding 2, but you're not adding that, the darker one, but they're all even, I mean they're all odd.	A recursive description of the pattern, drawing a connection with the sequence of odd numbers.

Ciaran's initial three comments during this discussion focused on the total number of tiles for terms in the pattern, and he attributed the property of odd or even to the totals rather than considering relationships between terms, or between terms and their positions. Daniel, in contrast, focused strongly on the rate of growth between terms in the pattern, repeatedly referring to "going up", and 2 as the rate of change between consecutive terms. The boys faced each other during this exchange, and the girls held a parallel discussion.

After Daniel's comment "3 and then another 2, and then another 2", Ciaran's attention seemed to shift, whereby he used the term "more", as opposed to "bigger" which he had used in the earlier extract, in reference to the difference between consecutive terms. This may indicate that Ciaran is beginning to refine his sense of the quantity of tiles which is added to construct each new term. In his comments from this point, he began to incorporate a comparison of consecutive terms for similarities and differences "each time you're adding 2, […] they're all odd". Tracking Daniel's thinking in a similar way, we can see that his initial observation was of the difference between successive terms, and he progressed from here to describe both the odd-numbered total of tiles in each term, and the difference between terms. The boys seemed to draw each other's attention to two salient details, the total number of tiles for each term, and the rate of growth between terms. In the following paragraphs, I analyse subsequent extracts from the children's discussions, which allow access to their thinking about Pattern 1. During their conversations, the children's observations of the structure of the pattern develop to include both recursive and explicit approaches.

It is pertinent to mention that within this group, an overly zealous consideration of whether total numbers of elements were odd or even at times dominated the children's discussions, of this pattern and subsequent patterns. Frobisher and Threlfall (1999) suggest that children may notice aspects such as that the total number of elements of a figure is odd or even, rather than the aspects we expect them to notice such as the rate of change, or relative rates of change. Aside from the odd or even aspect, the children seemed to be focusing strongly on recursive and numerical aspects of the pattern, whereby the term number had not yet been mentioned, and also little reference was made to the size or shape of terms, other than when Ciaran observed that "each time it gets bigger". Shortly after this exchange, and before construction of the terms using manipulatives, Grace suggested that "the number on the bottom, that eh, the number on the bottom, eh, each one on the bottom is less, and each one on the top is more". While this comment does not indicate any relationship between the number of tiles in terms and their position in the pattern, Grace did separate the two rows of the terms in a figural manner in order to investigate the structure of the pattern.

After the children commenced physical construction of the 5th term of this pattern, the following exchange took place:

Ciaran	Will I help you on 6 cos I'm done 5 now?
Daniel	I'm done 6.

Ciaran	No, you have to make it like this. You have to go diagonally see like this.
Daniel	Like a pattern.
Grace	It goes like that, it goes like that, it doesn't go straight.
Ciaran	Yeah, you have six of them going diagonally, and then you need to put a red one in the corner which is the weird one.
Daniel	Yeah.

It would seem that Daniel erred in the construction of the 5th term, and that Ciaran tried to explain the structure of the term to him by describing 6 diagonal pairs, with an additional top tile, as portrayed in Figure 5.10.

During this section of the interview the girls spoke to each other, at times almost inaudibly, and the boys continued to work together. Ciaran again referred to the tiles "kinda going diagonally", and Daniel added "oh yeah, because that one, that's sticking out, that's sticking out, so, and they're not like all in a line". When I asked the boys which terms they had constructed, Daniel identified his construction as the 6th in the pattern, but Ciaran corrected him, saying "No, I'll do 6, look, you need 6 of these going diagonally look, look, you need like this but do the 3 more, you see" and Grace joined in, describing her construction of the 6th term as "you know you do 7 squares, and 6 at the bottom".

Both Grace and Ciaran at this point are describing the terms explicitly – they do not refer to the relationship with previous terms, and the quantities of tiles they describe seem to be related to the position number of the terms. I next asked the children to describe the 12th term, as a near generalisation of the pattern. At this point Grace stated "it's obviously going to have 13, because term 5 had 6 squares on the top and 5 on the bottom, so term 12, so it's gonna be really easy", as presented earlier in this chapter. Ciaran did not offer such a confident description at the start, but he did construct the 12th term perfectly, during which the following exchange occurred with Daniel:

Ciaran	1, 2, 3, 4, 5, 6, 7, 8, 9, we need 4
Daniel	We need 3 more
Ciaran	No, do you remember we're making 12

Ciaran required 13 tiles for the top row of the term, indicating possibly that his thinking had broadened from the sets of pairs of tiles to a view of the terms as consisting of two rows, where the top row of a term contains n+1 tiles.

The questions underpinning this research are what strategies children employ in constructing general terms for shape patterns, and what environmental factors mediate their progress. Ciaran and Grace are referring to particular terms in this situation, and have not constructed abstract generalisations, but they appear to have applied an understanding of the commonality of this pattern, as they worked with "particular instances of the variable" (Radford, 2011). While not possible to isolate any contributing factor to the children's thinking, there is a change in how Ciaran and Grace think about the pattern after constructing the terms with the concrete manipulatives. It is plausible that the physical act of construction supported them in isolating elements of the pattern, and considering the relationship between quantities of elements in constituent parts of their constructed terms, and the relevant position number in the pattern.

Group interactions

In Chapter 5 I explored the role played by group interactions in supporting children when seeking to solve the patterning problems presented. In this section I discuss specifically the impact of verbal interactions between children on the relationships children explored within the structure of the patterns.

In the second interview group, Lily Rose seemed to begin by reasoning recursively when describing and extending all the patterns. For example she stated "I just added one more each on the top so all of them have three on them now", in explaining her drawing of the 6th term of Pattern 2. For reference, please see Pattern 2 in Appendix R. Following their construction of the 5th term, and drawing of the 6th, I asked Lily Rose's group to consider the 10th term in this pattern. Lily Rose's answers focused strongly on figural elements of the terms, and were quite vague, as in "it would be about that long" and "I think it would be this long and this wide". Her use of the 'hedges' "about" and "I think" may indicate some ambiguity in her thinking, and a desire on her part to present her ideas as proposals rather than conclusions (Rowland, 2007). Such ambiguity, or unwillingness to present an idea as fully formed, may be expected in this context, given the novelty of the tasks, but may also indicate that she has engaged with the alternative thinking of her group members, and is beginning to question, or rethink, her approach. During the group discussion the children's opinions varied, and I asked them to discuss the differences between their ideas. The children discussed the distribution of tiles on the legs of the x-shape, and the presence of "some middle square". During this discussion Lily Rose suggested that the 10th term would have five tiles on each leg, but neglected the central tile, and it was not clear whether she was using explicit thinking, or building from the 6th term to the 10th, as she followed "because half of 5 is 10 and then you give one each every time", with "it would have 5 going up and 5 going down. I think you might have 5 going across and 5 going left and right".

After further discussion about the 10th term, I asked the group to construct a far (75th) term for this pattern. At this point, Lily Rose demonstrated explicit thinking, as she sought without hesitation to halve 75, and adhered to the asymmetric nature by stating "you will have to give one side, one extra once. Like down at the bottom two, they might have one extra than the top two". Again Lily Rose used the plausibility shield "might" in presenting her construction, and while she was quick to suggest halving 75 as a strategy for identifying the number of tiles required for each leg of the x-shape, her language indicated that she remained tentative about the structure of the general term.

Discussion of this pattern with her peers, along with attention to the tasks presented, seem to have supported Lily Rose in fine-tuning her thinking from her early vague statements, to a very specific explicit strategy for describing the 75th term, with attention to the quantities involved and their positioning.

Strategies Involving a Whole-Object Approach

Lannin, Barker and Townsend (2006) identified a whole-object approach as follows:

> A term is used as a portion to identify a larger term by using multiples, e.g. the 10th term is calculated as twice the 5th term. The student may not compensate for the resulting term being inaccurate, when applicable.

Barbosa (2011) expanded upon this definition to differentiate between instances when children make no adjustment to their whole-object construction of terms, and when they make numeric or visual adjustments, as presented in Table 6.3.

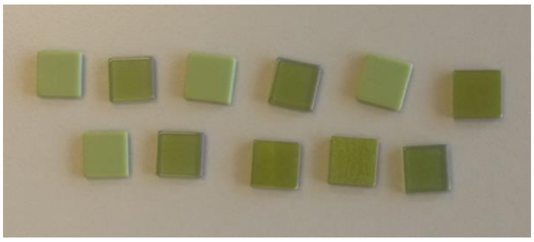
Table 6.3. Varying Whole-Object approaches as observed by Barbosa (2011)

No adjustment	Considering a term of the sequence as unit and using multiples of that unit.
Numeric adjustment	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on numeric properties.
Visual adjustment	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on the context of the problem.

In this section I will discuss the thinking of Emily, as an example of a child who applied whole-object thinking in her attempts to construct general terms for patterns. This is similar to the approach taken by Ciaran to Pattern 3. In this example Emily did not succeed in describing a correct far generalisation through the use of a whole-object approach, but it would be possible to do so if she had made an appropriate final adjustment. The approaches discussed in this thesis from the frameworks presented by Lannin et al. (2006), and Barbosa (2011) are not considered to be either appropriate of flawed in a particular context, but the focus is on whether the approach children chose provided them with the tools to solve the pattern successfully.

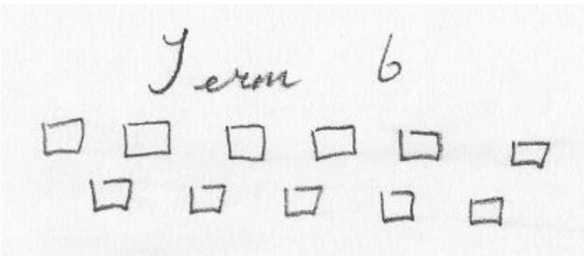
During the initial description of the first pattern, Emily remained quiet, making only one comment, when she sought to correct an error in Wyatt's thinking by saying "It's not going up in one, that's a three and that's a five, so the top goes up in one and the bottom goes up in one". From this comment it could be assumed that she was thinking recursively about the terms, and also incorporating figural aspects, as she separated the top and bottom rows of the Beams shape. Emily constructed the 5th term correctly from tiles, with 6 tiles in the top row, and 5 in the bottom row, as shown in Figure 6.3.

Figure 6.3. Emily's construction of the 5th term of Pattern 2



Emily proceeded however to draw the 6th term incorrectly, when she included 6 tiles in the top row, and 5 tiles in the bottom row, as shown in Figure 6.4.

Figure 6.4. Emily's drawing of the 6th term of Pattern 2



It is difficult to determine why Emily made this slip, as she spoke very little during construction and drawing of these terms. From the video footage, Emily appeared very relaxed and quite self-assured. She was the first to finish construction of the 5th term, and she sat back in her chair smiling at the other children. She pointed out to Orla that she was missing tiles in her construction, by gesturing to the worksheet and to her

Orla's 5th term. Emily was also the first child to complete her drawing of the 6th term, and again sat back in her chair smiling around at the other children. When all children had completed their drawing of the 6th term, I asked them to discuss with each other what the 12th term might look like. In response to this request, the following exchange took place:

AT	Talk to each other about what would the 12th term look like? You don't need to make it or draw it or anything just yet, but talk to each other about what it might look like.
Luigi	It would have 2 there
Emily	It would look like...
Wyatt	The 12th would have 13 on the top…
Luigi	Yeah, yeah
Wyatt	… and 12 on the bottom
Emily	Double 6
Luigi	14, it's be cos that's … [counts the tiles on the top row in his drawing of the 6th term]
Emily	You double 6, it would be 12 on the top, and then …
Luigi	And then it's 13, so the next one's 13
Wyatt	When it was term 4 there was 5 on the top, so 13
Luigi	Yeah, so it's 14

Wyatt took a strong explicit stance, and justified why he believed there would be 13 tiles on the top row of the 12th term, by referring back to the structure of the presented terms. Luigi seemed to waver between thinking that the top row should contain 14 tiles, or 13 tiles. Following this exchange I asked the children to justify their verbal constructions to each other. Luigi expanded upon his thinking, and it became clear that he believed that the 12th term would consist of 13 tiles in total, so he seemed to be experiencing some confusion, and may have been attempting to incorporate the thinking of his peers too much, rather than constructing a personal insight into the structure of the pattern terms.

In seeking to justify her position, Emily suggested "12 on the top, I think it could be since this one is 6, you double the top and that's 12, it could be 12 on the top row, then since it's 5 on the bottom, it could be 11 on the bottom. 12 on the top and 11 on the bottom." Working from her incorrect drawing of the 6th term, Emily applied a whole-object approach in doubling the top row of the 6th term to construct the 12th term. She however made a figural adjustment to her answer to produce a bottom row that was one less than the top row, rather than doubling the bottom row as well. I asked the children to construct the 12th term from tiles, encouraging them to discuss their thinking with each other and to tease out why they might be different. Emily justified her 12th term firstly as

> Mine's the same cos I took the 6th term and then I kept adding two, so I'd one on the bottom and one on the top, and that was term 7, and I kept doing that til it reaches 6, and I got the number that I said it was gonna be.

And shortly thereafter she used the following justification:

> Because 6 plus 12 equals, 6 plus 6 equals twelve, so I doubled the term 6, ehm, 12, so I doubled the top, 12 on the bottom, then put the one 13.

Unfortunately, due to her error in her drawing of the 6th term, her recursive approach did yield the same result as her whole-object approach, and an opportunity for her to challenge her own thinking was missed. Emily seemed comfortable with a situation where children's constructions differed from each other's. While assertive in justifying her own position, she did not seem to press other children on theirs, or seek to convince anybody that hers was the "correct" way, through arguing with anyone else's proposals. I would suggest that Emily may have assumed that there was a flexibility in this question which allowed for more than one correct solution.

When the children began to discuss the structure of the 86th term, Wyatt stated that the top row of the term would contain 87 tiles, and the bottom row 86 tiles. I prompted him to explain why he thought this, and he again referred back to the original terms presented to the group. Emily followed his comments by saying "I think it could be the ehm it could be the number, like, for all of them the number that you're on, sort of like, 86 on the top and 85 on the bottom because since ehm, since here the term for…". She spoke very slowly, and on the video footage she can be seen referring firstly to the terms she has drawn, and then to the original terms presented. When she examined the original terms, her voice trailed off, and she seemed unsure about what she should say. Ten seconds later, she stated "The bottom number is the proper number and the top is just one more than it". Emily repeated this assertion that each term consists of two rows wherein the top row contains n+1 tiles and the bottom row contains n tiles from this point forward, and makes no other reference to a whole-object approach at this time. However, following the group's discussion of the 86th term, the conversation returned to the 12th term, at which point Emily restated her opinion that the 12th term would contain 12 tiles on top "because it is double the 6th". While her construction was based upon a faulty 6th term, what is most interesting here was her justification, where she identified her construction as based upon, or utilising, a multiple of a previous term.

Strategies Involving a Recursive Approach

While much of the thinking exhibited by the children connected terms to their positions in the pattern, recursive reasoning was also present, particularly in constructing terms for Pattern 2. As mentioned at the beginning of the previous section, I did not consider a recursive approach to be in all cases inferior to other approaches, but I sought to examine whether recursive thinking supported children in successfully constructing near and far terms for the patterns presented. It is pertinent to highlight that none of the children involved in this research successfully constructed the far term of a pattern using recursive thinking, but the sample size is not large, and given the age of the children, it is possible that many of them did not possess the fluency in the combination of multiplication and addition required to solve patterns recursively. For example, in order to solve Pattern 2 recursively for the 75th term, it is necessary to consider multiples of the additional two tiles, and to add them to the quantity of tiles of an initial term, as demonstrated in Figure 6.5.

Figure 6.5. One possible recursive solution for Pattern 2, and the arithmetical computations required

> Largest term given: Term 4: a total of 9 tiles distributed as four legs of two tiles, and one central tile.
>
> Required: Term 75
>
> In order to construct term 75 from term 4 recursively: one must add 71 (75-4) multiples of 2 tiles to the 9 tiles of term 4, giving 151 tiles. In order to describe the shape of the figure, consideration must be given to symmetry between the left and right sides of the figure, and two possible options for the distribution between the top and bottom of the figure:
>
> > Option 1. An equal number of tiles is placed on the top as on the bottom;
> >
> > Option 2. The bottom has two more tiles than the top.
>
> Subtracting 1 from the numerical total of 151 to allow for the central tile, 150 tiles must be distributed across the four legs of the 75th term. 150 must be divided by 4, giving 37 tiles with a remainder of 2. As 150 is not a multiple by 4, option 2 must be employed, giving 37 tiles on each of the top legs of the x-shape, and 38 tiles on each of the bottom legs.

This computation would be marginally less complex, for children who viewed the x-shape as consisting of two strips of n, and n+1 tiles, or as two symmetrical sides containing n tiles with at most one additional tile below the central diamond (see Chapter 4 for an overview of the possible perspectives on the components of terms of this pattern). Nonetheless, there is a necessity to juggle multiplicative, and additive aspects of the structure of this pattern, along with the complex figural structure in order to solve for the 75th term. Many researchers have highlighted the cognitive demand placed on children's working memory when they seek to solve mathematics problems, and there are suggestions that cognitive overload, particularly in working memory, may present challenges for children, that are at times unsurmountable (examples are Kieran et al., 2016; and Lee & Fong, 2009). The demands on working memory, coupled with successes experienced in the use of explicit approaches may explain, at least in part, why recursive thinking was not more evident in children's discussions of far terms for the patterns.

One example of a child who did use recursive thinking was Christopher, as he sought to solve Pattern 2. Throughout his work on Pattern 2 Christopher seemed to struggle to broaden his thinking beyond the comparison of consecutive terms. He did not identify a unit, or use guess and check, but referred persistently to the "extra" 2 tiles, or to the terms growing, without specifying exact dimensions of near or far terms. All his comments when describing and extending the pattern related to "adding" and "extra" tiles, and at no stage did he mention the term number when aiming to construct a term. Christopher did not succeed in generalising from this pattern, and confident assertions which were present during his discussion of Pattern 1 and Pattern 3 such as "this is easy" or "I know what [this term] is" were absent from his contribution to the discussion relating to Pattern 2.

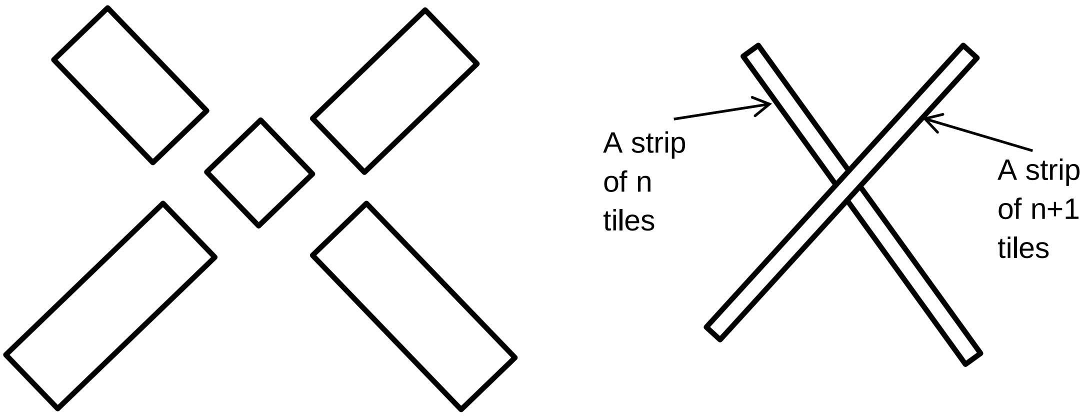
Lannin, Barker and Townsend (2006) discussed how children involved in their research were more likely to use recursive thinking when "the task provided a clear connection to incremental change" (p. 12). In their initial discussion when presented with Pattern 2, Christopher's group focused on the rate of change between terms, and Christopher described the pattern's growth as:

Christopher	It's like going up, so first it goes that, and then the two blocks is… and term two it goes there, and then for term three you add another one, the same again on the other four posts, on this side, and then I think that, that, term 5, that it's going to add onto the right side.

As highlighted in Chapter 4, the structure of this pattern is such that the rate of change remains numerically constant, as two tiles are added to each consecutive term. Figurally however, the tiles are added to the top or bottom of the figure on alternate terms. While all three patterns were presented as linear sequences of four consecutive terms, the structure of this pattern may have obscured the relationship between terms and their position more than was the case with the other patterns, and encouraged Christopher to focus on the ‘incremental change’ between terms.

Within Christopher’s group, Lily Rose and Jane both solved this pattern successfully by adopting an explicit approach, but Christopher did not seem to benefit from the discussions of his group about this pattern. A contributing factor to why Christopher did not build upon the thinking of his peers may have been the varying perspectives on this pattern adopted by the children in his group. Lily Rose and Jane described the terms as consisting of a central diamond with tiles on each leg, as Jane described the 10^{th} term as “I think, I think there would be 5 on each leg” and later included “some middle square”. Christopher worked closely with Danny during the extension of Pattern 2, and Danny identified two overlapping strips of n and n+1 tiles, as he described his construction of the 5^{th} term as “I put 5 across then I put 6 up”. Figure 6.6 presents depictions of the children’s descriptions of this pattern.

Figure 6.6. Depictions of Pattern 2, from the descriptions of Lily Rose, Jane and Danny

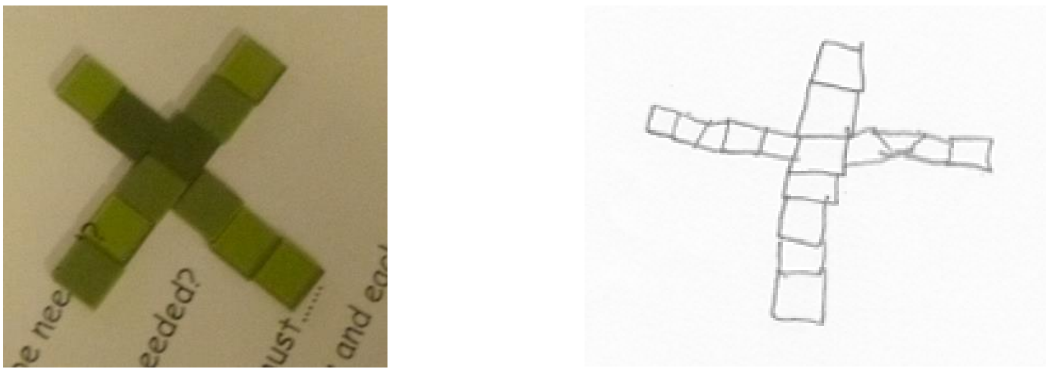


Lily Rose and Grace's description

Danny's description

In addition to their verbal comments, Danny and Christopher's construction of the 6th term, and Christopher's drawing of the 10th term, as presented in Figure 6.7, support this depiction of their thinking, as both demonstrate a separation of the strips rather than legs extending from a central tile, or any alternative structure.

Figure 6.7. Christopher and Danny's construction of term 5 for Pattern 2, along with Christopher's drawing of term 10



Christopher and Danny's construction of the 5th term of Pattern 2.

Christopher's drawing of the 10th term of Pattern 2.

Alternatively, Christopher may have reverted to recursive thinking when he encountered this pattern, due to the level of challenge he experienced, while with other patterns he applied connections between the terms and their position numbers to confidently tackle the patterning tasks. As discussed in Chapter 4, Pattern 2 involved a challenging requirement to generalize differently for odd and even numbered terms. While it may not be helpful to consider recursive and explicit thinking as hierarchical, it is very likely that Christopher was more familiar with recursive thinking, and that he was less secure in using an explicit approach.

Strategies Involving Counting

During their initial investigations of each pattern, many children counted the elements of constituent parts, as I mentioned from the overview section of this chapter. This 'counting' may be seen as natural and in many ways necessary. In contrast, the following discussion focuses on the use of counting as a strategy for the construction of far terms, when alternative strategies would have been more efficient. Cherry is an example of a child who used counting in the construction of a far term, and Luigi could also be seen to use counting in his exploration of near and far terms. When first presented with the pattern, Cherry correctly constructed both a 3-panel, and a 7-panel fence. From the video footage of her interview Cherry may be clearly seen counting the posts needed to construct the 7-panel fence, in response to my question of how many posts were needed. In retrospect, it may have been supportive of the children's thinking if I had prompted them to consider whether there was a way to quantify the number of posts without counting them individually, but I did not do so at the time.

When asked to describe the 9-panel fence, Cherry may again be seen counting posts on her construction, and she delightedly raises her arm with an answer before anyone else in her group. When asked how she calculated her total of 28, Cherry said "I got 22 in my head, and I don't know why I got this kind of number in my head and then I kept adding the sticks". Before asking the children to describe the 9th term, we had discussed the 7th term. My understanding is that Cherry either counted on 6 posts from the 22 posts, by using her construction of the 7 panel fence which was on the desk before her, or constructed a physical representation of the 9-panel fence by "adding the sticks" to this 7-panel fence. This understanding is based upon her verbal comments, and her counting gesture (using her construction of the 7-panel fence) which is evident from the video footage. Her comment mentioned above may indicate that she counted on from 22, and a later comment seemed to indicate that she constructed the 9-panel fence, as she said "I got nine of them ones in there and then I got some kind of, I think ten, or something" while pointing to her construction on the desk in front of her.

It is not possible to hear the numbers that Cherry is counting on the video footage, and she could have counted up in nine groups of three and added on an additional fence, or in nine pairs of horizontal posts, and added on 10 vertical posts. She did not however refer to groups of two or three, or to the quantity of posts within panels at any stage, however, which led me to conclude that she was counting rather than using a recursive or explicit strategy. The following is a transcript of the section of the interview when I asked Cherry about her calculations:

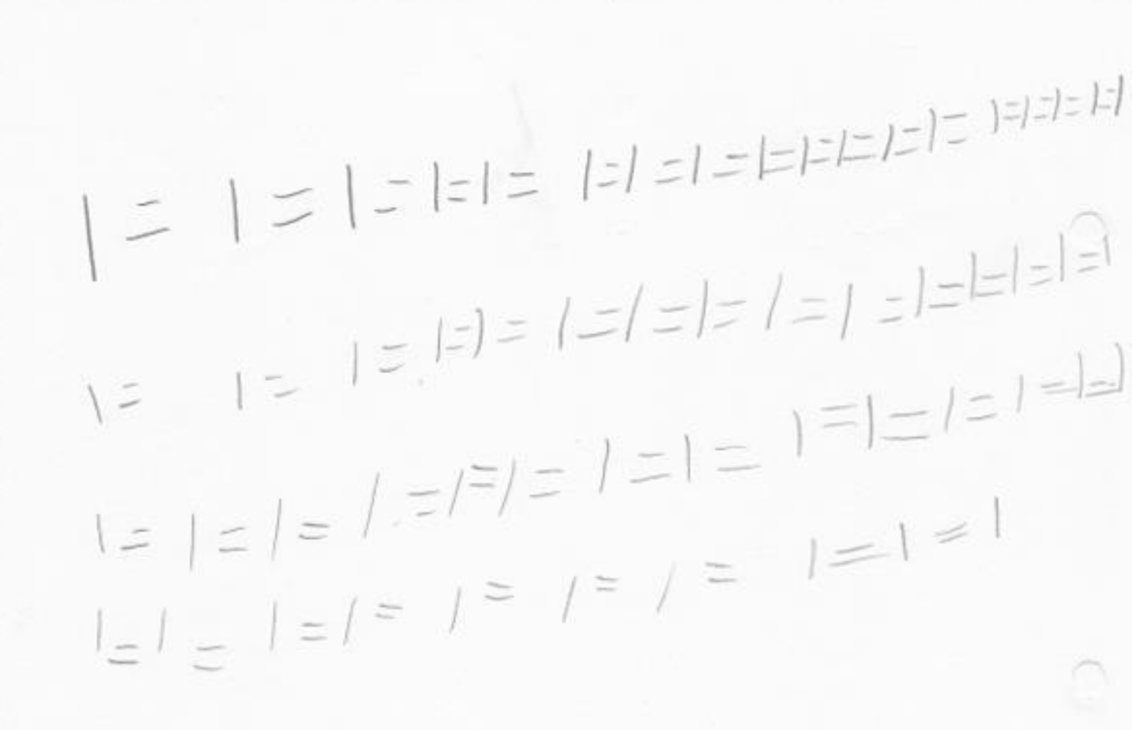
Cherry	I got, like, ehm, 22 in my head, and then I like, I really forget really, ehm, I got like this kind of number in my head and then I kept adding the sticks.
AT	OK, and how many sticks did you add for each panel, can you remember?
Cherry	[4 seconds silence] I don't know.
AT	You had said to me earlier that for… there was three sticks for each extra panel and four sticks for the one at the end.
Cherry	Oh yeah.
AT	So what did you do?
Cherry	I got nine of them ones in there and then I got some kind of, I think ten, or something, [points to her construction of the desk]
AT	OK and you had 22 posts here [points to Cherry's construction on the desk] and to make the next two panels how many more did you add on?
Cherry	6

Cherry	6, why 6?
Cherry	22 and 6 is 20 [2 secs pause]
AT	8, yeah, but why did you add on 6?
	[8 sec pause, Cherry does not respond]
AT	Can anyone else see why she might have added on 6?

As can be seen from the transcript, when I asked Cherry how she calculated 28 as the total number of posts for the 9-panel fence, she could not answer me. I prompted her by asking “how many sticks did you add for each panel?” and she responded “I don’t know”. I prompted her further by asking “you had 22 posts here and to make the next two panels how many more did you add on?”, to which she responded “6”. When I asked her why she added 6, she could not give me an answer, which seemed to support my supposition that she had counted the posts, either counting on from the 7-panel fence, or possibly by constructing the additional two panels, and counting all posts.

Cherry continued to use counting as a strategy when she was asked to quantify the number of posts required for far terms, and was motivated to draw a 56 panel fence in order to count the posts for this far term. Cherry’s drawing of the 56th term is presented as Figure 6.8.

Figure 6.8. Cherry's drawing of the 56 panel fence



While counting is, in this situation, a very inefficient strategy, Cherry is expanding upon the terms presented, and as such has observed the structure of the pattern. It is not possible to be certain that Cherry applied a commonality within the structure of this pattern, however, as she did not give any indication in her verbal responses that she could quantify elements of constituent parts of any term, beyond the initial terms presented. As mentioned above, it was unfortunate that I did not suggest to Cherry that an alternative strategy may have been more efficient, as I had a sense that her motivation to count all elements of the 56-panel fence, stemmed at least in part from her success with the 9-panel fence.

Conclusion

In this chapter I explored whether children involved in this research adopted explicit or recursive approaches in seeking to construct general terms for the patterns presented. I gave an overview of the approaches adopted by all children, and I discussed examples of children who used various approaches, be they explicit, recursive, whole-object, or counting.

As discussed in Chapter 2, it is probable that the children who participated in this research had not engaged in specific curricular content aimed at supporting them in adopting explicit approaches to pattern solving. However, some children demonstrated a willingness to use explicit thinking in solving patterning tasks, without suggestion or encouragement to do so. Mason (2008) contends that children demonstrate a facility in thinking algebraically from early childhood, and English (2011) warns that teachers and policy makers should not underestimate children's ability to take on and work with new ways of thinking. English states that children "have access to a range of powerful ideas and processes and can use these effectively to solve many of the mathematical problems they meet in daily life" (p. 491). In the forthcoming redesign of the Irish Primary Mathematics Curriculum it will be important to reflect the broad range of thinking strategies of which children are capable, as evidenced in this research.

Two important elements of the children's engagement with the novel tasks presented were the interactions within the groups, and the use of concrete materials in constructing terms of the patterns. Warren and Cooper (2008) found that concrete representation of patterning terms supported the children in their research in broadening their thinking to include an explicit approach, and I would posit that in my research the children's thinking also benefitted from their interaction with concrete and iconic representations of the pattern terms. Radford (2010) highlighted the "sophisticated rhythmic coordination of gestures, words and symbols" that underpins children's algebraic thinking, and emphasised that such gestures, words and symbols are more than the product of children's thinking, but also the building blocks. Children learn through describing, and discussing the patterns presented, and during my research children constructed understandings of the structure of the patterns from discussions, and also from their own gesturing.

CHAPTER 7: CHILDREN'S OBSERVATIONS OF FIGURAL AND NUMERICAL PATTERN STRUCTURE

Introduction

In seeking to construct general terms for shape patterns, children's opportunities for success improve when they attend to both the spatial and numerical aspects of the structure of the pattern (Radford, 2011). In seeking to extend a pattern children identify the position of pattern elements by attending to the spatial structure, and in order to quantify such elements they must attend to the relevant quantities within the pattern terms presented. In this section I explore the tendency of children participating in my research to attend to the spatial and numerical aspects of pattern structures, as they worked towards the construction of general terms.

As discussed in Chapter 2, Rivera and Becker (2011) identified two approaches utilised by children in seeking to construct general terms for geometric patterns. Some children in Rivera and Becker's longitudinal study focused solely, or primarily, on numerical aspects of the terms provided, and sought to use the numerical aspects observed in order to identify a commonality, extend the pattern and construct a general term. Such children the authors described as adopting a 'numerical' mode of generalising. In comparison, Rivera and Becker considered a child to have adopted a 'figural' approach if he/she used figural aspects of the pattern such as the shape of terms, or the position of elements, both within the term and relative to each other. A figural approach may include attention to numerical elements, but not in a manner that supersedes the child's perception of the spatial aspects of the pattern structure.

In this chapter, I present a summary overview of the approaches adopted by each child to the patterning tasks presented to them, focusing on their observations of the numerical and figural structure of the pattern. Supporting the presented overview, I explicate the decisions I made in coding children's comments, and the distinction I drew between children’s purely numerical approaches and approaches including observations of the figural structure. Following on from the overview, I compare the approaches adopted on all patterns by each group, and by all children on each pattern. In Chapter 5 I explored the role played by children’s interactions with each other, and in this chapter I focus more closely on the role played by the patterns, by comparing the engagement of one group across all three patterns.

In Chapter 6, I explored children’s observations of relationships within patterns which support their understanding of the structure. Such observations work in tandem with their numerical or figural approaches, and in this chapter I refer to children’s explicit, or recursive thinking when relevant.

Summary Overview of Children’s Responses

In this section, I present an overview of the children’s tendencies to articulate observations relating to figural and numerical aspects of patterns. This overview is a distillation of data gathered throughout the research process, including field-notes written during the children’s interviews, and when reviewing the interview footage, complimented by observations gathered during the transcription process, and from the interview transcripts. In preparing for the group interviews I felt it necessary to have a systematic approach to taking field notes. In Chapter 2 I presented a framework of growth points in algebraic thinking, and in Chapter 4, I described how I used this framework to produce an informal assessment spreadsheet which would allow me to record children’s progress on tasks presented to them. Appendix M includes a blank copy of this spreadsheet, plus examples from interviews. While it was challenging to juggle the roles of task facilitator and researcher, I aimed to record my initial response to children’s thinking. During analysis I watched the video footage, or listened to the audio footage, of each group and further developed my initial observations with regard to each child’s response to the patterns. I supported my observations by referring to artefacts from the interviews, such as children's workings on paper, and photographs of their constructions. Table 7.1 contains an overview of each child's response on each pattern. The group code given includes a reference both to the children's school and to their group within the school, for example group 2.3 was the third group interviewed in the second School.

Table 7.1. An overview of the mode of generalisation adopted by each child across all three patterns

	Group	*Numeric/figural mode of generalisation*
Christopher	2.3	Commented on figural aspects from description to generalisation for all patterns. Used figural aspects to isolate variables.
Danny	2.3	Some comments re figural aspects. Struggled to extend all patterns. Seemed to guess numbers.
Jane	2.3	Figural – used both position of diamonds/squares/posts and role of posts to support her construction of generalisations.
Lily Rose	2.3	Figural to the point of ignoring numerical, e.g. "I think it would be about this big [gestures]", but made reasonable estimates.
Ciaran	1.1	On Pattern 1, numerical at start, focused on odd/even numbers, but began to point to diagonal aspects of terms. Referred to top and bottom to calculate numerical answer for near and far. On Pattern 2 referred regularly to legs, top, bottom, worked through initial misconception re rate of change. On Pattern 3 referred to lines going down.
Daniel	1.1	Referred to extra one in line, but no reference to top or bottom. No reference to shape or position
Fiona	1.1	Difficult to determine due to some inaudible content, and a disinclination to voice her opinion. 2nd pattern refers to centre, and top/bottom as distinct
Grace	1.1	Pattern 1: Figural from the start even when others in her group steered conversation towards odd/even. Pattern 2: Referred to symmetric/asymmetric early on, and also centre, top and bottom as distinct,
Alex	1.2	Many numerical comments, but also made some references to 'top' and 'shaded one'. Referred to 'middle one' and 'legs', and

		referred to 'panels' and 'posts'.
Arina	1.2	Figural on all three patterns, for example differentiates middle square on describing Pattern 2, and distributes squares on sides, as presented in Figure 7.5 later in this chapter.
Cherry	1.2	Did not articulate any observations of Pattern 1 or Pattern 2. Pattern 3: differentiated extra post on last panel.
Jay	1.2	Pattern 1: referred to shaded, top and bottom. Referred to shape at start of Pattern 2.
Emily	1.5	Referred to figural aspects throughout.
Luigi	1.5	Referred consistently to figural aspects: top and bottom, 'going across' and horizontal, but also made comments which focused on numerical aspects.
Orla	1.5	Pattern 1: referred to top and bottom, Pattern 2: referred indistinctly to legs of the x-shape, as lines, but seemed to use figural aspects to construct the 10^{th} term. Made no verbal observations relating to Pattern 3.
Wyatt	1.5	Referred to figural aspects throughout.

To further analyse the children's thinking I examined each comment by each child, and coded each one as figural or numerical, using nVivo. I tabulated the quantity of comments coded as numerical or figural for each child, for each pattern, and for each task within each pattern. For clarity, I precede my presentation of the table of coded comments by summarising, and building upon, the account of my coding methodology from Chapter 4, in so much as it pertains to the children's perception of numerical and figural aspects of pattern structures.

Decisions made in coding children's statements

Rivera and Becker (2011) state that adopting a figural approach is to "figurally apprehend and capture invariance in an algebraically useful manner" (p. 356). When coding children's statements, I identified a statement as indicating a figural approach if it included reference to the position of an object within a term, by using words such as 'top', 'bottom' or the deictic 'there' along with an associated gesture (Radford, 2006).

An object in this context referred to a square, a diamond, a tile, a line, or any item which formed a constituent part of a term. Comments were deemed to indicate a numerical approach if no reference was made to the position of objects within a term. The term ‘growing’ was used regularly by children, and required some thought with regards to whether it indicated a figural approach. Typically, when mentioning growth children were referring to a sense of the terms’ shape growing in size, that is “selectively attend[ing] to aspects of sameness and difference among figural stages”, but I could not assume that this was always the case (Rivera & Becker, 2011, p 356). Rather, it was necessary to attend to some term, or deictic within a child’s comment, and to seek to determine the referent, which would indicate whether the child was referring to the shape as growing, or the quantity of constituent elements. Examples of children referring to growth are given in Table 7.2, where I also present my interpretation of whether the child was thinking figurally, or numerically. The term I am associating with the referent in each case is italicised.

Table 7.2. Examples of comments including a reference to 'growing', presenting the associated referent, or what was growing, and my decision whether the child was speaking from a numerical or figural perspective

Child	*Pattern*	*Comment*	*Referent*	*Approach*
Christopher	1	I think that's gonna be 87 on the top cos the bottom one's gonna be 36 because there, at the top is 2 bigger and then it grows and grows and grows and the number on the top always has to be bigger than the number on the bottom	The top row of the figure	Figural
Jane	1	*It's* growing in odd numbers. I think there's 13 at the top and 12 at the bottom and *it's* just like growing in numbers.	The entire shape. While Jane refers to the numbers, she indicates that 'it' the term is growing, rather than the numbers growing.	Figural
Ciaran	2	That's like, it's going up in two's each time, look cos *these* grow then *these* grow then *these* grow so it'd probably be like 16, at the other end	The legs of the x-shape	Figural
Danny	3	Every time *it's* adding six panels, *it* adds the sixth section and then, *it's* growing and *it's* bigger, and ehm, to the top, is 4 and the bottom is 4.	The fence, the entire shape of each term	Figural

When coding comments as numerical or figural, it was difficult on occasion to be certain about which aspect of the pattern a child was referring to in a single comment.

As described in my Research Methodology, in order to make a decision, I referred to previous and subsequent comments by the child. For example in discussing Pattern 3, Jane had two isolated comments where she referred only to the quantity of posts needed to construct the ninth term, in saying “altogether, twenty-seven” and shortly thereafter “twenty-eight”. This focus on numerical properties of terms could indicate a purely numerical approach. The following comments by Jane from the discussion of Pattern 3 suggested that she was in fact thinking figurally,

> The number of the amount of sections and the amount of poles are there but like adding just one.
>
> Ehm, the ones going across are the – there are the amount of the sections as the name says.
>
> And the poles are the same amount of sections and one extra.
>
> It would be ten vertical ones.
>
> Because all the fences, there’s one extra to keep the end together so that they’d be like this...
>
> …to keep it together
>
> So the panels, we double them and then the posts would be the number of the panels and one extra.

Among the comments presented, Jane referred consistently to the role that elements play in the shape of each term, in her use of the terms ‘poles’, and ‘sections’. She also referred to the position of posts as ‘vertical’ and ‘going across’. Therefore, her comments were not deemed to indicate a numerical approach, and no comment by Jane during this discussion was coded as ‘numerical’.

The distinction between a numerical and a figural approach

Rivera and Becker (2011) cite the case of Anna who, constrained by a numerical strategy, applied faulty reasoning to a patterning task, and did not succeed in constructing a valid generalisation. Similarly in my pilot research, Aoife attempted to make the pattern fit her numerical schema, rather than accepting fault in her schema, and aiming to reconstruct using figural aspects of the pattern's terms. These two examples demonstrate distinctly numerical thinking, where figural aspects are ignored, but most of the discussions held by the children in the group interviews of this research incorporated figural aspects of the patterns. There are incidents however, where numerical thinking becomes dominant in discussions. In coding, I aimed to differentiate between comments where both numerical elements are considered along with figural, from distinctly numerical, where there was no reference to the figure. The following are some of the comments of Alex on Pattern 2 which I considered to indicate numerical thinking, with no reference to figural aspects:

> I think there might be 23.
>
> 40.
>
> You could add the smaller… you could add numbers that are smaller than 75 but bigger than the term ten or term 6.
>
> I think it would be, like, 21 or 23.
>
> I've taken… two added on to term nine and four added on to term eight and twelve added on to term seven. And sixteen added on to term six.
>
> Because that's term four and there isn't 4 on that term.

During his group's discussion of Pattern 2, Alex did not refer to figural aspects of any term in the pattern, and he experienced subsequent difficulty in considering what might constitute a general term. In comparison, during the following excerpt from School 1 Group 5, the children incorporated figural aspects of Pattern 2, while referring consistently to the quantity of tiles in each term.

Luigi	I added 1 on the bottom then 1 on the, one on the… that's it, I only added one.
Orla	I did 4 on the bottom.
Luigi	Because there's 7 or 6.
Emily	I did it, I was like…
Luigi	I added two from here.

Emily	I think it has the one centre and then 3 coming out from each side of it.
AT	And why is that, do you think?
Luigi	3
Emily	Because the last one it had 3 coming out of just 2 sides, and 2 coming out of the other 2, but this one if you add 2 on to the one from the top it would have 3 on each side.

In this section of their discussion, the children consistently referred to figural elements of the terms, such as 'top', 'bottom', 'centre', and the deictic 'from here' when Luigi pointed to the bottom of the two legs of the x-shape. Immediately before this excerpt Luigi stated that he had added two tiles to the 5th term to construct the 6th term. In his first comment of the transcription presented above, Luigi stated that he added one tile, and he pointed to the left leg of his x-shape. He then paused as he pointed to the right leg, and finished his sentence by stating that he only added one, seeming to experience difficulty articulating where he had added the second tile, having described the first tile as on the bottom. He recovered however, and gestured to the bottom of both legs when he stated that he "added two from here". Figural aspects of the pattern, and manipulation of the concrete representation, seemed to support Luigi in clarifying his thinking about how this pattern extends beyond the presented terms. Similarly Emily adopted a figural approach when she described the 6th term by distinguishing an element of the pattern which remained constant, from elements which varied between terms.

A comparison of the comments coded as numerical and figural

Having coded the children's comments as figural or numerical when appropriate, I tabulated the quantity of comments within each group to facilitate comparison of the approaches taken by children within groups, and between groups. I also sought to explore whether children were more or less inclined to adopt either approach depending upon the pattern they were engaged with. In Table 7.3 I present a comparison of the quantity of comments coded as numerical or figural for each child, for each pattern. An example of the complete table for School 1 Group 1 is included as Appendix U.

Table 7.3. Comments of the children categorised as indicating figural (Fig) or numerical thinking (Num)

School 1, Group 1										
	Ciaran		Daniel		Fiona		Grace		Group 1.1	
	Fig	Num	Fig	Num	Fig	Num	Fig	Num	Fig	Num
Pattern 1	16	4	3	7	1	0	15	4	35	15
Pattern 2	18	8	18	4	6	0	16	8	58	20
Pattern 3	4	0	1	0	0	0	1	1	5	2
Total	38	12	22	11	7	0	32	13	99	36
School 1, Group 2										
	Alex		Arina		Cherry		Jay		Group 1.2	
	Fig	Num	Fig	Num	Fig	Num	Fig	Num	Fig	Num
Pattern 1	4	2	8	0	2	0	1	0	15	2
Pattern 2	4	8	9	1	0	0	4	1	17	10
Pattern 3	0	3	1	0	2	0	1	1	4	4
Total	8	13	18	1	4	0	6	2	36	16
School 1, Group 5										
	Emily		Luigi		Orla		Wyatt		Group 1.5	
	Fig	Num	Fig	Num	Fig	Num	Fig	Num	Fig	Num
Pattern 1	6	0	5	7	2	0	10	1	23	8
Pattern 2	25	0	13	3	3	0	18	0	59	3
Pattern 3	11	0	6	0	1	0	7	0	25	0
Total	42	0	24	10	6	0	35	1	107	11
School 2, Group 3										
	Christopher		Danny		Jane		Lily Rose		Group 2.3	
	Fig	Num	Fig	Num	Fig	Num	Fig	Num	Fig	Num
Pattern 1	8	0	4	0	5	1	5	0	22	1
Pattern 2	9	1	5	1	10	0	16	0	40	2
Pattern 3	5	0	4	1	5	0	1	1	15	2
Total	22	1	13	2	20	1	22	1	77	5

Looking at the balance between numerical and figural approaches indicated by comments made by the children, there is an overwhelming tendency by children in this cohort to use figural approaches in their thinking about the patterns. Alex in Group 2 is a notable exception to this pattern as he made more comments relating to numerical aspects than to figural, and his thinking is explored later in this chapter. While the proportion of comments deemed to be numerical could not be said to be an accurate representation of the proportion of time or thought children gave to numerical aspects, I felt that the generation of Table 7.3 could support me in analysing the children's strategies. Four children, Fiona, Cherry, Emily and Orla, used figural approaches throughout, without making any exclusively numerical comments. Most children however made comments that I interpreted as indicating a completely numerical approach, and also some comments indicating an inclusion of figural aspects. In the remaining sections of this chapter, I consider variations between children depending on the group they participated with, and variations between patterns.

Intra-group variation

As can be seen from Table 7.3, there is some consistency in places between what could be considered the overall approach of the group, and that of individual members. In Group 3 for example, the overall proportion of comments deemed to be numerical was 6%, and the corresponding proportion for group members varies between 4% and 13%. Similarly in Group 1, the group members referred exclusively to numerical aspects of the patterns in 27% of the comments they uttered, and three members of the group had proportions of 24%, 31% and 33%, which were similar to the overall 27%. Fiona, the fourth member of Group 1, was distinctly different in her approach as she made no comments which I deemed to be exclusively numerical in nature.

Looking at this data, a question emerged regarding how well some children's thinking was supported, or challenged, by the thinking of their group. Children whose thinking might merit consideration in this context are Fiona, Alex, and Luigi. Mueller et al., (2012) assert that many factors support collaboration among children when working on mathematical tasks. Some aspects which support collaboration include diversity of opinion, and also a balance between similarity and difference in background and attainment. As discussed in Chapter 4, these aspects were achieved to varying extents during the group interviews. Other aspects highlighted by Mueller et al. (2012) include "students' psychological perspectives" including the children's beliefs about their roles within their groups, and how they self-identify in terms of mathematics attainment. Fiona, Alex and Luigi all attend a primary school where children are streamed for mathematics lessons, and not streamed for any other subject. At the time of the interviews, Fiona and Luigi were attending the lowest of three streams, and Alex was attending the middle stream. As mentioned above Fiona used numerical thinking far less than her fellow group members, and both Alex and Luigi uttered far more numerical comments than their respective group members (Luigi uttered 29% compared to 0%, 3% and 0% for other members of his group, while Alex uttered 62% compared to 5%, 0% and 25% for other members of his group). In Chapter 5 I discussed how the thinking of children in their groups may have impacted on the strategies adopted by some children in seeking to construct general terms.

Inter-pattern variation

Among the children who participated in their study, Rivera and Becker (2011) felt that some children's inclination to adopt a numerical approach may have been due to the "predictive and methodical nature" of an established pattern-solving strategy in use within their schools (p. 356). While children in the Irish system may not have been introduced to a such an established strategy for the solution of patterning problems, some may seek to draw connections with the number system with which they are very familiar, when dealing with the novelty of shape patterns. In contrast, a figural approach requires a child to attend to figural elements within terms and between terms in order to reason about the pattern. In this section, I discuss whether there is any evidence, from the number of comments children made, to indicate whether the children tended to express more numerical thinking in relation to one pattern than another.

In comparing children's approaches across patterns, there is very little consistency in how each child approached all three patterns. For example, 21% of the comments uttered by Grace in relation to Pattern 1 were deemed to be numerical, whereas the corresponding proportion for Pattern 2 was 33%, and for Pattern 3 100%. Fewer than 10% of the total comments made by each of nine of the children were deemed to indicate a numerical approach. The numbers of comments made are presented in Table 7.3, and it may be seen that of the remaining seven children, some had so few comments on specific patterns deemed either numerical or figural to render comparisons trivial. Ciaran and Danny remained consistent across all three patterns, where most of their comments indicated a figural approach, but some comments referred only to numerical aspects. Grace, Alex and Jay had a higher proportion of numerical comments on Pattern 2, than on Pattern 1, and on Pattern 3 than on Pattern 2. For these children, their tendency to comment on purely numerical aspects of the pattern terms increased as they worked through the three patterns. In contrast, Daniel and Luigi expressed fewer numerical comments as a proportion of their total comments in later patterns than in the first pattern they encountered. Overall, no pattern emerges in the proportions of comments deemed to indicate numerical thinking, and therefore it cannot be said that any one of the three patterns motivated the children to adopt numerical or figural thinking more or less than the other patterns.

Inter-Pattern Variation in the Approaches Adopted

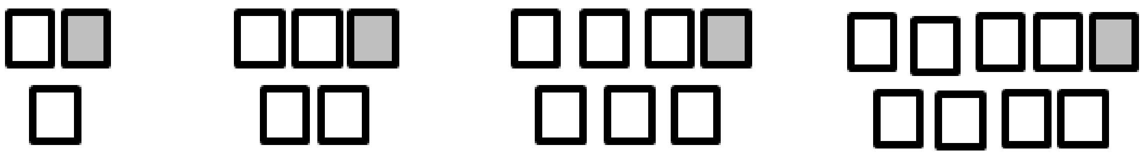
Rivera and Becker (2011) analysed the thinking of children when seeking to construct general terms for shape patterns. They found that in some instances, the children seemed to find the numerical strategies for pattern solution with which they were familiar more "easy to use". In Chapter 2 I drew attention to the analysis by Rivera and Becker (2011) of the significant challenge children experienced in adopting a figural approach to the construction of general terms for shape patterns. Rivera and Becker (2011) highlight the need for children to achieve 'cognitive perceptual distancing' whereby children would firstly "figurally apprehend and capture invariance in an algebraically useful manner" and proceed to "selectively attend to aspects of sameness and differences among figural stages" (p. 356). Rivera and Becker (2011) suggest that when children only attend to numerical aspects, they are grasping the commonality within the structure of the pattern at a superficial level, even when they successfully generalise. In this section I explore whether some selected children identified invariance between terms, and also whether they succeeded in distinguishing elements of terms which differed, or remained the same. I consider whether their thinking incorporated figural or numerical aspects of the patterns, and whether the figural supported them in arriving at solutions to the patterns.

In order to explore the strategies used, I focus in on Group 2, rather than taking a broad overview of all children. I have selected Group 2 for closer inspection, because this group includes Alex, who seemed very numerical in his thinking, and Arina whose thinking appeared to be predominantly figural, based upon my coding of their comments, as presented in Table 7.3. The other group members were Cherry and Jay. Cherry's comments were predominantly figural, but she made fewer comments during discussions than the other children, and she experienced significant challenge in solving the tasks presented. Jay was less consistent than the other children in terms of the approach he adopted. He also seemed to experience some challenge in solving the patterning tasks, and while contributing a lot earlier in the interview, he tended to disengage somewhat as the interview progressed, making 9 comments about Pattern 3 when he had made 34 about Pattern 1.

Pattern 1

Pattern 1 is a beams pattern consisting of two horizontal strips of n+1 tiles as a top row, and n tiles as a bottom row, as presented in Figure 7.1.

Figure 7.1. Pattern 1



In describing Pattern 1, Jay was very quick to attend to the structure of the presented terms, saying “Don’t count the odd one out, 1 four and 1 four, cos 3, 3, odd one out; 2, 2, odd one out”. Jay attended to the invariance of the shaded square, and explored the difference between terms in the pattern. By comparison, the comments made by Alex were quite general, for example “it’s getting bigger, it’s adding two or three terms” and he did not seem to separate what is changing from what is staying the same. During this initial discussion about the pattern, Arina and Jay both isolated the shaded tile at the end of the top row of tiles. Arina commented “The shaded one is probably an added always one at the end”, and Alex responded to this by suggesting “Oh, so we only add one, so… [trails off]”. It is possible here that Alex misinterpreted Arina’s comment to indicate that the total number of tiles increases by one from one term to the next. Alex may have held to this misconception throughout his exploration of this pattern, as when first encountering Pattern 2, he commented “They keep adding two on instead of one this time “. Alex proceeded to count the total number of tiles between all four terms presented, arriving at a total of 24 tiles. He didn’t comment as to why he counted the total number of tiles, but it may suggest that his focus was not on distinguishing “aspects of sameness and difference” between terms.

When constructing the fifth term of Pattern 1, the boys, Alex and Jay, worked separately from the girls, Arina and Cherry. The boys included five tiles in the top row, and four in the bottom, a replica of the fourth term. The girls constructed a correct fifth term. I asked the children to look at each other's, and to discuss whether there were differences. Jay immediately corrected his fifth term. While it is possible that Jay deferred in some way to a belief that the girls would be more likely to be correct than he would, it may also be the case that on seeing the girls' construction he realised that he had forgotten one tile on each row. Jay's initial description of the pattern would suggest that he had a strong grasp of the structure, and should not have had difficulty in constructing the next term. When I asked Jay why he was changing his term, Alex replied "I don't know", and Jay did not answer my question. I next asked the children to draw the 6^{th} term, and Alex drew eight tiles along the top row, rather than seven, as presented in Figure 5.13 in Chapter 5.

When asked why he drew eight, he replied "Because I thought because it was a higher and new term it should have higher and new numbers". There is no indication that Alex was in any way differentiating elements within the structure of the pattern in order to support him in drawing terms, even though by this point he had described, and extended the pattern among a group of children who were consistently referring to numerous figural aspects of terms, such as 'top', 'bottom' 'shaded one'.

When I asked the children to construct a near generalisation of the 12^{th} term, Alex again used very vague language in suggesting "It would have a whole lot of squares". While Alex did not quantify the tiles at this stage, his comment suggests that his focus remained on the quantity, rather than the position, of the tiles in the term. The children seemed to struggle to imagine the twelfth term, and I asked them to look back over the original terms presented, and their constructions. I prompted them to consider whether there might be a relationship between the terms and their position in the pattern, to which Alex replied “I think we added or shaded whenever they move on to a new term... [interruption]… skips a number, like it skipped two or four, like they [...inaudible…] skip three. I think it’s the shaded numbers.” While still focusing strongly on the quantity of tiles within terms, these comments could indicate that Alex was beginning to consider separately the variable and the constant within terms. However Alex’s next comment undermines this assumption as he proceeds to state the following:

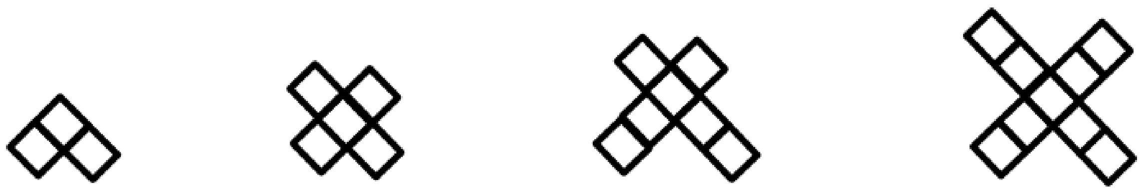
> It might be easier if you leave out the shaded squares. Let’s see how many are here if you leave out the shaded squares. [Jay and Arina discuss other terms] There’s twenty if you leave out the shaded square but if you count the shaded squares with them there’s twenty-four.

This excerpt seemed to indicate that Alex continued to view the four presented terms as a single entity, as he counted all the tiles contained within the four terms, with and without the single shaded tile in each term. After this comment, Alex can be seen on the video footage to remain quiet for two minutes, during which time Arina described the 12th term as “I think it’s going to be 12 blank squares and so it’s 13 squares at the top [inaudible], and the bottom, 12,” and Cherry and Jay both agreed with Arina. I asked Alex whether he thought Arina’s idea made sense, and he replied “I do feel it makes sense like, you’re just adding… just like less than the top one and it must be that the top one only has that and the shaded one is that one more.” This was the first time during the discussion when Alex referred to elements of the terms, by differentiating the top from the bottom, and the shaded one as in some way added on. I proceeded to ask the children to consider the 86th term. Again Arina and Jay led the conversation, with Arina stating confidently “87 at the top and 86 at the bottom, yeah?” where her question “yeah?” seems somewhat rhetorical as it is asked very assertively, and followed with “yeah, it’s easy”. Jay demonstrated some uncertainty, asking “How about 88 and then 86 at the bottom”, and Arina answered him with a rebuttal of “No, no, the pattern, the pattern is like that”. At this point, Jay agreed with Arina, but Arina stressed her point by repeating “It’d be 87 at the top and 86 at the bottom. Because look… [gestures to the terms presented on the worksheet]”. At this point Alex contributed his opinion, stating “So term 86 yeah would be 86 at the top and 85 at the bottom, but the 86th square on the top would be the shaded one.” Alex’s construction is numerically incorrect, as he is suggesting that each row contained one fewer tile than it should. He was however attending to the various elements of the term, and demonstrating some grasp of the structure of the pattern. Taking into account Alex’s justification of the 12th term, and his construction of the 86th, I would infer that Alex is at this stage taking an explicit approach to the construction of these terms, and succeeding in generalising. Also in differentiating between different elements of the terms, he is utilising figural aspects to grasp the structure of the pattern.

Pattern 2

Pattern 2 is a representation of 2n+1, where each term consists of a central tile and four legs of tiles. The tiles on the legs are distributed whereby the number of tiles on the left legs sums to n, and the number of tiles on the right legs sum to n. For odd-numbered terms, the bottom legs contain one more tile than the top legs, as shown in Figure 7.2.

Figure 7.2. Pattern 2



When the children commenced their conversation about Pattern 2, they explored the structure of the pattern, in order to describe the pattern and extend to the next term. The following transcript contains the children’s observations from when I introduced Pattern 2:

Jay — Guys, guys, guys, the first one is not the x, that one’s an x, but that one’s kind of and that one’s an x

Arina	But I think it's… it's like the… it's like the shaded one
Alex	It's like the shaded one as well
Arina	Yeah I think it's like the shaded one, you see like…
AT	OK. Don't talk about the previous pattern. Try to forget about that because that might confuse you, so just try to forget about the previous one.
Arina	OK. It's like when it's term one it's like it's in the middle square it's like the middle square is kind of like the odd one out and when it's term one there's a square here and in term two there's squares over here and in term three it's three here.
Alex	They keep adding two on instead of one this time. There's only, there's only, there's three there but then they add on two, then they add on another two, then they add on another two.
Jay	That's what I'm talking about. That's what I'm talking about. Hey, guess what….
Arina	So it's like, just like adding two but like up here. If you take…so in the middle, like, you see that square in the middle.
Jay	That should be shaded.
Alex	[Inaudible, but included some reference to wiping memory]
Arina	When it's term four, like on term four…
Cherry	Yeah go on
Arina	When it's term 4, it's like, up here, Like 1, 2, 3, 4.
AT	Right, would you like to make term five? And again you can make one each or you can make it together, whichever you'd like.
Alex	I think I know how much that one… [trails off]
Alex	1, 2, 3, 4, 5, something, 8, 9.
Alex	Wait, I know, term five would have eleven. If the fifth term … [becomes inaudible]
AT	OK, That's interesting Alex
Arina	If the fifth term…if you take term 4.
Alex	If you add two 5s, you add half 2 on to that. That would make 11.
Alex	Let's make a big, big, big…
Arina	If we're making term five. Like only two up here and three up here.

Cherry	Yeah.
Arina	Yeah 1, 2, 3, 4… OK, now we need to do the x. So ehm, think the… [trails off]
Cherry	Oh yeah.
Alex	So that's half 11 on. We all add two on to one, we add two on to that, it's 11. So half 9?
Cherry	You guys are copying off us?
Jay	No we're not.
AT	The point is that you share your ideas. So copying is perfectly allowed.
Jay	Will we be leaving when it's lunchtime?
Arina	I think we're done. Oh wait no, we're not. We need to add one more.
Alex	1, 2, 3, 4, 5, 6, 7, 8, 9. Wait, 10.

As can be seen at the beginning of this transcript, Arina articulated her observation of the central tile, in stating:

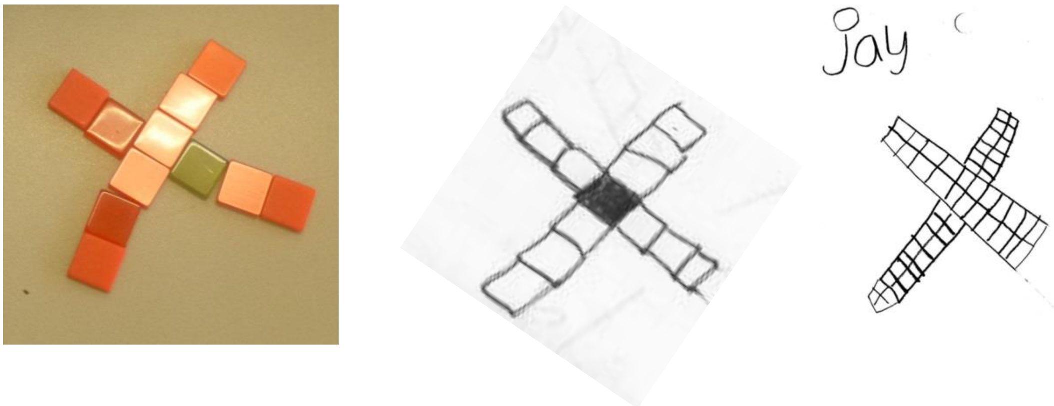
> OK. It's like when it's term one it's like it's in the middle square… It's like the middle square is kind of like the odd one out and term one, when it's term one there's like a square here and in term two there's squares over here and then in term three it's three here.

Arina's comment may indicate that the pattern seems straightforward to her, as she distinguished a constant from variables within the structure, and connected the quantity of tiles in each term, to the position number. Alex replied confidently with his observation that "they keep adding two on instead of one this time. There's only, there's only, there's three there but then they add on two, then they add on another two, then they add on another two." Arina then repeated her assertion drawing the other children's attention to the rate of change, and the role of the central tile, while Cherry and Jay agreed with her. Alex meanwhile counted the number of tiles in each term, and when I asked the children to construct the fifth term, he stated "Wait, I know, term five would have eleven. If the fifth term… [inaudible]". He continued to comment on the total number of tiles for the fifth term, saying “If you add two 5s you add half 2 on to that. That would make 11”.

At this point, Jay’s attention seemed to waver, as he asked “Will we be leaving when it’s lunchtime”, and when he drew the sixth term of Pattern 2, his drawing bore no resemblance to the correct sixth term, as presented in Figure 7.3. For comparison his construction of the 5th term for this pattern, which he constructed with Alex, and an example of a correct 6th term, are also presented.

Figure 7.3. Jay’s constructions of terms from Pattern 2

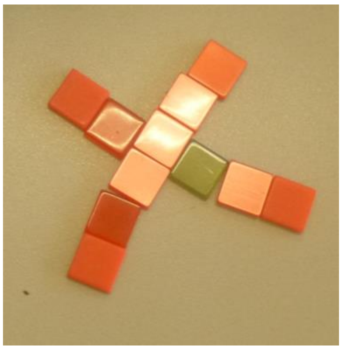
Correct 5th Term, constructed by Jay and Alex	Example of a Correct 6th Term	Jay’s 6th Term



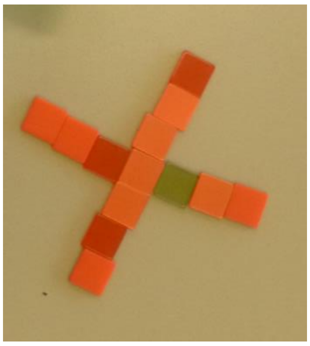
In contrast, Alex constructed the 6th term for this pattern perfectly using the concrete materials available to him, and his construction is presented in Figure 7.4, along with his 5th term.

Figure 7.4. Alex's constructions of terms from Pattern 2

Correct 5th Term, constructed by Jay and Alex



Alex's 6th Term constructed individually



The children's conversation about the construction of the 6th term followed on from the transcript presented above. A complete transcript of this group's discussion of Pattern 2 is included as Appendix V. It is evident from the video footage, and also by comparing Alex's two constructions, that he added two individual tiles to his 5th term in order to construct the 6th term. When he had completed this he sat back in his seat, and seemed to feel very proud of his strategic approach. When asked by Arina whether he was finished, he replied "Yeah I just needed to add two on to that. So just make, so just put like, just make an x with your… [inaudible]". Continuing on from his observation of the difference of two tiles between terms, Alex used a recursive approach to correctly construct the 6th term. While Alex extended the pattern as opposed to constructing a generalisation, his approach is in contrast to the explicit approach he adopted in his identification of a general term for Pattern 1.

Following the children's extensions of this pattern to the 5th and 6th terms, I asked them to construct the 10th term as a near generalisation. The following is a transcript of the exchange which took place among the children in response to my request:

AT And now we're going to look at the tenth term. So what I want you to do first of all is describe it, think about it, what would it look like? And then tell me about how many tiles would you need for the tenth term.

Jay Erm…

Arina	21
AT	Now Arina why? Why 21 for the tenth term?
Arina	Because for the tenth term, like, I… for term four there's like, two and then like the middle square is kind of like I forget the square is not there and then I add these. So it's like…
Jay	I think it might be 23.
Arina	I added 5 and 5, so 20 and then I added the middle square so it'll be 21.
AT	21, OK, lovely. Anybody else have any other ideas?
Alex	I think there might be 23.
Alex	I agree with Arina. But, I'm not sure so I agree with Arina.

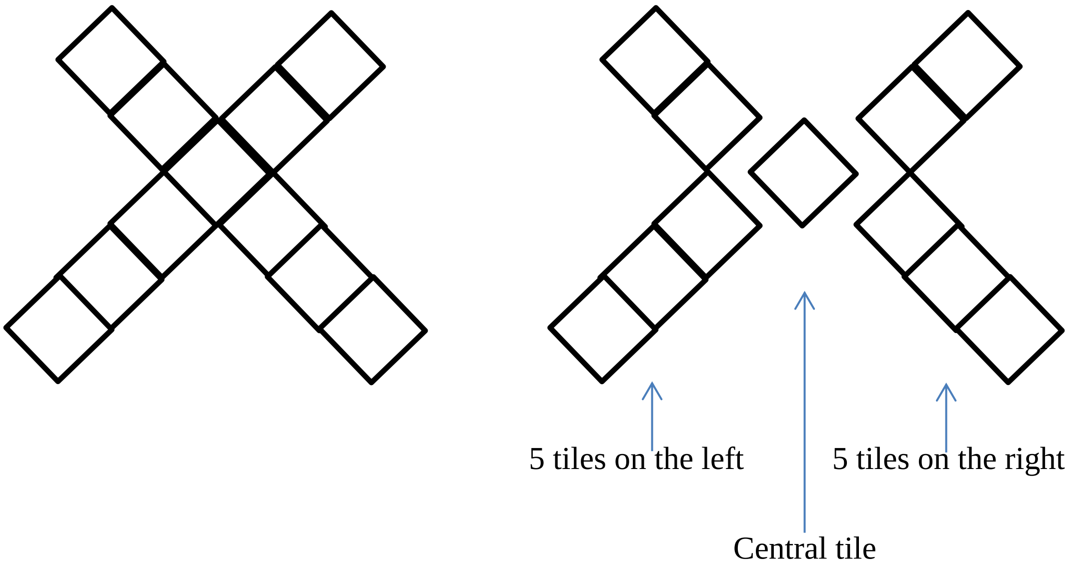
Arina suggested, and justified a total number of 21 tiles, whereas Jay suggested 23 with no justification. Alex agreed with Jay and then changed to agree with Arina, saying "I think there might be 23. I agree with Arina. But I'm not sure so I agree with Arina." I then asked the children to consider the 75th term, and they discussed this far term at great length, a transcription of which is included within Appendix V.

In considering the 75th term, Alex suggested 40 as the total number of tiles. When asked why, he replied "You could add the smaller… you could add numbers that are smaller than 75 but bigger than the term ten or term six, I think". When I probed his thinking at this point, asking him for more specific numbers, and to consider the structure of other terms in the pattern, Alex answered with "I've taken… two added on to term nine and four added on to term eight and twelve added on to term seven. And sixteen added on to term six." Repeatedly during the following discussion Alex answered all questions with total numbers of tiles for terms, and did not mention elements of terms, or adhere to figural aspects. While he seemed very capable of constructing the numerical generalisation of 2n+1 for this pattern, I sought to probe the children to consider how the tiles would be distributed. Arina explained that there would be 75 tiles on each side with a central tile. In doing so, she also mentioned that there would be 75 on each leg. I am confident that this was a slip in her explanation, and that both her gestures, and description, indicated that she saw the structure of the general term as n tiles distributed between the top and bottom leg on the left, plus n tiles distributed between the top and bottom leg on the right, plus one central tile, as depicted in Figure 7.5.

Figure 7.5. The structure of Arina's description of the 75th term, illustrated using the example of the 5th term

A typical presentation of the 5th term in Pattern 2

The 5th term in Pattern 2, presented as two 'sides' of 5 tiles, plus one central tile



Arina demonstrated figural thinking in her description of the 75th term, but the other children in the group did not demonstrate similar approaches. When Arina seemed to indicate that there would be 75 tiles on each leg, I took the opportunity to present this counterexample to the group, in contrast to the smaller terms to see how they would respond, by saying, "So if term four has 2 on this leg do you think that term 75 will have 75 on that leg?" Alex replied that this would not be true, and stated that "Because that's term four and there isn't 4 on that term" and later "So I think there are 2 on two legs but 3 on the two other legs" when discussing the 5th term. However, when I asked him to generalise from what he could see and manipulate to the general case of the 75th term, he replied "It would be 2 added onto the 74th term." Unlike Pattern 1, Alex seemed to persist with very numerical thinking when dealing with this pattern, and did not succeed in describing either the 10th term, or the 75th term, even though when prompted to do so, he did attend to figural elements of earlier terms when describing them.

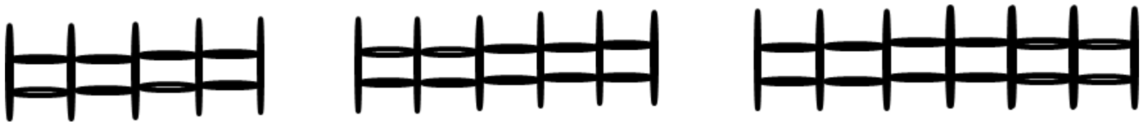
The children's engagement with Pattern 2 presents a contract between the approach of Arina which supported her in engaging with the pattern, and the approach of Alex which seemed to be limited in supporting his construction of general terms. Arina could deconstruct the term into constituent elements, quantify the components, and consider general cases. Alex, in contrast, while demonstrating at times a clear understanding of aspects of the pattern, seemed to estimate the quantity of tiles for the 75th term without any attempt at describing their figural presentation. Many other elements of Alex's perception of the pattern could have hindered or supported him during the group's work on Pattern 2, and it would not be appropriate to suggest that if he had focused more intently on figural aspects he would then have generalised successfully. It may be said, however, that the numerical approach did not support him in succeeding to construct a general case, as evidenced by his justification for his suggested total of 40 tiles for the 75th term:

> you could add the smaller… you could add numbers that are smaller than 75 but bigger than the term ten or terms six, I think".

Pattern 3

Pattern 3 is a sequence of fences consisting of n+1 upright posts, and 2n horizontal posts, as shown in Figure 7.6.

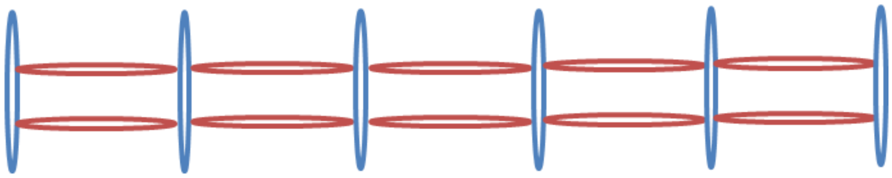
Figure 7.6. Pattern 3



Term 4 Term 5 Term 6

When the children began their deliberations about Pattern 3, their responses were similar to those from Pattern 2. Arina remained very quiet at the beginning of the discussions, but succeeded in constructing the 56^{th} term using an explicit approach. She didn't articulate her thinking enough to confirm whether her generalisation was based upon a numerical or figural mode of generalising, but she could be seen on the video footage counting up in twos to quantify the horizontal posts, and adding on a number equivalent to one more than the term number. Figure 7.7 presents an illustration of how Arina may have been quantifying the number of posts for terms in this pattern.

Figure 7.7. Arina's perception of the structure of the Fences pattern



5^{th} Term in the Pattern: n=5

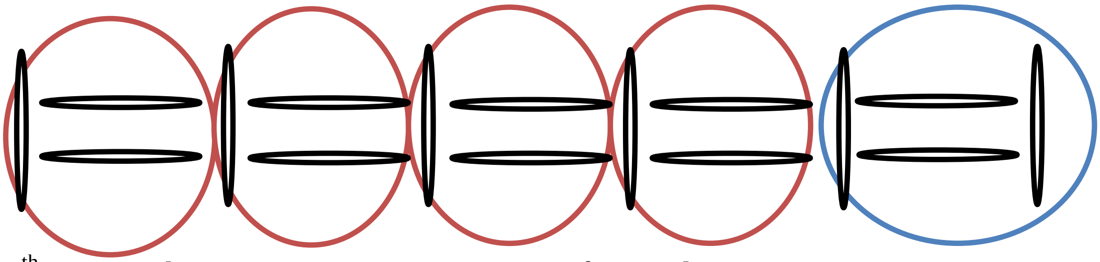
Horizontal Posts: 2n   Vertical Posts: n+1

Total number of posts: 2n + (n+1) = 3n+1

Arina's approach seemed to be hampered by an arithmetical limitation, as she seemed to count up in twos rather than doubling the number of panels to calculate the total of horizontal posts. I prompted her to consider that she could double 56, rather than counting up in twos to find what 56 twos were, and she was more successful in constructing the 56^{th} term than the 9^{th}, where she could be seen counting up in twos and adding on the number of vertical posts.

By comparison, Cherry seemed to be referring to the difference between most panels, and the final panel in each term, when she said "There's like… that matches with that one so four and then everyone they add on one post when they make it" and "You need four posts on the third one". In Figure 7.8 I present a diagram to illustrate my interpretation of Cherry's perception of the pattern, where each panel consists of a group of 3 posts, except the final panel which consists of four posts.

Figure 7.8. Cherry's perception of the structure of the Fences pattern



5th Term in the Pattern: n=5. Term consists of n panels.
Total number of posts = (n-1)x3 + 4
Total number of posts: 3n+1

I interpreted Cherry's perception of Pattern 3 as including figural aspects, as she referred regularly to 'posts' and grouped the posts into groups of three or four, as appropriate. However, Cherry's approach to this pattern was dominated by a counting strategy, an analysis of which is presented in Chapter 6. As outlined in Chapter 6, this counting strategy did not support Cherry in constructing far generalisations. For example, in seeking to construct a far generalisation, Cherry drew 56 panels, and began to count the number of posts needed. Her drawing is presented as Figure 6.8 in Chapter 6.

Alex's inclination towards a numerical approach was again present in his contributions to the discussions about Pattern 3. He did demonstrate an attention to the context, and elements of the terms, when he commented "To make the 3 panel fence you would have to take away 3 posts from the 4 panel." However, he did not seem to utilise this observation in seeking to construct other terms, as he reverted to counting to quantify the number of posts needed for both the 7th and 9th terms. It is possible that he added 6 posts to the 7th term in order to arrive at the 9th, but as he demonstrated some uncertainty regarding the total number of posts required for the 7th term, it is difficult to be certain about his thinking at that point.

As highlighted earlier in this chapter, Jay contributed little to the discussion of Pattern 3, as he seemed to lose his focus on the mathematics. Both Cherry and Arina demonstrated figural thinking, and Alex wavered between numerical and figural. It could be said that Alex prioritised numerical aspects, while Cherry and Arina prioritised figural as all three children sought to construct the factual generalisations of the 9th and 56th terms. This distinction between the aspects prioritised is insufficient however in exploring children's progress in constructing a general term, or in describing any general aspect of the pattern. While Alex's disinclination to explore the figural aspects of the pattern inhibited his progress, Cherry also struggled to construct the 56th term. Both children reverted to counting, and did not succeed in describing a general term. Only Arina demonstrated a robust grasp of the pattern structure in such a way as to support her in describing a general case. The comparison of the approaches of the three children may suggest that children's generalisations depend upon the interplay between their observations of relationships within the pattern structure, their attention to numerical and figural aspects, and also their multiplicative competence, and confidence in that regard. This point is elaborated further in Chapter 6.

Conclusion

In seeking to construct a general term for a shape pattern, it is beneficial for children to attend to figural aspects of the pattern terms presented. In this section I discussed the strategies adopted by children in my research and whether these strategies included attention to figural aspects, or a singular focus on numerical aspects of the patterns concerned. I considered the role played by the group within which children worked, and I further discussed this aspect of the interviews in Chapter 5. I found there to be no distinct difference between the patterns presented, in terms of children's tendency to attend to figural aspects.

I discussed the thinking of one group, and considered whether the children tended to think numerically or figurally about each pattern, and what implication that seemed to have on their grasp of the patterns' structure. Arina adopted figural thinking throughout her engagement with the patterns, and this approach could be seen to benefit her in constructing general terms for the patterns. By contrast, Alex's thinking did not seem to include many references to figural aspects of the terms, and some of the tasks posed considerable challenge for him.

CHAPTER 8: FINDINGS AND CONCLUSION

Introduction

The purpose of this thesis in its entirety is to "tell the complicated story of [my] data in a way which convinces the reader of the merit and validity of [my] analysis" (Braun and Clarke, 2006, p. 92). In Chapters 1 and 2 I presented a case for the relevance and literature base for this study. In Chapters 3 and 4 I drew attention to the methodological principles I followed and the decisions I made in implementation. Chapters 5, 6 and 7 analysed the thinking of the children in selected interview groups, and in this concluding chapter I distil my analysis in order to answer the research questions that I outlined in Chapter 2:

Question One What strategies do children employ in seeking to construct general terms from shape patterns?

Question Two What factors contribute to the strategies children adopt, and to the progress they make in their constructions of general terms?

To foreground a presentation of the research findings, I commence with a summary overview of the research study, including a synopsis of the progress children made in seeking to construct general terms for the shape patterns presented. In keeping with the phenomenological approach adopted throughout this study, subsequent sections address both research questions, exploring the strategies adopted and the contributing factors. The main findings of the study are presented and explored, including the following:

- Children demonstrated a surprising propensity for explicit thinking including many children who applied explicit thinking proficiently in constructing general terms. Given the absence of functional thinking or generalisation in the presentation of the Irish PSMC, and previous suggestions of children's inclination to reason recursively this finding is highly pertinent to the field of academic thinking, and the framing of curricula for algebra in primary schools.

- In order to explore children’s understanding of structure within patterns, there is a necessity to move beyond consideration of whether children attend to figural along with numerical aspects of the patterns. Rather than identifying children as figural or numerical in their approach, it is more appropriate to see their approach as sitting on a continuum ranging from strictly numerical to wholly figural.
- Attending to figural aspects of a pattern supported many children in constructing general terms. This research has also drawn attention to the thinking of some children, who while attending to figural aspects experienced challenges in progressing through the patterning tasks. In exploring children’s thinking it is pertinent to highlight the interplay between a child’s numerical or figural observations, and his/her observations of relationships between terms and their position in the pattern.
- Focusing on issues relating to task design in supporting children’s developing understanding, and for the purposes of assessment, it is noteworthy that within this study no single pattern supported children in observing figural and numerical aspects more or less than the other patterns.
- Within this research study, the context of a task-based group interview contributed to the algebraic thinking observed. Some of the contributing factors to children’s success in constructing general terms for patterns were their manipulation of concrete materials, and their interactions with their peers, with facilitator prompts playing a less identifiable role.
- The interplay between a task-based group interview as research instrument, and a hermeneutic phenomenological research approach afforded me rich insight into children’s algebraic thinking. In this chapter I highlight the potential of both

task-based group interviewing, and hermeneutic phenomenology for research into children's thinking, while drawing attention to relevant constraints.

- The development of a framework of growth points in algebraic thinking was central to this research study. In this chapter I position this framework within the field of algebraic thinking, and also draw attention to its relevance for curriculum development.

Following the synthesis of my research and discussion of findings, I attend to the implications of this research for teachers, teacher educators and policy makers. Having considered the implications of the research, I attend to the scope and limitations of the study, I highlight germane areas of study which this research identifies as appropriate for future investigation, and I conclude with a summary of the findings presented.

Overview of Research

The focus of the research described in this thesis is children's observations of structure within patterns. I conducted task-based group interviews with eleven groups, each of which had three or four participants. Seven interviews took place in School A in October 2014, and four interviews took place in School B in November 2014. The children were attending Fourth class, and had a mean age of 9.83 years at the time of the interviews. To facilitate strong child agency during the interviews, I encouraged the children to share and explain their ideas with each other, and to explore differences between their approaches without verification from me (Howe et al., 2007). Four interviews were selected for in-depth analysis, based upon criteria outlined in Chapter 4.

Children were expected to apply functional thinking to the context of shape patterns, as they were prompted to reason about co-varying quantities and their relationships, and also to represent them. In my analysis of the children's discussions, constructions, drawing and gesture, I drew from two established categorisations of strategies typically adopted by children to shape patterns. Firstly, drawing from the work of Rivera and Becker (2011) and Becker and Rivera (2006), I explored the observations of the children in terms of their attention to numerical and figural aspects of the patterns. Secondly, drawing from the work of Lannin (2004), Lannin, Barker and Townsend (2006) and Barbosa (2011) I explored the tendencies of the children to attend to relationships between terms, or relationships between a term and its associated position in a sequence. Lannin (2004) stated that when children begin to explore the structure of patterns, their natural tendency inclines towards a 'recursive approach', that is the examination of the mathematical relationship between consecutive terms in a sequence. In the development of a child's reasoning, it may be necessary for a teacher to encourage the child to consider an 'explicit approach', whereby the child identifies a rule for the relationship between a term and its position in the pattern (Lannin et al., 2006).

Adopting a hermeneutic phenomenological approach to the study, I aimed to explore what supported children in employing differing strategies. From my field notes recorded during the interviews, my engagement with the children's comments during transcription, and during coding of individual comments, I identified tentative patterns in how children constructed understanding about the structure of the patterns presented to them. In coding I examined children's comments for incidents when children changed their mind, seemed to be striving to make sense, justified their thinking or seemed to learn from others. I explored excerpts of transcripts in the context of each complete interview in order to investigate further the influences on children's thinking as they sought to construct specific, generic and general pattern terms. When coding and during analysis, I strove to find the 'best fit' interpretations for the children's thinking as I made inferences from their verbal comments, drawings, constructions and gestures.

Throughout this research study I was very struck by the variety of approaches children adopted in response to the patterning tasks. In Table 5.1 I presented examples of a number of ways in which children described general terms for the patterns presented to them, and in Table 7.1 I drew attention to the approaches adopted by each child. The variety in these approaches points to the observation of Rivera and Becker (2011) when they stated that we all see patterns differently, to which I would add that an individual may see, and therefore approach, two patterns in entirely different ways. Along with the generalisations highlighted in Tables 5.1 and 7.1 children also made observations of relationships and structure within patterns which spanned a continuum from numerical to figural, and included examples of explicit thinking, recursive thinking, counting and whole object strategies both with and without final adjustments.

Children's Strategies Based upon Their Observations of Relationships within the Pattern

In this section I summarise my comparison of the inferences regarding the children's thinking to the framework of Lannin, Barker and Townsend (2006) in order to categorise and explore children's strategy use. Lannin et al. presented a framework of four key approaches in their observations of children's thinking about pattern structure: a) explicit, b) whole-object, c) chunking and d) recursive. Barbosa (2011) built upon this framework when she drew particular attention to the numerical or figural final adjustment made by children to their whole-object approaches. In synthesising my research I highlight the propensity of children in this study to apply explicit thinking in their construction of general terms for patterns.

In considering the algebraic thinking demonstrated by the children who participated in this study, a question arose as to whether it would be reasonable to anticipate that children would identify explicit relationships in the structure of the patterns. Radford (2011) presents algebra as a cultural construct and states that children's skill development will benefit from facilitation by the children's educational environment. Similarly, Lannin (2004) contends that most children will intuitively reason recursively, and Lannin et al. (2006) recommend that alternative perspectives on the structure of patterns may require intervention. In Chapter 2 I drew attention to the treatment of Algebra within the Irish PSMC, and highlighted the absence of both shape pattern tasks and generalisation. It is improbable that the children who participated in this study had experienced any instruction or facilitation in observing structure within shape patterns prior to their engagement with the patterns in the group interviews. However, Mason (2008) contends that children demonstrate a facility in thinking algebraically from early childhood, including specializing and generalizing, and I expected therefore that a small number of children might demonstrate explicit thinking allowing for the absence of necessary educational interventions as described above.

As outlined in Chapter 4, I coded the children's utterances under the framework of Counting, Whole Object, Final Adjustment, Recursive or Explicit, and the numbers of comments are presented in Table 6.1. Taking the numbers of coded comments as indicative of a pattern, it may be said that children expressed more ideas of an explicit nature, than of any other, when describing, extending and generalizing from the patterns presented to them (180 comments in total). It may be considered surprising that an explicit approach seemed to dominate the groups' thinking about the patterns presented to them, considering previous research in the area, and also the approach to patterning within the PSMC, as discussed earlier in this chapter (Lannin 2004; Government of Ireland 1999). Twelve of the sixteen children demonstrated explicit thinking at some stage during their engagement with the patterns, while the remaining four children did not apply explicit thinking in order to construct terms.

As there seem to be indications from research that recursive thinking is likely to be the intuitive approach of many children, I found the inclination of children to demonstrate explicit thinking to be of particular interest, and I consider it to be evidence of original high-order thinking in response to novel cognitively challenging tasks. Later in this chapter I draw attention to the implications of this finding for the field of algebraic thinking, and also the development of curriculum for children attending primary school.

Children's Strategies Due to Their Observations of Figural and Numerical Aspects of the Patterns

In this section, I assert and justify that children's approach to generalising could be viewed as existing on a continuum ranging between strictly numerical and wholly figural. The continuum I present accommodates children who made figural observations, and described figural aspects of the structure of the patterns, but did not apply these observations in successfully constructing general terms. I also synthesise my analysis to build an argument for the consideration of children's figural or numerical thinking in collaboration with other elements of their approach, such as whether their thinking included an explicit or recursive approach, or depended upon counting. I suggest that figural thinking did not lead inevitably to successful generalisation for the children who participated in this study.

In seeking to explore the strategies used by children, and the reasons underpinning their strategy choice, I firstly considered an overview of whether each child's approach across all three patterns focused on numerical or figural aspects of the pattern structure, as presented in Table 7.1. In summary, Figure 8.1 presents an overview of my interpretation of the children's thinking, as it pertained to the balance between numerical and figural observations.

Figure 8.1. An overview of the children's thinking, as it pertained to their focus on numerical and figural aspects of patterns

Wholly numerical: all comments referred to numerical aspects of patterns.	Alex
Largely numerical, some comments focused on figural aspects, or observed figural aspects but not in a manner which seemed to support an understanding of the structure of the pattern.	Daniel, Luigi
Somewhat figural, but experienced confusion, or remained quiet during large parts of the interview. Some comments indicated confusion due to 'loyalty' to numerical aspects.	Cherry, Orla, Danny, Fiona, Jay
Largely figural, and comments indicated the use of numerical aspects to gain a strong understanding of a pattern's structure.	Ciaran, Grace, Emily, Arina, Jane, Wyatt, Christopher Lily Rose
Exclusively figural: all comments referred to figural aspects of patterns.	

In collating and analysing this data, I referred to the comments children made in the context of the exchanges they participated in. I also referred to field-notes made during the interviews, photographs of the children's constructions, and the children's drawings and jottings which I had retained as artefacts of their thinking during the interviews.

As can be seen from Figure 8.1, this small group of sixteen children span the continuum, from children who made comments focusing largely on numerical aspects, through children who commented on both numerical and figural aspects in meaningful ways, to children who referred largely to figural aspects. In considering the aspects children were attending to, I sought to distinguish between children's observations which supported their thinking, and observations which they made and did not build upon or apply in order to construct a general term.

While attending to figural aspects may have supported some children in constructing general terms, figural observations did not lead inevitably to generalisation. In order to compare children's tendency to observe figural aspects with their success in constructing factual or contextual generalisations, I drew on the scoring rubric for children's progress towards the construction of a general term outlined in Chapter 7. I generated a total score for each child and calculated the mean score for each cluster of children identified in Figure 8.1 above. I found that the cluster of children identified in Figure 8.1 as making "largely figural" observations while applying numerical observations succeeded well, achieving a mean score of 8.6, where a score of 9 would equate to, for example, factual generalisation of all three patterns. In contrast the group of children who also made many figural generalisations, but expressed some confusion fared considerably less well, achieving a mean score of 3.2, where a score of 3 would equate to extending each pattern correctly, but not making any progress in describing a general term.

Rivera and Becker (2011) suggest that when children only attend to numerical aspects of a pattern, they are grasping the commonality within the structure of the pattern at a superficial level. They demonstrate how children who participated in their research encountered difficulties in sense-making relating to the multiplicative structures of the expressions generated from the shape patterns they worked with. Rivera and Becker highlight how such difficulties were resolved when the children made a "shift" from a numerical to figural mode of generalising (p. 357). In analysing the comments made by the children in this research study, however, it may be seen that one cluster of children attended to figural elements but did not succeed in generalising. I would contend therefore that some children who attend to figural aspects of the pattern, may persist with a limited and superficial understanding of the structure of the pattern, and that a figural perspective may not lead inevitably to successful construction of a general, or generic term. Other perceptions of the pattern structure are required in tandem with observations of both figural and numerical aspects, such as children's propensity to think explicitly, recursively, or to use whole-object or counting approaches.

I found that no single approach precludes or determines another, and I posit that it is possible for a child to reside on any place on the continuum from 'Strongly Figural' to 'Strongly Numerical' while counting or while observing recursive, explicit or whole object relationships within the pattern. Equally a child who demonstrated explicit thinking on one pattern may apply recursive or whole object thinking on another.

Factors Contributing to Children's Construction of General Terms

As described above during my analysis of the strategies children used opportunities arose to investigate where children broadened their observations or altered the strategy they were employing. I sought to take advantage of these opportunities to explore the aspects of the interview that supported children in interrogating, and at times improving upon their initial approach. In this section I summarise the primary reasons I identified for such changes in direction, being the role of concrete representations, peer interactions and facilitator prompts.

Concrete representations

Goldin (1998) emphasises that opportunities to construct external representations of their thinking contributes to the development of children's thinking, and during the interviews I facilitated children in representing their thinking concretely using tiles and match-sticks. In Chapter 5 I have drawn attention to the role of concrete manipulatives in the thinking of Jay who missed the opportunity to draw or construct a correct extension term for Pattern 2, and in Chapter 6 I highlighted the thinking of Ciaran and Grace who moved from recursive thinking to explicit having constructed pattern terms with tiles. Also in Chapter 6 I described my understanding of Luigi's interaction with the concrete representations and how they supported him in imagining terms that were not made visible through presentation on the worksheet. These examples may point to the role of embodied cognition in the children's exploration of these novel tasks. In Chapter 4, I presented the thinking of Pouw et al. (2014) and Kyttala and Lehto (2008) who identify the role of embodied cognition in supporting children in internalising concepts and transferring understanding from one context to another. I suggest that in this case the physical manipulation of the concrete objects also supported the children in broadening the observations that they made, and thus facilitated them in grasping more fully the structure of the patterns.

Peer interactions

The tasks presented to the children were both novel and cognitively challenging. It was pertinent therefore to facilitate the children in exploring the mathematics within the highest cognitive ranges of their ZPD. Endeavouring to support such high order thinking, I selected group interviewing as a research instrument and in all three analysis chapters the role of peer interactions was central to the children's developing understanding. I presented the examples of Wyatt interrogating Emily's whole-object approach; Lily Rose's attention to an explicit rule for the pattern following the discussion of her peers; the interaction between Alex and Arina which seemed to nudge Alex to broaden his perspective while he remained hesitant and overly focused on numerical aspects. I also drew attention to the examples of Orla who seemed to gain less from the interactions of the peers around her, Daniel when his perspective differed greatly from Ciaran's, and Christopher when his perspective differed significantly from that of his peers. Jordan (2004) refers to the need for a shared understanding of the activity at hand in order to facilitate co-construction, and from the interviews analysed there appears to be a necessary minimum overlap between the perspectives of children who shared their thinking with each other in order for co-construction to occur, and for children to build upon each other's thinking in a meaningful way. Where this necessary minimum overlap did not occur, children seemed to work in parallel, rather than in collaboration. Children's capacity to articulate their ideas may at times lag behind their mathematical thinking, and it may be that the children lacked some of the mathematical language that would have supported them in expressing their thinking clearly for each other (Dunphy et al., 2014). Equally, as mentioned in Chapter 2, and later in this chapter, it is possible that the children involved in this research have not experienced sufficiently consistent opportunities to improve their collaborative skills during mathematics lessons. Mercer and Littleton (2007) point to the necessity for children to hone the skills of exploratory talk, without which their group interactions may remain superficial, rather than interrogative and robust. Nevertheless, this study has drawn attention to the constructive role played by peer interactions in supporting and stretching children's mathematical thinking, within the context of the task-based group interview.

Facilitator prompts

During this study I aimed to minimise my input, and to thus facilitate the children in constructing understanding without my opinion dominating their thinking, drawing from the research of Mueller et al. (2012) and Howe et al. (2007). When I did suggest an idea, the impact on children's observations and strategy use did not seem to be significant. In Chapter 6, I presented examples of specific prompts I made to encourage the children to broaden their observations of the pattern, and to draw their attention to alternative aspects of the structure from those they had observed. Such prompts included the set questions accompanying each pattern, wherein children were asked to identify a connection between terms and their position in the pattern. From my analysis of the children's comments, and constructions after my prompts the children did not seem to build upon my inputs. Possibly, as with peer interactions above, there was no overlap between my suggested perspective and their existing viewpoints. Alternatively, it may have required time for the children to mull over the suggestions that I made, and if I had asked the children to discuss my suggestion with each other, they may have made more sense of the prompts and built upon them in their thinking.

Summary

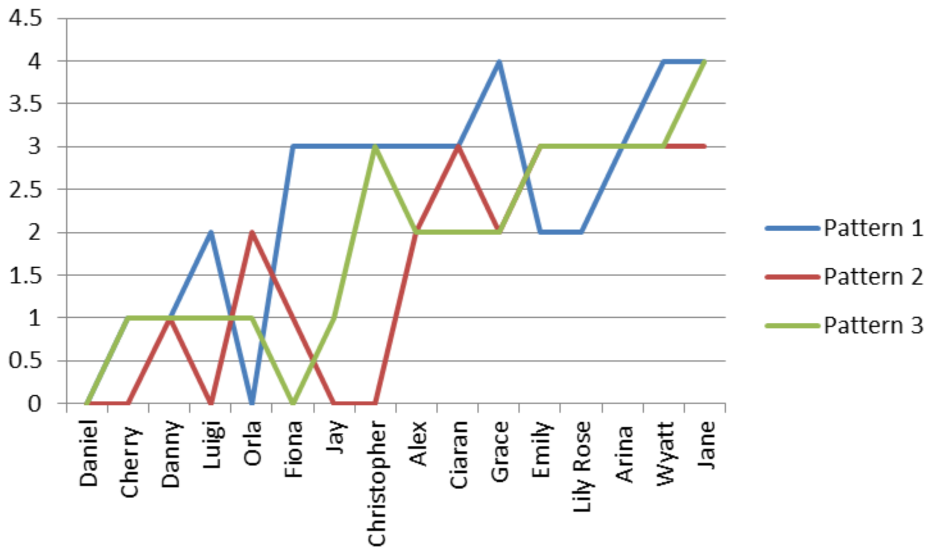
In this section I have drawn attention to some elements of the context within which the children worked when engaging with the novel tasks of constructing general terms from shape patterns. As highlighted above, the children demonstrated a surprising propensity for original thinking, and robust application of their ideas in order to solve the tasks presented to them. I contend that the elements of the setting high-lighted here played an important role in the thinking of many of the children, and later in this chapter I outline the implications of this finding for research, and also for the teaching of mathematics.

Inter pattern comparisons

The tasks underpinning an assessment are central to the propensity of the assessment to both stretch and elucidate children's thinking (Burkhardt, 2007). When analysing the strategies children employed in seeking to construct general terms for shape patterns, I paid attention to the role played by the tasks, and as there was no variation between tasks for each pattern, I sought to explore consistencies or variations across the three patterns. I scored all children on a rubric relating to the progress they achieved with the patterning tasks (as presented in Table 7.2). Within this rubric, a score of 0 indicated that the child did not correctly extend the pattern, a correct extension of the pattern scored 1, some description of general terms scored 2, factual generalisation scored 3 and a contextual generalisation scored 4. While the sample of scores was too small to warrant statistical analysis, I compared the scores for all children on each pattern through generating average values and plotting graphs where I compared each pair of patterns. A summary of this data is presented as Figure 8.2.

Figure 8.2. A comparison of scores achieved on the three patterns

	Count of children	*Sum of scores obtained*	*Average score obtained*
Pattern 1	16	38	2.375
Pattern 2	16	26	1.625
Pattern 3	16	30	1.875



As discussed in Chapter 7, Pattern 1 appeared to be slightly less challenging for most children, and Patterns 2 and 3 seemed to be of reasonably comparable challenge. The relative challenge of Pattern 2 might not be surprising given the asymmetrical nature of the rate of growth of the legs comprising the x-shaped terms, as highlighted in Chapters 2 and 3. As discussed in Chapter 7, some of the lower scores obtained on Pattern 3 seem to be due to children's fatigue as they had engaged in novel cognitively challenging tasks for more than 40 minutes before commencing Pattern 3.

Furthermore, in exploring the articulated observations and comments of the children, I was interested in exploring whether any one pattern elicited more or fewer figural comments than the other patterns. Identifying features of patterns that drew children's attention to figural aspects may be very constructive in supporting children who are tending to focus exclusively on numerical aspects. As described in Chapter 7, and earlier in this chapter, I had coded all of the children's comments in order to capture comments that referred to figural and numerical aspects of the patterns. From the quantities of comments coded numerical or figural, I investigated whether any one of the three patterns motivated the children to refer to numerical or figural aspects more than the other patterns. While the proportion of comments deemed to be numerical could not be said to be an accurate representation of the proportion of time or thought children gave to numerical aspects, I felt that such comparisons could support me in analysing the children's strategies. The comparisons largely supported my observations contained in Figure 8.1 above.

For each pattern, more comments referred to figural aspects than purely numerical, and while Pattern 2 generated far more comments overall, the proportion of all comments coded as referring to numerical aspects alone was similar for all three patterns. No pattern emerged in the proportions of comments deemed to indicate numerical or figural thinking for any one pattern, and therefore it cannot be said that any one of the three patterns motivated the children to adopt numerical or figural thinking more or less than the other patterns.

Task-based group interviews as a research methodology

The purpose of this study was to unpick how children responded to patterning tasks that required them to generalise from given terms to a general case. It may be conceivable that such a study would involve large numbers of children in paper and pencil tests that are marked and analysed quantitatively. However, I deemed it valuable to access the reasoning underpinning children's work towards generalisation, and I adopted a phenomenological approach in order to capture and examine both the 'how' and 'why' of children's thinking about patterning (Creswell, 2013). I would attest that this approach yielded rich and rewarding insights into children's constructions including their work towards generalising and the various ways of seeing the mathematics. If my approach had not been phenomenological but rather had sought to classify or categorise children's strategies without exploring the contributing factors, the findings presented in this chapter would have been lacking both in detail and in depth.

Task-based group interviews have the potential to accede the agency in mathematics to the participating children, and hermeneutic phenomenological methods ensure that the children's thinking is laid bare so that strengths and achievements of all children are visible. In particular, in this study I found that the research methods adopted gave me access to the interplay of children's predisposition to notice aspects of tasks, the interview setting, and the tasks themselves. Rather than collecting data relating to the product of children's thinking I was afforded access to the process of their thinking as it unfolded throughout each interview. This research methodology may have particular relevance in unpicking children's higher order thinking with novel or less familiar mathematics, where children will engage most productively if facilitated to work within the highest cognitive ranges of their ZPDs.

There is an element of risk-taking intrinsic to field-based research, as the researcher has less control over the factors that act upon the participants during data collection, and the eventual data collected (Creswell, 2013). In choosing to utilise task-based group interviewing as my research instrument, I anticipated that this risk was heightened as not only would the children engage with a dynamic situation, but in order to maximise child agency I had committed to intervening as little as possible with their thinking. However, I was motivated by my conviction that children are capable of high levels of mathematical thinking, and that the environment I constructed would scaffold them sufficiently for them to engage with the tasks at each child's personal level. I would caution however, that the rich data I collected, and the findings I have presented here are dependent upon rigorous fidelity to a well-defined research method. In particular, I adhered to the principles of task-based interviewing outlined by Goldin (2000), and presented in Chapter 2. In addition I was committed to facilitating child agency, and this was emphasised in the introduction to each interview, and in my prompts to the children to refer to each other for clarification, or feedback throughout the interviews.

There were many occasions during the children's discussion when children interrogated each other, clarified their thinking through justification and demonstrated understanding independently after listening to the perspective of others. I have discussed examples of these constructive events in the analysis chapters and synthesised them in the section of this chapter entitled 'The role of group interactions'. There were also occasions during the interview when a child seemed to repeat opinions previously expressed by their peers, and in some cases evidence supported my understanding that the child fully agreed with the opinion, while in others evidence indicated that s/he was giving what s/he perceived the 'answer' to be, rather than the solution to the task. There was not a clear dichotomy between times when children actively constructed understanding, and parroted the thoughts of others, but rather incidents occurred when children seemed to contribute some thinking while still relying on others' opinions. When analysing the thinking of children in the context of a group interview it is critical that attention be paid to this continuum between robust collaboration wherein children build towards an understanding in tandem with each other, and situations whereby one child acquiesces to the viewpoints of other(s) without interrogation.

In addition, there are ethical considerations relating to the experience for the children who participate. In a task-based interview that involved novel and cognitively challenging tasks, children will experience challenge that for some is insurmountable. In this research study I made decisions during planning and during the interviews to not place children in positions of potential embarrassment. This led inevitably to situations where the thinking of some children was not stretched, and these children may not have demonstrated their full potential within the interview. Equally I may have been unaware of times when a child or children felt inadequacy in comparison to their peers due to the relative ease with which some children appeared to navigate the tasks at hand. Also as the tasks involved groups of children, I did not respond to the individual needs of children to the extent that I may have if the interviews were individual, for example when children demonstrated fatigue.

In this section I have drawn attention to the affordances and constraints of task-based group interviewing coupled with hermeneutic phenomenology as a research methodology. Strengths of this approach include the rich potential to explore, in depth, the mathematical thinking of children when their thinking is mediated by tasks, concrete materials, peer interactions and facilitator prompts. Constraints include the risks associated with the dependence upon rich peer interactions, ethical issues relating to the experience for each child and also the challenge in differentiating each child's original thinking from when children accede their thinking to that of others.

Framework of growth-points in algebraic thinking

There are many ways in which the development of children's reasoning may be conceptualised as it progresses over time. Dunphy et al. (2014) draw attention to the historical context of describing developmental pathways as many academics and policy makers have now moved beyond the clear distinctions of Piaget's developmental stages to a more fluid and multi-faceted "landscape of the learning" (Fosnot and Dolk, 2001). In this research study, I have outlined a framework of growth-points in patterning, wherein I have drawn heavily from research in the fields of algebraic thinking, and theory relating to developmental pathways. Within the study, I have applied this framework in the development of the task-based interview and during analysis.

The potential applications of such a framework are more far-reaching however. In the redesign of the Irish PSMC those tasked with the curriculum design could potentially make use of the synopsis of research in the field of algebraic thinking contained within this framework of growth points. Also, beyond the Irish curriculum, this framework may support the research of academics in the field of algebraic thinking who are exploring the progression of thinking from young children's informal 'powers' of imagining, expressing, generalisation, and specialising to the robust algebraic thinking required of students in secondary school (Mason, 2008). In exploring the algebraic thinking of children, the framework of growth points in patterning presented in this thesis may support researchers in positioning children's thinking, and thus preparing appropriate tasks to support learning, or explore thinking.

Implications of Research

The findings presented in this chapter indicate that the development of children's proficiencies in the topic of patterning is a complex endeavour where multiple aspects are interwoven. Children's choice of strategies rests upon their observations of figural and numerical aspects of the pattern structure along with observations of relationships and connections within the pattern, and proficiency in multiplicative thinking. Assumptions should not be made about children's potential to reason in novel ways when their thinking is mediated by peer interactions, representations including concrete materials and cognitively demanding tasks. In this study the children had not previously engaged in formal learning situations involving generalisation from shape patterns yet many demonstrated an impressive adeptness in figuring out the tasks presented to them, and in expressing their thinking in general terms, albeit informally.

Stretching children's algebraic thinking

Mason (2017) emphasizes the need for research to inform teacher actions, and for such actions to broaden children's observations of structure. The analysis presented here is consonant with Mason's position, and demonstrates how facilitating children in observing structure in multiple ways supports their success in generalizing from shape patterns. The value inherent in children constructing understanding is largely accepted in mathematics education research, but teaching approaches underpinned by transmission persist in many classrooms in Ireland (Dooley, 2011; Nic Mhuirí, 2013). The success achieved by some children in this research, on novel high-order tasks demonstrates the efficiency of discovery methods, and points to the benefits of specific catalysts to support children's observations of structure. Equally English (2011) warns that teachers and policy makers should not underestimate children's ability to take on and work with new ways of thinking. English states that children "have access to a range of powerful ideas and processes and can use these effectively to solve many of the mathematical problems they meet in daily life" (p. 491).

On the basis of this study, I make the following four recommendations for the forthcoming redesign of the Irish Primary Mathematics Curriculum. Firstly, at every class level, children should be facilitated in generalising, both in dedicated content relating to algebraic thinking, and also in other content areas. This is particularly true in areas of number, such as the zero-properties of operations, but is also relevant to the generalisation of properties of shapes, for example in exploring whether the sum of the angles of all quadrilaterals in 360°. Secondly, specific content relating to functional thinking, from shape patterns and other appropriate contexts, should be included from First class. Such content should include focus on description of pattern structures, along with pattern extension and also justification and expression of general terms. Teachers should be guided to facilitate children in broadening their observations of structure within patterns and functions through as broad a range of methods as possible, such as discussion, construction, drawing, and tabulating. Thirdly, the revised curriculum content should aim to avoid placing upper levels on children's thinking, or the expression of their thinking. Teachers should be encouraged to consider the potential of all children to engage in high order thinking in response to cognitively challenging tasks, and guidance should be afforded to teachers to scaffold children in expressing abstract thinking such as generalisations of functions. It is pertinent to ensure that teachers remain cognisant of children's understanding exceeding their capacity of expression, and that at times the teacher's role will include facilitating and stretching children's thinking that they are not yet articulating clearly (Dunphy et al., 2014). Fourthly, task-based group work including concrete materials should be embedded as a signature teaching approach in order to mediate children's high-order thinking. While referring specifically to the Irish PSMC in this instance, I contend that these recommendations would be equally relevant to curricula in many jurisdictions.

Formative assessment using developmental pathways

Within this research study, I presented a framework for the shape patterning domain of functional thinking (Table 2.3), and I drew from the framework when developing tasks for the interviews, observing the children during the interviews, and reviewing the children's progress during analysis. The analysis presented in Chapters 5, 6 and 7 and synthesised above evidence the necessity to consider such frameworks as suggestive rather than predictive. Many children demonstrated varying levels of progression across the three patterns, and this research depicts a snapshot in the children's education, with no basis for determining how the children may progress into the future. Taking the progress across the three patterns as a crude comparison, one might expect children to achieve greater progress on Pattern 2 or 3 having engaged with the novelty of generalisation from shape patterns in Pattern 1. In reality only 3 children, Orla, Emily and Lily Rose, achieved their lowest score on Pattern 1, and for 7 of the 16 children the score achieved on Pattern 1 was the highest. The developmental pathway outlined in Chapter 2 should therefore be considered a theoretical synopsis of research in the field of algebraic thinking. If applied to practical situations in classrooms, certain constraints should be foregrounded.

However, the presentation of a framework can provide teachers with a continuum indicating a possible pathway children could navigate in developing understanding in shape patterning, and functional thinking. Dunphy, Dooley and Shiel (2014) state that "research-based conceptual frameworks which describe mathematical thinking in terms of levels of sophistication" support teachers in interpreting children's thinking. Such frameworks may also inform teachers if they are less familiar with a mathematical topic than might be ideal. For example, shape patterning, and more broadly functional thinking, may be included within the forthcoming revised Primary School Curriculum for Mathematics, and many teachers in Irish primary schools will not have encountered this area of mathematics during their teaching career. Teachers will need additional support in navigating the purpose of the content objectives presented, and in interpreting how activities may be chosen to support children's developing thinking. The framework of growth points presented in this thesis could serve as a basis for teachers to interrogate potential learning pathways in patterning for their pupils.

When frameworks are presented in this way, it would be preferable if accompanying materials emphasise the potentially non-linear routes navigated by many children, and also advise teachers regarding teaching approaches. If the framework is interpreted as a step-by-step route, whereby children complete a lesson for each step in order to 'learn' a procedure or approach, the outcomes for children in terms of the breadth and depth of conceptual understanding may be compromised. Throughout the pathway from Pre-formal Pattern to Abstract Generalisation, children should be facilitated in observing broadly all change and relationships within patterns and their constituent elements.

Formative assessment through task-based group interviews

Referring again to the quote above from Dunphy, Dooley and Shiel (2014), when educators wish to assess children, attention must be paid to the means by which children's mathematical thinking may be elicited, and thus assessed. As highlighted in

Chapter 2, paper and pencil assessments are limited in terms of what and how they assess. As mathematics education internationally moves towards an understanding of learning as active, participatory and child-led, summative assessment through once-off paper and pencil tests are in stark contrast to this child-centred approach (Schoenfeld, 2015). Schoenfeld states that assessment plays a major role as "a potential lever for change" during this time of transition for mathematics education (p. 192). He expresses concern about the capacity of teachers who have heretofore focused on procedural understanding as they seek to assess not only understanding, but also the multiple elements comprising mathematical proficiency as presented in Chapter 1 as Figure 1.1.

In contrast to paper and pencil tests and worksheets which may have been prevalent in traditional classrooms, task-based group work affords opportunities for children to develop thinking, and also for teachers to assess the many elements of children's mathematical proficiency presented here. It would not be practical or necessary for teachers to analyse children's thinking to the levels presented in this thesis, but rather to develop and employ rubrics based upon developmental pathways as discussed above. While more research is required to develop rubrics, or guidelines for teacher development of rubrics, my research study presents evidence of children broadening and developing their thinking while being assessed by me, during their engagement in a task-based group interview.

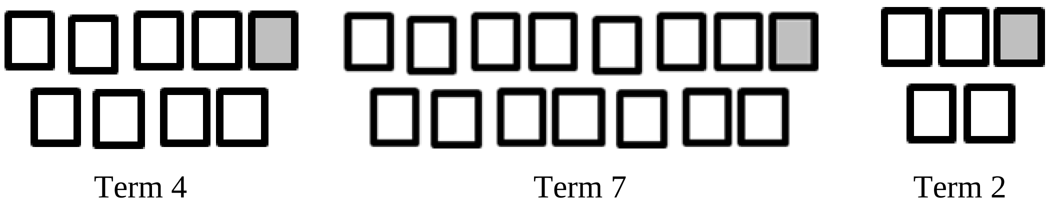
Scope and Limitations of Research

Marshall and Rossman (2011) state that "all proposed research projects have limitations" and that identification of such limitations gives context to a study, and inform readers that the study is not generalizable (p. 76). Cresswell (2012) concurs adding that the identification of a study's limitations supports recommendations for future research, and informs researchers who may choose to replicate the study.

Tasks

In exploring the interplay between task design and children's problem-solving Goldin (1997) emphasises that tasks should be chosen or designed based upon a strong theoretical basis. To this end, the tasks I used when interviewing the children involved in this study were taken from similar research studies by established authors in the field of algebraic thinking. While the tasks I chose were appropriate and fulfilled the suggested criteria and format of the research cited above, my presentation of the patterns was linear in each case, beginning at term x and showing the following two or three terms. Non-linear presentations (such as the version of Pattern 1 presented in Figure 8.3) would have the potential to further engage children's thinking (Mason, personal communication, July 27, 2016).

Figure 8.3 Non-linear pattern presentations



From my existing analysis of the children's thinking, and through discussion with Mason I would suggest that my strictly linear presentation may have led children towards recursive approaches more readily than if I had presented non-consecutive terms. Additionally, I used the same structure for each pattern, and if I had used alternative questioning techniques for each pattern, I might have supported some children to observe structures that eluded them as they answered the questions I set. For example, in one pattern I could have given a suggested near term and asked the children whether it was correct, an alternative term, or did not belong to this pattern. Such questioning might have scaffolded some children in their observations of what contributed to a term's fit within a pattern, and the connections between terms and their position.

My role as researcher

Howe et al. (2007) outline the contentions of Dewey and Piaget that within a group, children are motivated to pool their individual perspectives, and in the ideal scenario may strive to collaboratively solve a problem by mutual consideration of each other's ideas. If an adult intervenes, the contribution of this perceived more knowledgeable presence may supersede the contributions of the children and reduce their motivation to actively engage with each other's thinking in order to arrive at a solution. Thus, Howe et al. emphasise that children collaborate best with minimal input from teacher, and in conducting the task-based group interviews I aimed to stand back from the children's conversations to allow them to engage more purposefully with each other. I did choose to intervene when children were pursuing what I felt may be a fruitless avenue, if children were becoming distracted by non-mathematical topics, to draw out children who were less forthcoming with their ideas, or to ask children to compare their thinking with each other.

While I aimed at all times to support child agency, and to encourage the children to think independently, I found it challenging at times to not intervene. In my role as primary school teacher, I have spent years supporting children in various contexts to make progress within tasks, by scaffolding, and probing their thinking. During the interviews, I felt a strong urge to support children in constructing general terms, and in progressing through the tasks, particularly if I felt that there was a crucial observation that would support them. At times, this led me to intervene more than I should have. As I look back at the video footage I see examples of situations where the children's thinking would have benefitted from less input from me. Added to the occasions when I felt that I intervened too much, there were also situations within groups when I feel in hindsight that the group discussion may have been richer if I had moved the children's conversation on more quickly.

The balance between recursive and explicit thinking

Prior to the task-based group interviews, I had engaged with much literature regarding the typical developmental progression of children with regard to their observations of structure within pattern. While Lannin, Barker and Townsend (2006) emphasise that the approaches of explicit, whole-object, recursive and counting should not be seen as hierarchical, Lannin (2004) also states that to identify explicit relationships within patterns may require intervention from a teacher. My focus therefore was on whether children who could reason recursively would also reason explicitly. Watson et al. (2013) however caution against such an approach and emphasise that for robust functional thinking to be present, a child must be adept at thinking both explicitly and recursively, and deciding when each would be appropriate. Among the children whose thinking I have highlighted, Jane only reasoned explicitly throughout her group's interview and in hindsight I would have asked her to demonstrate recursive thinking to ensure that she was also proficient in reasoning recursively, and could see when recursive thinking might be more appropriate than explicit. In the future, when conducting assessments of this nature on the functional thinking, I will include patterns that are more easily solved with a recursive approach, and ask participants to identify which approach is appropriate in the solution of each pattern.

Limiting children through reduced expectations

When planning and preparing the task-based group interview schedules, I was guided by the writing of Lannin (2004) and Radford (2011) when they suggested that there are elements of algebraic thinking, relating to patterning, that for many children require teaching interventions. Given the absence of patterning and generalisation from the PSMC I assumed that the tasks I would present would be novel for many, if not all, of the children participating (Government of Ireland, 1999). Furthermore within the PSMC children are not facilitated in using symbols to represent varying quantities. A frame is used to represent an unknown from Third Class but variables are not introduced before Sixth Class, and even then the examples suggested point towards an unknown with single value, rather than a general number. I therefore anticipated that many of the children would not be fluent in expressing their mathematical ideas using abstract notation. Accepting the lack of familiarity the children would have with shape patterns, generalisation and symbols, I aimed to not limit the children's participation by facilitating construction of both factual and contextual generalisations. Having analysed the discussions from the interviews however, I feel that I did limit the responses of some children. In developing children's proficiency with abstract notation Hewitt (personal communication, February 3, 2017) referenced Tahta (1989) when he suggested that it may be constructive for teachers to volunteer to "look after the signs", allowing the children to express their thinking verbally while the teacher notates. Hourigan and Leavy (2015) demonstrate how this might occur in their study involving children's construction of general terms for shape patterns. The children in Hourigan and Leavy's study, who were of comparable age to the participants in my research, were facilitated in expressing their generalisations symbolically. For example in an excerpt cited in Hourigan and Leavy (2015) a teacher asked "If *n* means the number of storeys, how would you describe the number of windows?" (p. 37). I would hypothesise at this point that at least Jane and Wyatt may have succeeded in expressing their thinking using abstract symbols with this form of scaffolding.

Collaboration

As described in Chapters 2 and 4, a key aspect of this research was the novelty of the tasks for the children participating, and the consequent necessity that they would work within high cognitive ranges of their Zone of Proximal Development (ZPD) (Vygotsky, 1978). In order for this to occur a co-construction of understanding was necessary. Initially I planned to interview children individually but I found that it was improbable that I, as stranger researcher, could facilitate a co-construction supportive of children attaining these high cognitive ranges.

Allowing that a group setting *could* facilitate children in working within high cognitive ranges of their ZPDs, it was not possible to ensure that this would occur. In planning and facilitating the interviews I paid close attention to issues of group composition, collaboration and child agency, as described in Chapter 4. Additionally, in Chapter 4 I outlined my attempts to focus my analysis on groups where there was the greatest possibility of ‘internal diversity’ as an indicator of high levels of collaboration (Davis and Simmt, 2003). I remain concerned that for some children who were included for fine-grained analysis, their engagement with their peers within the task-based group setting might not be considered collaborative. For example, Orla seemed to feel isolated when she asked “can I join in?” during her groups’ discussion of Pattern 2. Moreover Cherry and Fiona tended to contribute very little to their groups. The participants of Group 3 in School 2 were Jane, Lily Rose, Danny and Christopher. All four members of this group also contributed far fewer comments than the average of 103, but the distribution of comments uttered by children across the group may indicate a parity of contribution that did not exist in the groups of Orla, Fiona and Cherry. In seeking to explore the strategies children adopted in constructing general terms for shape patterns it would be ideal if all children collaborated, and thus achieved a co-construction of understanding with their peers, but it is not certain that this occurred during this research study.

Mercer, Wegerif and Dawes (1999) and Mueller, Yankelewitz and Maher (2012) draw attention to the role of the teacher, and the normalised classroom practices in the facilitation of exploratory language among children, and of collaboration wherein children draw from and build upon each other’s thinking. In the Irish context, it may not be assumed that all children are facilitated in developing the skill of solving a problem or completing a project through collaborative discourse with a partner. Powell (2006) defines this skill as 'negotiary discourse' and states that it is the means through which children's thinking is mediated by the thinking and interactions of another. In contrast Shiel, Kavanagh and Millar (2015) refer to the tendency in many Irish classrooms towards whole-class teaching and urge a swift uptake of recommended approaches under the forthcoming revised curriculum, such as the promotion of a focus on rich discussion among children in the classroom.

Scale of research

This study focused on the thinking of 16 children in four groups. All children were selected from Fourth class in Irish primary schools. As described in the Research Methodology and Research Methods chapters, I endeavoured to include a range of abilities within this small sample, but findings could not be deemed to be generalisable. Marshall and Rossman (2011) state that while qualitative research studies are not generalizable, "their findings may be transferable" (p. 76). It is the intention, therefore, of this research study to shine a light on how some children engaged with shape patterns, and how they thought about the tasks presented to them wherein they sought to construct general terms in collaboration with their peers. This study will contribute to the knowledge base about how some children may respond to similar tasks, but anticipation of how children respond should not be limited to the findings presented here. Rather researchers and teachers should remain open to the potential of children to demonstrate original observations and creative strategies.

Further Areas of Study

As highlighted above from the observations of Mason (2017) all research relating to mathematics and mathematics education is of greatest relevance when it is made manifest in classrooms. In order to support children in broadening and varying their observations of structure, teachers will need themselves to be cognisant of the many variations of structure within patterns. As no research has been conducted with teachers or student teachers in this regard in the Irish context, research will be necessary to explore this population's understanding of pattern structure, generalisation and functional thinking more broadly.

Intervention to support children's developing functional thinking

On the basis of this study, there is the potential for a research intervention in relation to generalisation from shape patterns, and functional thinking. Specifically it would be useful to compile a programme of patterns with accompanying classroom approaches, and to research teacher's implementation of such a programme as part of their mathematics lessons. Such an intervention could explore the challenges experienced by teachers in supporting children in broadening their observations of structure. Teachers could also be encouraged to facilitate children in expressing their ideas through the use of symbols, adopting a scaffolded approach where the teacher firstly models by 'looking after the signs' and paraphrasing children's verbal expressions through the medium of abstract symbols.

Patterning is a context for functional thinking, and many other contexts are in use in primary schools internationally, for example function machines and exchange games or context based questions (Ng, 2017; Chimoni, 2017). In exploring formal functions in secondary school, there is the potential for children to apply the thinking they have developed in the context of patterns to multiple contexts for functions. I would caution against assuming that this application will be straightforward for all children, and research is merited to investigate how children who have developed functional thinking in one context apply it in another.

Task design principles for patterning

Repeatedly throughout this thesis I have referred to the centrality of the task design. Anthony and Walshaw (2007) emphasise that tasks guide and support the thinking that children undertake. I aimed to select tasks which would allow the children to "access important mathematical concepts and relationships, [and] to investigate mathematical structure" (ibid., p. 2). While I contend that the children did engage with challenging mathematical concepts, and explore the structure of the patterns, only three tasks were utilised in the task-based group interviews. In outlining implications of this research, I have suggested that teachers may need guidance in navigating content objectives of the forthcoming curriculum that involve functional thinking and shape patterning. Relevant tasks that support a range of learners while allowing all to engage in broad observations of structure will be required.

In discussion relating to patterning as a context for algebraic thinking at the Algebraic Thinking Thematic Working Group of CERME10 a question was raised by Christof Weber regarding the need for design principles for patterning tasks. I interpreted his question to suggest a possible categorisation of patterns for scaffolded use by teachers. My research would indicate that it is not necessary to engage children with a succession of tasks in an excessively incremental approach. It may nonetheless by useful for teachers to have groups of tasks which they could target at children who are, for example, engaging in excessive counting, or are limited to recursive or numerical approaches in their observations of structure. Further research would be necessary to compile a large collection of tasks, and to research how children interact with them so that judgements could be made about the ways in which identified task attributes support specific approaches.

Conclusion

Cognisant of omissions in the current Irish PSMC relating to generalisation and functional thinking, I commenced this research study to investigate what proficiencies Irish children would have when asked to construct a general term for a shape pattern. I found resonance between my personal and professional experience of children's thinking and the assertions of Mason (2008) and Hewitt (2009) who advocated for children's nascent strengths in reasoning. Acknowledging the potential novelty of shape patterning tasks to the children who participated in this study, I anticipated that they would draw from their informal understandings and emerging thinking in navigating the patterning tasks, while remaining alert to the possibility that some children would struggle to overcome the novelty of what I was asking of them.

The children's creative and original responses to the patterning tasks impressed me greatly, and re-invigorated my enthusiasm for the mathematical potential that resides within all children. In addition, it was wonderful to witness the enjoyment the children derived from the cognitive challenge presented during the interviews. As teachers, teacher-educators, policy-makers and curriculum developers endeavour to support all children to achieve their potential, it is incumbent upon us all to aim high in our expectations of children, scaffolding sufficiently when appropriate, but not limiting children's potential, or denying any child access to the appropriate cognitive challenge.

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APPENDIX A: PLAIN LANGUAGE STATEMENT (CHILDREN)

Project Information for Children

I would love if you could help me out with some maths!

You will be looking at mathematical problems together with other children from

your class. Some problems may be tricky because there are problems included for primary school and early secondary school. We will stop as soon as the problems become too difficult for you.

I will be video-taping you doing maths. I will be showing video clips to people who want to know more about teaching maths. I will ask your teacher to share with me

the results of Mathematics tests you have sat in school. I will not use your full name or the name of your school when I am using the videos.

You can decide to leave this project at any time, and you will not get into trouble for doing this

Hopefully we will have lots of fun and learn a lot together!

I hope you will take part!

Thank you,

Ms. Twohill

APPENDIX B: INFORMED CONSENT FORM (CHILDREN)

My name is ________________________________

I would like to take part in this maths project.

Yes_______  No________ 

I am happy to be video-taped doing maths.

Yes_______  No________ 

I am happy for teachers to look at some of my written maths work.

Yes_______  No________ 

I am happy to talk about my ideas in mathematics.

Yes_______  No________ 

I am happy for the result of my Mathematics tests to be shared with Ms Twohill.

Yes_______  No________ 

I understand that I can stop doing the project at any time, and I will not be in trouble for it.

Yes_______  No________ 

I know that my name will be kept secret at all times if people are talking about the lessons.

Yes_______  No________ 

Thank you 

APPENDIX C: PLAIN LANGUAGE STATEMENT (PARENTS)

COLÁISTE PHÁDRAIG
ST PATRICK'S COLLEGE
DROIM CONRACH | DRUMCONDRA

Plain Language Statement
October 2014

Dear Parents or Carers

As a PhD research student in St Patrick's College I am researching whether children in Irish primary schools are developing skills in thinking mathematically, particularly in the area of algebra. This project is concerned with studying children's thinking and as such will involve individual and group interviews to see how children respond to mathematical problems and tasks. First of all your child will take part in a group interview, which will take between forty minutes and an hour. When I have looked back over everything the children did in the group interview, I might have questions for some children. If so, I will meet them individually or in pairs the day after the group interview to ask them what they think about something that came up during the group interview. I may also invite your child to be interviewed individually on a new task, when all the group interviews have taken place. With your consent, the consent of your child and the consent of your child's school, interviews may be videotaped and audiotaped. Your child's school has kindly agreed to facilitate this project.

During the interview, I will ask your child to solve some maths problems with me. I will support him/her to ensure that your child does his/her best. I will not exert pressure on your child. With your consent, I would appreciate access to your child's results in school based standardised tests in Mathematics.

I will use videos to aid reflections on the children's responses to the problems, and videos will be analysed and presented for research purposes only. I will try not to disclose the name of the school but cannot promise that I will be able to do this. I will not say the name of the school or give children's full names to the people who are watching the videos. While I will not say the name of the school in any presentations, there is a small chance that the school can be identified (for example, the building or the uniform might be recognisable). In reports that I write about the project the school's name or your child's identity will not be stated. The videos will be at all times securely stored and will be password protected. From the videos I will make clips which I may use at presentations. The video clips will only be shown to people who are involved in studying the teaching of mathematics. Videos and video clips will not be available on-line.

Your child does not have to take part in the project. With your consent I will meet with your child and ask him/her to take part. If your child does agree to take part, he/she can withdraw from the project at any time.

This research project has been reviewed and approved by the Research Ethics Committee of St. Patrick's College. I have been Garda vetted through St. Patrick's College. If you have concerns about this study and wish to contact an independent person, please contact:

The Administrator,
Office of the Dean of Research and Humanities,
Room C214
St Patrick's College,
Drumcondra,
Dublin 9.
Tel +353-(0)1-884 2149

When all interviews in your child's school have been completed, I will be happy to meet with you and your child to discuss his/her engagement with the mathematics problems. If you have any questions about this project, I would be very happy to meet you. Please feel free to contact me on 086 6002823. I would be most grateful if you would complete the enclosed form indicating that you give permission for your child to participate in a task-based interview.

Thank you in anticipation

Aisling Twohill

Michael Jordan Fellow

APPENDIX D: INFORMED CONSENT FORM (PARENTS)

COLÁISTE PHÁDRAIG
ST PATRICK'S COLLEGE
DROIM CONRACH | DRUMCONDRA

Informed Consent Form

October 2014

Research Project: Assessing algebraic reasoning skills in young children

Purpose: To research whether children in Irish primary schools are developing skills in thinking mathematically, particularly in the area of algebra.

Requirements of Participation in Research Study
This project is concerned with studying children's thinking and as such will involve group interviews to see how children respond to mathematical problems and tasks. Interviews will take between forty minutes and one hour. With your consent, the consent of your child and the consent of your child's school, interviews may be videotaped and audiotaped. Your child's school has kindly agreed to facilitate this project.

During the interview, I will ask your child to solve some maths problems in a group with other children from his/her class. I will support him/her to ensure that your child does his/her best. I will not exert pressure on your child. Interviews will be carried out in an open environment which allows full visibility of the participant and researcher at all times. With your consent, I would appreciate access to your child's results in school based standardised tests in Mathematics.

Confirmation that involvement in the Research Study is voluntary
Your child does not have to take part in the project. With your consent I will meet with your child and ask him/her to take part. If your child does agree to take part, he/she can withdraw from the project at any time.

Arrangements to protect confidentiality of data,
I will use videos to aid reflections on the children's responses to the problems, and videos will be analysed and presented for research purposes only. I will try not to disclose the name of the school but cannot promise that I will be able to do this. I will not say the name of the school or give children's full names to the people who are

watching the videos. While I will not say the name of the school in any presentations, there is a small chance that the school can be identified (for example, the building or the uniform might be recognisable). In reports that I write about the project the school's name or your child's identity will not be stated. The videos will be at all times securely stored and will be password protected. From the videos I will make clips which I may use at presentations. The video clips will only be shown to people who are involved in studying the teaching of mathematics. Videos and video clips will not be available on-line. All original videos will be destroyed within ten years of recording.

Please complete the following (Circle Yes or No for each question).

Have you read or had read to you the Plain Language Statement?	***Yes / No***
Do you understand the information provided?	***Yes / No***
Do you have questions or would you like to discuss this study?	***Yes / No***
If yes, have you had an opportunity to ask questions and discuss this study?	***Yes / No / NA***
Have you received satisfactory answers to all your questions?	***Yes / No / NA***

I give permission for my child, ________________________, to take part in a task-based interview.

I give permission for the use of what my child may say or write for research purposes	***Yes / No***
I give permission for Aisling Twohill to be informed of my child's results in standardised tests in Mathematics.	***Yes / No***
I give permission for the interview with my child to be videotaped (with his/her consent) for future analysis and research purposes.	***Yes / No***
I give permission for the interview with my child to be audiotaped (with his/her consent) for future analysis and research purposes.	***Yes / No***

I have read and understood the information in this form. The researchers have answered my questions and concerns, and I have a copy of this consent form. Therefore, I consent to take part in this research project

Signed ____

Name in Block Capitals ____

Date ____

APPENDIX E: PLAIN LANGUAGE STATEMENT INCLUDING CONSENT FORM (TEACHER)

Coláiste Phádraig
St Patrick's College
Droim Conrach | Drumcondra

October 2014

Dear

As a PhD research student in St Patrick's College I am researching whether children in Irish primary schools are developing skills in algebraic reasoning. This project is concerned with studying children's thinking and as such will involve group interviews to see how children respond to mathematical tasks. Interviews will take between twenty and forty minutes. With the consent of the school, the participants and the participants' parents, interviews may be videotaped and audiotaped. With your consent and that of the children's parents, I would appreciate access to the standardised test results of the children who are participating.

I will use videos to aid reflections on the children's responses to the tasks and videos will be analysed and presented for research purposes only. I will try not to disclose the name of the school but cannot promise that I will be able to do this. While I will not say the name of the school in any presentations, there is a small chance that the school can be identified (for example, the building or the uniform might be recognisable). The videos will be at all times securely stored and will be password protected. From the videos I will make clips which I may use at presentations. The video clips will only be shown to people who are involved in studying the teaching of mathematics. Videos and video clips will not be available on-line.

I will not say the name of the school or give children's full names to the people who are watching the videos. Direct comparisons will not be made between classes or class groups and teachers' identifications will at all times be confidential. If there is a need to refer to a teacher in my thesis or in a presentation, a pseudonym will be used and I will ensure that the teacher cannot be identified through connection with a school.

This research project has been reviewed and approved by the Research Ethics Committee of St. Patrick's College. I have been Garda vetted through St. Patrick's College. If you have concerns about this study and wish to contact an independent person, please contact:

The Administrator,
Office of the Dean of Research and Humanities,
Room C214

St Patrick's College,
Drumcondra,
Dublin 9.
Tel +353-(0)1-884 2149

When all interviews have been completed, I will meet with you, if required, to discuss the children's engagement with the interviews. I will also discuss with you some ideas and resources which could support the school's work in the area of mathematics.

Children do not have to take part in the project. With the consent of the school management, I will approach parents and children and obtain their informed consent before commencing the research. Every effort will be made to ensure that participation will be a positive experience for child, teacher, and school.

Thank you in anticipation

Aisling Twohill

Michael Jordan Fellow

St. Patrick's College, Drumcondra.

……………………………………………………………………………………

I give permission for children in the class I am teaching to participate in a project entitled 'The Development of Algebraic Reasoning in Irish Primary Schools'.

Yes No

I give permission for Aisling Twohill to be informed of the standardised test results of the children in my class who are participating in the research.

Yes No

Signed ______________________________________

APPENDIX F: PLAIN LANGUAGE STATEMENT INCLUDING CONSENT FORM (PRINCIPAL/CHAIRPERSON OF THE SCHOOL BOARD OF MANAGEMENT)

COLÁISTE PHÁDRAIG
ST PATRICK'S COLLEGE
DROIM CONRACH | DRUMCONDRA

October 2014

Dear Principal/Chairperson [one addressee omitted as necessary]

As a PhD research student in St Patrick's College I am researching whether children in Irish primary schools are developing skills in thinking mathematically, particularly in the area of algebra. This project is concerned with studying children's thinking and as such will involve group interviews to see how children respond to mathematical problems and tasks. Interviews will take between twenty and forty minutes. With the consent of the school, the participants and the participants' parents, interviews may be videotaped and audiotaped. With your consent and that of the children's parents, I would appreciate access to the standardised test results of the children who are participating.

I will use videos to aid reflections on the children's responses to the maths problems and videos will be analysed and presented for research purposes only. I will try not to disclose the name of the school but cannot promise that I will be able to do this. While I will not say the name of the school in any presentations, there is a small chance that the school can be identified (for example, the building or the uniform might be recognisable). I will not say the name of the school or give children's real names to the people who are watching the videos. The videos will be at all times securely stored and will be password protected. From the videos I will make clips which I may use at presentations. The video clips will only be shown to people who are involved in studying the teaching of mathematics. Videos and video clips will not be available on-line.

This research project has been reviewed and approved by the Research Ethics Committee of St. Patrick's College. I have been Garda vetted through St. Patrick's College. If you have concerns about this study and wish to contact an independent person, please contact:

The Administrator,
Office of the Dean of Research and Humanities,
Room C214
St Patrick's College,
Drumcondra,
Dublin 9.

Tel +353-(0)1-884 2149

Children do not have to take part in the project. With your consent, I will approach parents and children and obtain their informed consent before commencing the research. Every effort will be made to ensure that participation will be a positive experience for child, teacher, and school.

I would be most grateful if you would complete the form below indicating that you give permission for the school to participate in this project.

Thank you in anticipation

Aisling Twohill

Michael Jordan Fellow

……………………………………………………………………………………

I give permission for the school ________________________ to participate in a project entitled 'The Development of Algebraic Reasoning in Irish Primary Schools'.

Yes No

I give permission for the standardised test results of the participants to be shared with Aisling Twohill

Yes No

I give permission for interviews with the participants to be videotaped, with consent from the participants and their parents/carers.

Yes No

I give permission for interviews with the participants to be audiotaped, with consent from the participants and their parents/carers.

Yes No

Signed _______________________________________

APPENDIX G: TRAJECTORIES OF EARLY PATTERNING COMPETENCIES

Repeating Patterns

In exploring how children develop thinking with regard to patterning, Threlfall (1999) explores two strands which comprise children's development in working with repeating patterns, namely increasingly complex patterns and sophistication of perception of how patterns are constructed. In considering how patterns progress in difficulty for young children, he cites Vitz and Todd (1967, 1969) who established the following order of complexity of patterning, which I will refer to as the Vitz-Todd array:

ababababab

aaabbbaaabbb

aabbaabbaabb

aabaabaabaab

aaabaaabaaab

abcbcbcabc

aaabbbcccaaa

aabbccaabbcc

acccbcccaccc

aaabcaaabc

aabcbcaabc

aabbcaabbcaa.

When assessing children's developing thinking in solving repeating patterns, it will be necessary to remain cognisant of this order of difficulty. For example, if a child successfully expands a pattern of the form ABAB, but fails in the expansion of AABAAB, it may be necessary to consider AAABBBAAABBB, rather than assuming the child's patterning ability is limited to step 1 of this sequence.

Similarly Threlfall (1999) considers the immature thinking of children in relation to pattern expansion and cites Rustigan (1976) who identified the following steps in how children utilise prior elements of a pattern in order to produce the subsequent element: 1) a random choice with no reference to prior elements; 2) repetition of the last element; 3) incorrect order of prior elements; 4) a symmetrical reproduction of the sequence; 5) "a deliberate continuation of the pattern, involving glances back to the start" (p. 24). Threlfall (1999) contends that the growing sophistication of children's approaches to patterning indicates development in their perception of how patterns are comprised. He suggests that "there seems to be a major developmental threshold, concerning whether or not the children are aware of the pattern as a whole being related to a unit of repeat" (p. 25). His thinking in this regard is mirrored by Radford (2011) who stresses that for activities to be algebraic, children's ability in identifying commonalities must be developed. Threlfall (1999) advises that in order for children's thinking to grow in sophistication in the identification of the unit of repeat, positive adult intervention may be productive in asking children to describe the pattern and verbalise their impression of how patterns are constructed.

Clements and Sarama (2009) advise that the first growth point encountered by very young children is to recognise patterns. At early stages children may have an inaccurate understanding of the term, incorrectly including decorative designs which do not incorporate a repeating unit. From this point, children should progress to a stricter understanding of "pattern" as a sequence wherein there is repetition of a shape, colour or other item (ibid.). The NSW Continuum of key ideas in Maths Curriculum advocates that children would "recognise, describe, create and continue repeating patterns," continue increasing and decreasing patterns plus "use the term "is the same as" to describe equality of groups" (Early Stage 1, Patterns and Algebra). The Northern Territory Curriculum Framework (NTCF hereafter) advises that children should count elements of a pattern to help in identifying and repeating patterns. Counting may be a more efficient strategy of identifying the required number within an item and I would suggest that children may progress from using a one-to-one matching strategy to counting as their mathematical thinking develops. The following framework of growth-points is based upon the patterning trajectory of Clements and Sarama (2009a) and incorporates the counting strategy as suggested by the NTCF.

It is pertinent at this point to introduce suggestions from Garrick, Threlfall and Orton (1999) that the assumed trajectory wherein children copy patterns before learning to create their own is not indicative of how children learn. Garrick et al. contend that there is no substantive evidence in research that children need to receive patterns from a locus of control before devising their own. They point rather to natural inclination of some children to arrange many everyday items into patterns from a very early age. I have personally witnessed such activities with two of my own children, where at two years of age my daughter routinely arranged books or envelopes across the floor in patterns of clear, straight rows, and my son at three arranged pegs or blocks into lines of colour. Both children engaged in these activities spontaneously without any direction or without the necessity of being "taught" about patterning. Similarly, in the research of Garrick et al. children could be seen to develop patterning skills through "self-directed investigations" when they were provided with appropriate play materials and without the need for adult modelling (p. 16). The research does show that some children need support in developing skills in patterning, and often take direction from the more eager patterners in their peer group. A belief that children's learning trajectory in patterning commences with duplication "seems from the research evidence to be misleading, and to miss the potential that exists" (p. 16). That said, for the purposes of this learning trajectory which is being devised in order to frame an assessment instrument, Garrick et al. do agree that while children are capable of producing their own patterns without the preliminary step of duplicating others', they do experience a challenge in recognising and copying completed patterns with which they are presented. In contrast, Aubrey (1993) conducted research with 16 children in their first term of school, with a mean age of four years and six months. In her research, Aubrey found that of the 16 children none could produce a simple regular pattern when provided with suitable materials. Fewer than half of the children involved could copy or extend either a pattern of the form ABAB or ABCABC, but in this research some children did succeed to copy and extend who could not, when prompted, create a pattern. The Level 1 framework of growth points in patterning commences with a preliminary step of creating a pattern before the child is asked to attend to a pattern produced by the researcher, but there is a possibility that this element of the framework will require review after piloting.

Create a pattern with two colours;

Fills in missing elements of an ABAB pattern;

Duplicate an ABAB pattern (may have to work close to the original pattern);

Extend an ABAB pattern;

Duplicates a repeating pattern of a form other than ABAB, with the pattern alongside, e.g. AABAAB;

Duplicates a non-ABAB repeating pattern, with the pattern at a remove from where the child is working;

Extends a simple repeating pattern;

Isolate the smallest unit in a repeating pattern (ibid.).

From this point, Clements and Sarama (2009a) progress onto growing patterns and suggest that children should "observe, copy and create patterns that grow," focusing in particular on patterns of growing squares and triangles, "noting the geometrical and numerical patterns that they embody" (p. 198). I wonder whether this would in fact be beyond the conceptual reach of children with heretofore a relatively limited appreciation of number patterns. In my experience, the numerical pattern underlying square and triangular numbers prove difficult, even for children with advanced addition skills. Successive square numbers are achieved either by squaring the position in the sequence or by summing consecutive odd numbers, for example the 2nd square number is $2^2=4$, the sum of 1 and 3. The third square number is 9, which is obtained by squaring 3, summing 1, 3 and 5 or by adding 5 to 4. Assuming that the repeating pattern work as identified within the first level of the patterning trajectory would apply most likely to children in the age range 4 to 7, it is unlikely that this numeric patterning would be the most natural next stage in a developmental pathway.

Arithmetic Progressions: Concrete, Figural and Numeric

Progressing from repeating patterns, children could be facilitated in considering patterns which either increase or decrease by repeated addition or subtraction of a constant (Clements and Sarama, 2009a, NTCF). In this section, I will identify such patterns as 'arithmetic progressions' and physical arithmetic progressions are those where the children are working with concrete objects and have not yet encountered the numeric representations. The NTCF advocates that children should recognise and continue physical arithmetic progressions, express progressions numerically, express numeric progressions physically, apply simple rules to generate patterns and continue and complete numeric arithmetic progressions.

Preliminary growth points in concrete arithmetic progressions

1. Fill in missing elements of a physical arithmetic progression, e.g. A, AA, AAA, AAAA.

2. Duplicate a physical arithmetic progression.

3. Extend a physical arithmetic progression.

4. Describe a growing pattern: describe a term in relation to its position in the pattern, e.g. respond correctly to the question – why is there five counters here?

5. Generate a pattern from a set of instructions.

6. Express patterns as a number sequence by counting elements (NTCF);

7. Express a number sequence as a physical pattern;

8. Generate number patterns;

9. Describe a far term in a growing pattern.

Preliminary growth points in numeric patterns

1. Count forwards and backwards by ones within 20.

2. Extend a linear number pattern, that increases or decreases by twos.

3. Extend a linear number pattern, that increases or decreases by 3, 5, or 10.

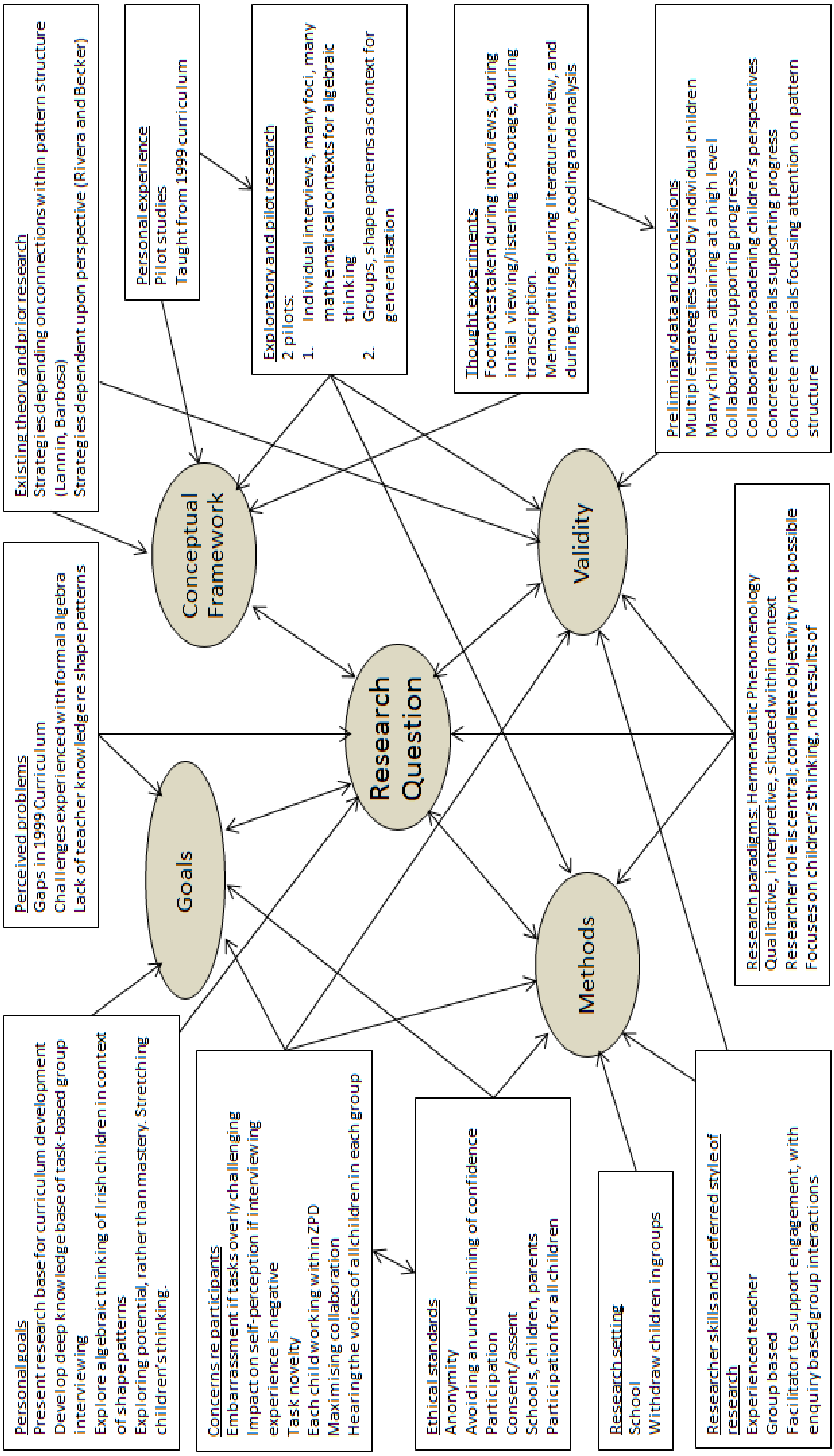
4. Use everyday language to describe patterns e.g. "add 3 each time."

5. Represent a given number sequence using manipulatives.

6. Discover missing numbers in familiar number patterns.

7. Recognise odd and even numbers.

APPENDIX H: UNDERPINNINGS OF THE RESEARCH DESIGN



APPENDIX I: EXAMPLES OF QUESTIONS ASKED DURING THE TASK-BASED GROUP INTERVIEWS

School 1, Group 1, Pattern 1

Now, what terms have you together there?

just pause for a second there, look back at the 12th term. The 12th term is on the table in front of you. The 12th term, how many tiles were there?

Ciaran explain to everyone why you're adding 86 and 86

can everybody describe for me what the 86th term would look like? [1030]

and Fiona, how many tiles would there be along the top row of the 86th term?

could you work it out?

School 1, Group2, Pattern 2

Now Arina why? Why 21 for the tenth term?

OK. What about the 75th term? [30:20]

Why 75 plus 75, Jay?

And why would you add 75 plus 75?

OK, Alex, why? Look back at the terms at the top of the page and tell me why… what will you need to do to work out how many tiles are needed with seventy fifth term?

In term four there's 2 on each leg. On term three there was 1 here and 2 down here. Term two there's 1 on each leg. So for term six can you tell me again how many, Cherry would you tell me how many again, how many diamonds would be on each leg of term six? [37:15]

OK. And have you any ideas what numbers they would be? Can you explain to me, Alex, how the tenth term is made up? What would the tenth term look like? And why?

OK. Very interesting. Right for the… now I'm not too worried about the total number for the 75th term, I'm more interested in how you're thinking about it. Arina, tell me how you thought about the 75th term? What it would look like?

OK. And Cherry, do you think… would you agree with that?

So if term four has 2 on this leg do you think that term 75 will have 75 on that leg?

No, why not?

OK. Now, Alex can you use that same thinking to tell me about the 75th term? What would the 75th term look like?

That's OK, they're very new, different kind of questions to what you usually come across, aren't they? OK, Arina, what do you think?

I can help you that then, I can tell you that half of 75 is 37 and a half. Hopefully that helps you to think about it.

OK, so describe term 74 to me.

Well Arina, you know, the number, the answer doesn't really matter, I'm just interested in how you are thinking about it. So you are adding 37 four times, why is that?

OK. Any more ideas? Will we take a look at the last two questions? Right, finish this sentence: to fit into this pattern a term must, what?

And does that ever change?

Always symmetrical? Which way is it symmetrical?

And is there always? OK so you're saying that there's a line of symmetry here, so this will follow here, is there always a line of symmetry there?

OK. Arina, anything else you wanted to say?

Any connection? Cherry, can you see there? Cherry any ideas what a term would have to have to fit into this pattern? You can think about it for another few minutes, I don't want to put you under pressure because there's no need for it, we don't have anything we need to have done, I just want to hear what you think. So can you talk to me about a connection between the term number and the term? Now Jay keep looking here, keep looking at the terms, because everybody might find a different connection, everybody might see a different thing. A connection between the number and the term, and the number and the term, and the number and the term. Would you like your own copy, Jay, to look at? You look in with Alex.

APPENDIX J: NARROWING THE FOCUS TO FOURTH CLASS

In preparing for my pilot interview, I decided to include tasks which would be appropriate for children from Junior Infants to Sixth Class and possibly beyond. Some of the tasks I included had been used with middle school students from 6th to 8th grades (Knuth, Alibali, McNeil, Weinburg and Stephens, 2011; Lannin, 2005). I included tasks where participants were required to copy, and extend repeating patterns. I also included visual spatial patterns which I asked participants to generalise, and context-based patterns from which I asked participants to devise an expression. I included computations where I asked participants questions to assess their inclination to think algebraically in arriving at an answer and tasks designed to assess participants' success in dealing with variables (Knuth et al., 2011; Radford, 2011; Warren and Cooper, 2008; Lannin, 2005; Carpenter, Franke and Levi, 2003). Such a spread of tasks created difficulties in terms of interview timing and also depth of research.

The field of research in repeating patterns and early mathematical activities merits research in its own right, as does the area of equivalence and that of variable understanding and use. It will lead to far more in-depth research and analysis if I narrow my focus to the area of generalisation of visual spatial patterns, as discussed in my introduction and literature review. Both during the interviewing and when reviewing the data from my pilot, I felt that the visual spatial patterns facilitated discussion about generalisations in a manner that was accessible to most participants. Many participants seemed engaged by the visual spatial patterns, and described them as "like a puzzle", and "way different". Equally, the visual spatial patterns proved to be challenging and many participants struggled to extend the patterns. As such they present an opportunity for interesting questions about what the children see and how they design a subsequent term to those presented.

In administering the clinical interview during my pilot research, I aimed to keep the interviews shorter than 25 minutes for Junior Infants, 35 minutes for First class and 45 minutes for all others. I exceeded my planned time limit in ten of the fifteen interviews and in all interviews I felt time pressure exerting influence on the questions I asked. I felt that I did not have sufficient time to ask participants how they thought about an answer. I did not want to interrupt the interview in order to explore an interesting response and I did not succeed in progressing to some sections with some participants. In my main research, therefore, I will use my pilot transcripts and video recordings to gauge how long participants may take on a given task, and aim to allow time to discuss children's thinking with them in as much detail as possible.

Participants' Expressive Language

Zevenbergen (2000) states that there is a "particularized vocabulary" associated with mathematics (p. 205) and gives many examples of ways in which this specific vocabulary presents a challenge to learners of mathematics. In carrying out the pilot research, I found it quite striking how articulate some children were and how much some children struggled to express themselves, or were disinclined to do so. Specifically in working with patterns, Frobisher and Threlfall (1999) discuss children's "limited powers of description" in engaging in patterning tasks and caution that children's understanding of patterns may be poorly represented by their ability to articulate descriptions of patterns. One striking example is Nikki, a participant from Fourth class. Nikki used the expression "I don't know" on twelve occasions throughout the clinical interview, while succeeding well with almost all tasks, and replied "I don't really know anything about it" when asked what she could say about the pattern 1, 2, 4, 8, 16. In choosing to focus my research on a senior class, I aim to minimise the implications to my findings of children's difficulties in expressing their mathematical ideas. While some children in Fourth class will undoubtedly struggle to express their thinking clearly, the proportion of children among the sample should be less than if children from junior classes were also included.

APPENDIX K: DETAILS OF THE CHILDREN WHO TOOK PART IN THE TASK-BASED GROUP INTERVIEWS

Pseudonym	*Gen -der*	*Age on day of interview*	*Recent Standardised Assessment Result Mathematics*	*Recent Standardised Assessment Result Algebra/ Reasoning*	*In receipt of school-based support for Literacy*	*Maths Stream*	*E A L*
Ciaran	m	9.34	70	48		Top	
Grace	f	9.82	Did not assent to sharing the results of tests				
Daniel	m	9.69	61	41		Middle	
Fiona	f	10.01	new	new		Lower	
Arina	f	9.86	42	34		Middle	
Cherry	f	9.42	new	new		Middle	
Jay	m	9.69	12	21	Yes	Lower	
Alex	m	9.83	39	33		Middle	
Amber	f	9.79	55	41		Middle	
Billy	m	9.83	99	67		Top	
Willis	m	10.16	96	62		Top	
Jamie	f	9.91	68	44		Middle	
BillyBob	m	9.41	87	56		Top	
Alice	f	9.75	99	73		Top	
Claire	f	9.91	99	68		Top	
James	m	9.85	82	53		Top	
Orla	f	8.84	25	29	Yes	Lower	Y
Luigi	m	9.77	27	28	Yes	Lower	
Emily	f	10.21	99	69		Top	
Wyatt	m	9.63	97	63		Top	
Kay	f	10.20	90	59		Top	
Adam	m	10.43	98	66		Top	
Tom	m	10.07	54	87		Middle	
Lucy	f	10.12	55	39		Middle	

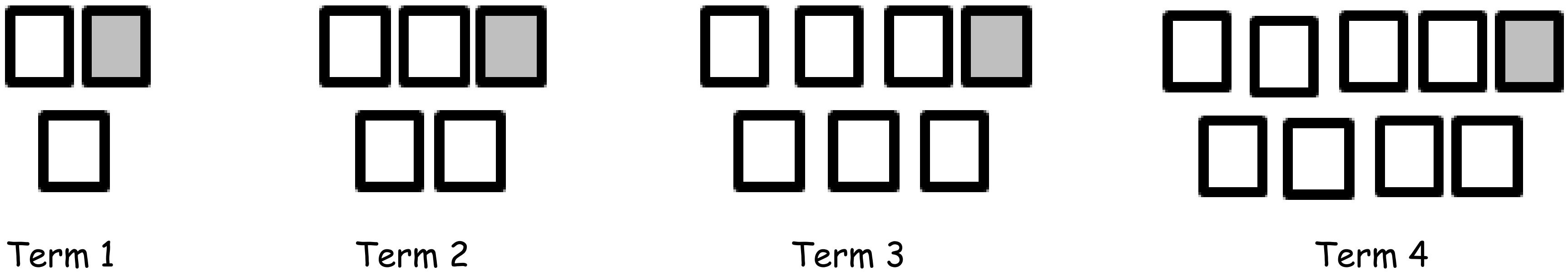
James	m	9.59	70	48		Top	
Stephen	m	10.54	90	81		Top	
Shane	f	9.74	27	28	Yes	Lower	Y
Simon	m	9.92	23	67			
Jane	f	9.75	55	50			
Courtney	f	10.00	68	67			
Robert	f	10.58	32	67			
Lloyd	m	9.75	1	33	Yes		
Desmond	m	9.75	92	67			Y
Melody	f	9.58	63	83			
Angelina	f	9.92	81	100			
Christopher	m	9.25	53	100			
Danny	m	9.50	23	17			
Jane	f	9.75	73	67			Y
Lily Rose	f	9.42	66	67			
Trip	m	10.25	47	33			
Peace	f	9.67	42	50			
Precious	f	9.75	27	33			

Notes:

Children in School 1 were streamed for mathematics lessons, and the stream to which they had been assigned is indicated in Table 3.3. Children in School 2 were not streamed for mathematics lessons.

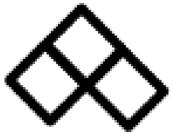
APPENDIX L: THE WORKSHEETS PRESENTED TO THE CHILDREN

Group Task Sheet 1

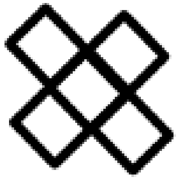


1. Talk to each other about this pattern. Try to mention everything you notice about it.
2. Make the 5th term from the tiles on the table.
3. Draw the 6th term.
4. Describe the 12th term.
5. Construct the 12th term from tiles.
6. How many tiles are needed for the 12th term?
7. Describe the 86th term.
8. How many tiles would be needed for the 86th term?
9. Finish this sentence: To fit into this pattern, a term must……
10. Can you describe a connection between the term number and each term?

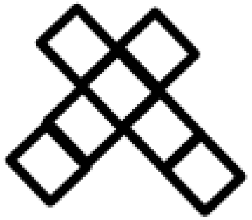
<u>Group Task Sheet 2</u>



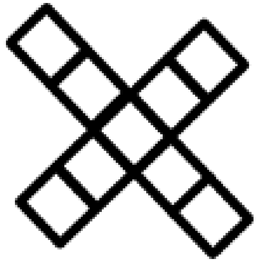
Term 1



Term 2



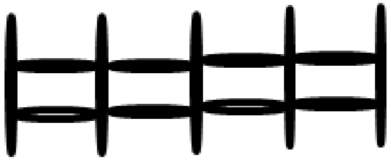
Term 3



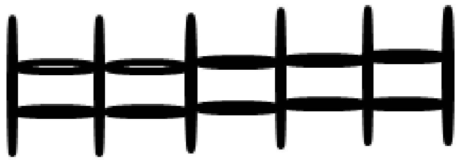
Term 4

1. Talk to each other about this pattern. Try to mention everything you notice about it.
2. Make the 5th term from the tiles on the table.
3. Draw the 6th term.
4. Describe the 10th term. How many tiles would be needed?
5. Construct the 10th term from tiles.
6. Describe the 75th term. How many tiles would be needed?
7. Sketch the 75th term.
8. Finish this sentence: To fit into this pattern, a term must……
9. Can you describe a connection between the term number and each term?

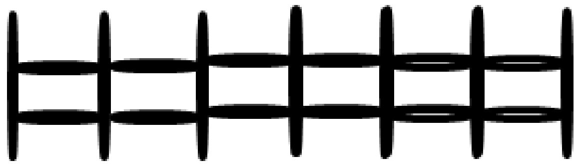
Group Task Sheet 3



4-panel fence



5-panel fence



6-panel fence

1. We have a sequence of fences. Talk to each other about this pattern. Try to mention everything you notice about it.
2. Construct the 3-panel fence. How many posts are needed?
3. Construct/draw the 7-panel fence. How many posts are needed?
4. Describe the 9-panel fence.
5. How many posts are needed to build the 9-panel fence?
6. Describe the 56-panel fence.
7. How many posts are needed to build the 56-panel fence?
8. Tell me a rule to describe the number of posts needed for any fence.

APPENDIX M: GROWTH-POINT INFORMAL ASSESSMENT SPREADSHEET

Name: ________________________________ Date:_______________ Time: ____________________

Growth Point	Characteristics
GP 0:Pre-formal pattern	Does not have a formal understanding of "pattern." Cannot identify a repeating term in a pattern.
GP 1: Informal pattern	Overview: Can identify a commonality and demonstrate understanding of pattern by copying, extending, inputting missing term, in visual spatial, numeric, repeating patterns. GP1.1: Can copy pattern of the form ABAB GP1.2: Can extend pattern of the form ABAB. GP1.3: Can extend pattern of the form AABBCAABBC and recognise that patterns contain a unit of repeat GP1.5: Can extend a repeating visual spatial pattern.

Strategy Description

Counting (C)		Drawing a figure and counting the desired elements.
Whole-object	No adjustment (W_1)	Considering a term of the sequence as unit and using multiples of that unit.
	Numeric adjustment(W_2)	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on numeric properties.
	Visual adjustment(W_3)	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on the context of the problem.
Difference	Recursive (D_1)	Extending the sequence using the common difference, building on previous terms.
	Rate - no adjustment(D_2)	Using the common difference as a multiplying factor without proceeding to a final

		adjustment.
	Rate - adjustment(D_3)	Using the common difference as a multiplying factor and proceeding to an adjustment of the result
Explicit (E)		Discovering a rule, based on the context of the problem, that allows the immediate calculation of any output value given the correspondent input value
Guess and check (GC)		Guessing a rule by trying multiple input values to check its' validity

Growth Point	Characteristics	2x+1	Diam	Fences
GP 2: Formal pattern	Overview: Can describe a growing pattern verbally. GP 2.1 Can identify next term in a growing visual spatial pattern maintaining shape or quantity but not both. GP 2.2 Can identify next term in a growing visual spatial pattern maintaining both shape and quantity.			
GP 3: Generalisation	Overview: Can correctly identify a near term. Can describe a pattern explicitly. Can offer a possible far term with reasoning. GP 3.1 Counting GP 3.2 Whole-object no adjustment GP 3.3 Numeric adjustment GP 3.4 Visual adjustment GP 3.5 Recursive GP 3.6 Rate – no adjustment GP 3.7 Rate – adjustment GP 3.5 Explicit DG GP 3.6 Explicit CNG GP 3.7 Explicit CG			
GP 4: Abstract generalisation	Can describe a pattern explicitly, describe the rule as an expression in symbolic notation and utilise the expression in order to generate a far term.			

APPENDIX N: SKIMA 5-POINT SCORING SCALE

General coding principles	*Score*
Full solution with convincing and rigorous explanations (not necessarily using algebra)	4
Correct solution with small errors, explanation acceptable but not completely convincing	3
Correct in parts, incorrect in parts	2
Partial and incorrect solution	1
Barely attempted, no progress towards a solution	0

APPENDIX O: AN EXCERPT FROM THE INTERVIEW OF GROUP 2 IN SCHOOL 1 WITH THE CODES I ASSIGNED TO EACH CHILD'S COMMENT

Facilitator S2G3
Pattern 1 Far Generalisation
OK. What do you think Danny?

Danny
Pattern 1 Far Generalisation
That the top one is bigger than the bottom.

Facilitator S2G3
Pattern 1 Far Generalisation
OK great. I think I might show you the next pattern because you worked through that one very quickly. Ehm, could you give me a hand, give me back the pattern sheets and put the tiles back into the boxes and get ready for the next one.

Facilitator S2G3
Pattern 2 Description
Now take a look and again the first thing is describe everything that you can see out nice and loud to each other? [15:09] [Children talking about the pattern, Jane can be heard counting]

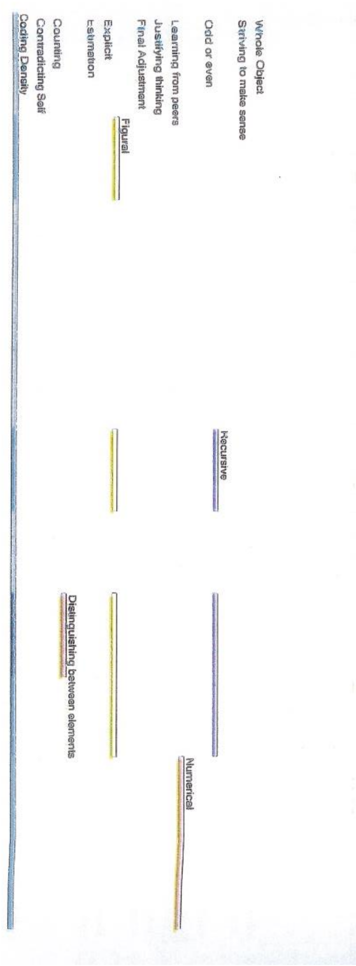
Christopher
Pattern 2 Description
It's growing that and that, [..inaudible..] another one here and then another one here.

Lily Rose
Pattern 2 Description
It's growing bigger and bigger

Christopher
Pattern 2 Description
First it grows that side then it grows that side

Jane
Pattern 2 Description
The first one has a row here and here and the other one adds two [0:15:53.7].

Christopher
Pattern 2 Description
That means three, five, seven, nine.



Jane

Pattern 2 Description

[...] and then it grows here

Danny

Pattern 2 Description

2 to the top, 2 to the bottom, 2 to the top and then 2 to the bottom

Facilitator S2G3

Pattern 2 Description

OK so would anybody like to tell me out nice and loud what they noticed?

Danny

Pattern 2 Description

It's two to the top and two to the bottom.

Facilitator S2G3

Pattern 2 Description

What do you mean?

Danny

Pattern 2 Description

Like there is two squares in the top and two squares in the bottom.

Lily Rose

Pattern 2 Description

I get it now

Facilitator S2G3

Pattern 2 Description

OK great. Christopher?

Christopher

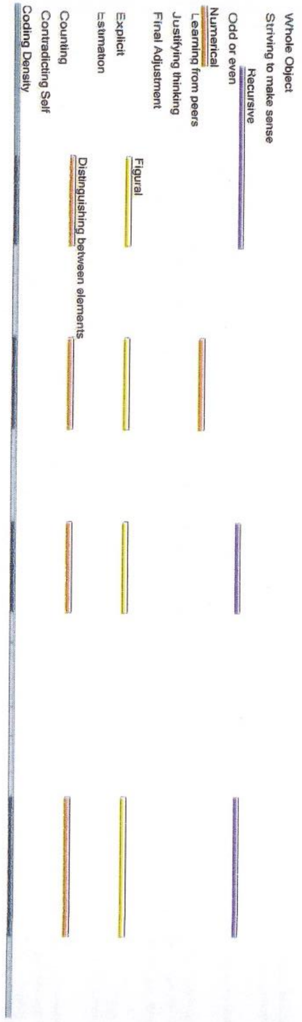
Pattern 2 Description

It's like going up so first it goes that and then the two blocks is and term two it goes there and then for term three you add another one the same again on the other four posts, on this side and then I think that that term 5 that it's going to add onto the right side [0:16:48.9].

Facilitator S2G3

Pattern 2 Description

OK great. What do you think Beatrice, anything different?



Jane

Pattern 2 Description

I think that like it adds two to the top and then like two to the bottom and then two to the top and then the next one probably will turn two to the bottom.

Facilitator S2G3

Pattern 2 Description

Oh great and Lily Rose anything different, did you notice anything else

Lily Rose

Pattern 2 Description

Ehm [followed by pause]

Facilitator S2G3

Pattern 2 Description

? Everybody else has mentioned a lot so don't feel like you have to say anything different but if there is anything I'd like to hear it.

Lily Rose

Pattern 2 Description

Ehm, I think you're just making a bigger times sign every time you just keep trying to make a big ehm kind of x thing bigger and bigger.

Facilitator S2G3

Pattern 2 Extension

OK great. OK and now the very same as last time I'm going to ask you to make term five so that we use the tiles to make the fifth term in this pattern [17:49].

Pattern 2 Extension

[children create terms, you can hear discussions of colours]

Christopher

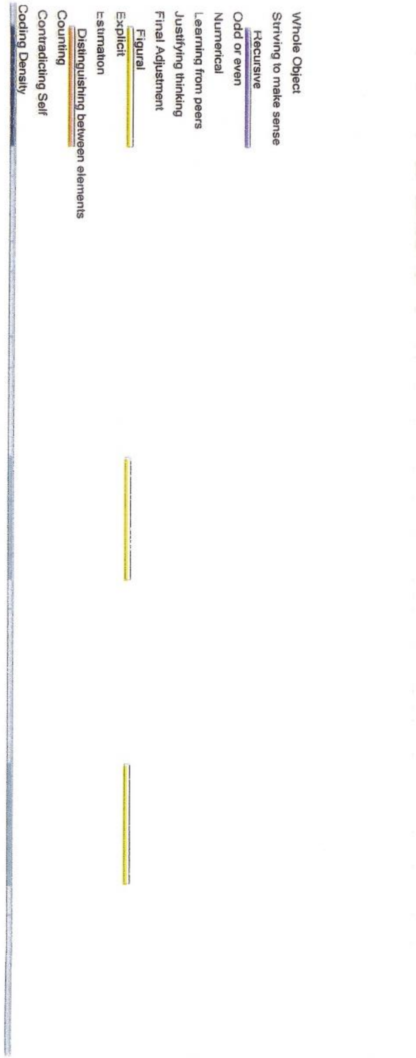
Pattern 2 Extension

This here then another one up [0:18:08.6]... then one down here, then another one on here, and one...another one on top and then another one on the bottom.

Lily Rose

Pattern 2 Extension

This one?



APPENDIX P: MEMOS GENERATED DURING TRANSCRIPTION, REVIEW OF TRANSCRIPTS AND PRELIMINARY ANALYSIS

- Concrete materials and cognition
- Radford (2010) Algebraic thinking from a cultural semiotic perspective
- Embodied and embedded cognition
- Does physical construction support children in seeing constituent parts, and thus classify elements as standalone quantities?
- Discuss agency due to interpersonal characterisitcs – Alex and Arina for example, Orla’s can I say something? Mueller Yankelewitz and Maher
- Compare tendencies of children to self-correct, or not with Maher’s structure, and discuss implications of absence of group work.
- Children did think explicitly with non-teacher catalyst, e.g. discussion and manipulatives.
- Radford vs Mason
- Also focus on interplay between strategies and supporting factors (catalysts)
- Which should come first: algebra/arithmetic?
- How did my theoretical lens support me?

- Strategy use gives researcher insight into observations of structure, and observations of structure support strategy adoption and collaboration broadens perspectives leading to more observations.
- One aspect of nVivo that skews stats is children's tendency to repeat themselves.
- Balance between being too open, and accepting all answers, or too inclined to lead If a diagnostic interview is a formative assessment, children should be guided away from incorrect thinking, e.g. Joe when he constructed the 5th term of Radford incorrectly, could I justify spending a few minutes then to support him in exploring the structure of the terms more?
- My role as teacher/facilitator/researcher: While I aimed at all times to support student agency, and to encourage the children to think independently, it was difficult for me to not intervene. I felt a strong urge to support them in constructing general terms, and in progressing through the tasks. At times, this lead me to intervene more than I should have. As I look back at the video footage I see examples of situations where the children's thinking would have benefitted from less input from me, for eg. Give quote then from Siún's ICME paper re zooming out to zoom in, and examining own practice.
- Zooming out and zooming in

- Uses of patterns: Could discuss what each pattern encouraged children to think, and how different types of patterns could fit into a schema for work. E.g. would pattern 2 be a good pattern for multiplicative recursive thinking? Did Pattern 1 or Pattern 3 support more explicit or figural thinking? Why?
- Did any child attend to figural after prompt from other child, or from me?
- Was all inter-child motivation from modelling, rather than direction?
- Success of interview as assessment
- Mueller, Yankelewitz and Maher 2012: Define and describe different discursive practices and give ways to assess collaboration to ensure these practices are facilitated and achieved. Discuss and compare in light of low levels of group work in mathematics. Also maybe discuss authority within groups and impact on children’s thinking.
- Working towards a concensus, times when children modified their thinking in order to agree, fit in with the thinking of a peer. See Daniel and Ciaran pattern 1 explicit/recursive chapter.
- Find readings on purpose of dialogue

- **Prompts:** Do I ask more suggestive questions/ give more group guidance/ ask more individual questions in some groups than in others – does this correlate with the group succeeding in progressing through the tasks, unpicking the mathematics and constructing strong ideas?
- Presence/absence of guesses and nonsensical answers. e.g. Ciaran and Daniel in School 1, Group 1
- nVivo – references to sense making
- Limitations of research: Anne Watson re need for recursive thinking too. Jane is always explicit in her thinking. Would have been helpful to have a task where recursive was needed to see if she would struggle/adapt.; Linear presentation supports recursive thinking - should have given eg 3^{rd} and 10^{th} terms, asking children to look at a suggested 14^{th} term and to identify and justify whether it was correct/incorrect.
- If children only think recursively during warm-up stages, how will they apply explicit thinking in constructing general terms?
- If children are supported to think explicitly during warm-up stages, they then have the option to do so when constructing general terms.
- Geometric elements - Do geometric elements support children in isolating what is changing from what is staying the same?
- Back to Mason, Graham-Wilder – children need to identify relative rates of change, quantify, and express them. Do geometric elements help all children, no, cos Alex, and also Luigi.

- Overview of counting strategy - One of the challenges encountered by children in constructing terms for shape patterns results from the need to think multiplicatively (Rivera, personal communication, March 12, 2013).
- Look at Amy Ellis model. In findings consider developing a model, and critiquing the use of models.
- Re use of symbols: Rivera and Becker – "Closure in mathematical activity via the construction of a direct formula" Rivera and Becker, p. 330
- My prompts and questions
- Example on page 95 of Mercer (1999) of conversation not being educationally appropriate, and of need for knowledge to be publicly accountable.
- **Task design principles for patterning:** Question raised by Kristophe [find full name] at CERME 10. In my research maybe identify aspects of tasks for children who are too inclined to focus on recursive/numerical.

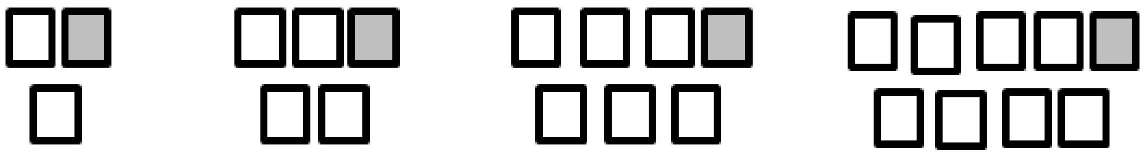
APPENDIX Q: EXAMPLE OF A TABLE GENERATED IN WORD ISOLATING UTTERANCES OF EACH CHILD RELATING TO EACH PATTERN

Jane	Pattern 2	The first one has a row here and here and the other one adds two … [inaudible]
Jane	Pattern 2	[…] and then it grows here
Jane	Pattern 2	I think that like it adds two to the top and then like two to the bottom and then two to the top and then the next one probably will turn two to the bottom.
Jane	Pattern 2	We made it all by the one block.
Jane	Pattern 2	Yeah.
Jane	Pattern 2	Yeah.
Jane	Pattern 2	Ehm, the term 5 was added to the bottom so this time I added 2 to the top.
Jane	Pattern 2	It's, ehm, all the legs are the same.
Jane	Pattern 2	Yeah.
Jane	Pattern 2	6, 7.
Jane	Pattern 2	7, 8.
Jane	Pattern 2	I think, I think there would be 5 on each leg.
Jane	Pattern 2	Yes.
Jane	Pattern 2	Well some middle square.

Jane	Pattern 2	21?
Jane	Pattern 2	‘cause the same way in term 10, it was like half of 10 was the amount of tiles on the legs. So probably half of 75 would go on the amount of the legs.
Jane	Pattern 2	I think there would be an extra one at the top on each line at the top…
Jane	Pattern 2	It must have at least sort of like the number of the term.
Jane	Pattern 2	Yeah.
Jane	Pattern 2	I think the same thing as Lily Rose said.
Jane	Pattern 2	Yeah
Jane	Pattern 2	Ehm, I think like, ehm, [pause] in one row there has to be like the number of the term and one extra

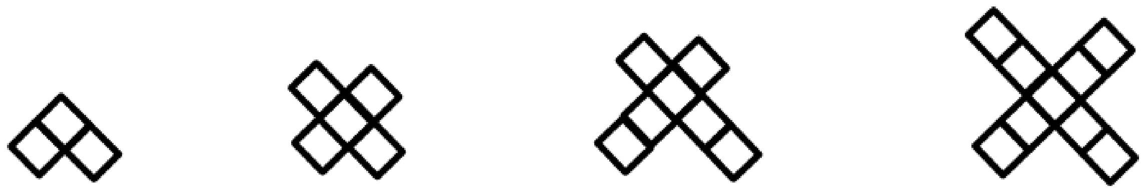
APPENDIX R: SHAPE PATTERNS PRESENTED IN TASKS

Pattern 1



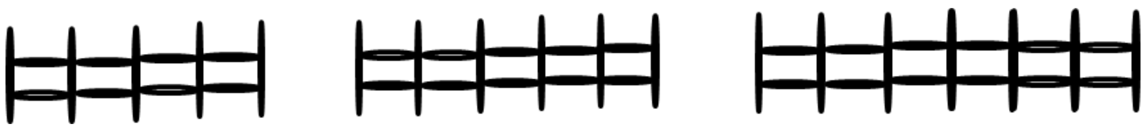
Term 1 Term 2 Term 3 Term 4

Pattern 2



Term 1 Term 2 Term 3 Term 4

Pattern 3



Term 4 Term 5 Term 6

APPENDIX S: SUMMARY OVERVIEW OF EACH CHILD'S RESPONSE TO EACH PATTERN

School 2 Group 3	*Pattern 1*	*Pattern 2*	*Pattern 3*	*Numeric/figural mode of generalisation*	*Recursive or explicit approach*
Christopher	Complete and valid 86th term.	Did not construct near generalisation. Incorrect extension	Numerically incorrect but valid reasoning.	Commented on figural aspects from description to generalisation for all patterns. Used figural aspects to isolate variables.	Explicit on the first and third patterns, and succeeded in constructing generalisations. Recursive on the second and did not succeed.
Danny	Did not construct near generalisation.	Did not construct near generalisation.	Did not construct near generalisation.	Some comments re figural aspects. Struggled to extend all patterns. Seemed to guess numbers.	Doesn't connect term to position even though discussion in group prompts him to do so.
Jane	Complete and valid, generalised for all terms.	Complete and valid, 75th term.	Complete and valid for all terms.	Figural – uses both position of diamonds/squares/posts and role of posts to support her construction of generalisations.	Initial descriptions of Pattern 1 and Pattern 2 tend to be recursive but from near generalisation of Pattern 1 also connects terms to position and generalises successfully for all patterns.
Lily Rose	Complete and valid near	Constructed far generalisation but	Complete and valid with prompting – "is	Figural to the point of ignoring numerical, e.g. "I	Recursive in descriptions but explicit in near generalisation from

	generalisation.	experience some confusion regarding the central diamond.	that all you need?" and ideas from the group.	think it would be about this big [gestures]", but estimates a reasonably suitable number.	Pattern 1 onwards.
School 1 Group 1	*Pattern 1*	*Pattern 2*	*Pattern 3*	*Numeric/figural mode of generalisation*	*Recursive or explicit approach*
Ciaran	Extended Radford correctly, thinking inhibited by obsession with odd/even	A lot less robust in thinking re pattern – could it be a circle. Lots of numbers offered, and expresses "I don't understand" regularly, but achieves good approximation in the end, and inhibited by difficulty in halving 75	Works more with Grace, and constructs correct near but incorrect far generalisations	On 1st Numerical at start, obsesses with odd/even numbers, but begins to point to diagonal. Refers to top and bottom to calculate numeric answer for near and far. Maybe gets idea to consider top and bottom from Grace. On 2nd refers regularly to legs, top, bottom, works through initial misconception re rate of change. On 3rd refers to lines going down.	Explicit on 1st, 2nd, recursive on 3rd near gen and whole object, no adjustment on 3rd far gen.

Daniel	Didn't extend independently, hazy sense of what pattern is, does it have to be an x, doesn't feel a need to conserve the shape	Difficult to say – very indefinite, seems to throw out numbers, and also "maybe you could make like one line longer and one line shorter and like the rest maybe long and medium or shorter" early on	Similar to 2nd, performs ad-hoc calculation, but then accepts Ciaran's in place of it when Ciaran suggests 179.	Refers to extra one in line, but no reference to top or bottom. No reference to shape or position	Going up in 2s, "there is no 12th one", also obsesses with odd/even. Makes no genuine attempt to generalise.
Fiona	Generalised, but not sure whether constructed independently, no improvement in VSR	Extended perfectly. Became less confident and difficult to hear with further terms	Very quiet	Difficult to determine due to disinclination to speak aloud. 2nd pattern refers to centre, and top/bottom as distinct	Difficult to determine due to disinclination to speak aloud.

Grace	Extended and generalised Radford perfectly	Succeeded in constructing near generalisation, but didn't find numeric answer – very distracting conversations.	Appears to reason recursively, and fails to generalise	1st: Figural from the start even when boys obsess with odd/even, finds it easy and confident. 2nd: Refers to symmetric/asymmetric early on, and also centre, top and bottom as distinct	Explicit on 1st. and 2nd, struggles to construct far generalisation on 2nd. Recursive in extension on 3rd, doesn't succeed in constructing far generalisation correctly
School 1 Grou p 2	*Pattern 1*	*Pattern 2*	*Pattern 3*	*Numeric/figural mode of generalisation*	*Recursive or explicit approach*
Alex	Very indistinct language at times but generalised well.	Describes the pattern well for the early terms. Seems to be thinking explicitly and figurally, but defers to Arina.	Doesn't complete any term, but describes how to extend and how to construct the 3rd. Makes correct addition to incorrect 7th to generate 9th.	Refers to top and shaded one, refers to middle one and legs, and refers to panels and posts	Whole object on near gen on 1st, ends up with 24 squares, including 4 shaded. Correct far gen, and describes well. Tries to solve Warren with recursive approach, but numbers overwhelm him, recursive on 3rd
Arina	Extended and generalised perfectly	Extended and generalised perfectly	Extended and generalised perfectly, but doesn't describe	1st: Figural, 2nd: differentiates middle square immediately, and	Explicit on al three

			why	distributes squares on sides	
Cherry	Very quiet, didn't extend correctly or offer thinking for any element of discussions. Didn't ask any questions or express confusion. More engaged with task 3 but couldn't explain thinking.			3^{rd}: differentiates extra post on last panel.	3^{rd}: Added 6 onto 7^{th} term to construct 9th
Jay	Works well through steps, but adds 86 to 85, instead of 87 to 86 at end	Doesn't engage well, extends incorrectly, becomes distracted	Doesn't engage well, begins to express an idea but tapers off	1^{st}: refers to shaded, top and bottom; Refers to shape at start of 2^{nd}.	1^{st}: draws connection between term and position no, but fails to apply it to generate future terms

School 1 Group 5	*Pattern 1*	*Pattern 2*	*Pattern 3*	*Numeric/figural mode of generalisation*	*Recursive or explicit approach*
Emily	Gets confused between whole object and explicit "The bottom number is equal to the term" "The top one is still 12" Even though she says that the bottom number is the term no, she still insists that the top row must be 12, because the top row of a previous term was 6, mistakenly taking that to be the 6th term instead of the 5th.	Perfect near generalisation, struggles with halving 75.	Generalises perfectly, and quickly	1st: refers to top and bottom from start 2nd: figural 3rd: figural	2nd: explicit 3rd: explicit
Luigi	Unclear how much of what he says is repeating other children's statements. Changes from 87+86 to 86+85 depending on whether Emily or Wyatt has spoken last	Asks others a lot of questions, and follows Emily's line of thinking, including when she makes mistakes.	Again parroting Emily. Difficult to know how much he is constructing	1st: Refers to top and bottom 2nd: Refers to top and bottom 3rd: refers to going across and horizontal	1st: begins to make connection between no of tiles on bottom row and term no, 2nd: demonstrates tendency to use term number

			himself, does not justify why he would do as he suggests		3rd: As with 2nd
Orla	Says very little, but doesn't extend correctly	Connects to template for cube. Extends correctly, and makes mistake in drawing near term but describes what it should be	Articulates very little	1st: refers to top and bottom,	
Wyatt	Constructs near and far generalisations easily,	Constructs near and far generalisations, falters a little on double counting central tile but learns from comments of others	Constructs near and far generalisations easily	Figural	Explicit

APPENDIX T: AN OVERVIEW OF THE CHILDREN'S COMMENTS CODED IN TERMS OF RELATIONSHIPS THEY IDENTIFIED WITHIN THE PATTERN STRUCTURE, WITH ACCOMPANYING FIELD NOTES

	School. Group	*Counting*	*Whole Object*	*Final Adjust-ment*	*Recursive*	*Explicit*	*Field notes from observations during interviews, and during transcription*
Ciaran	1.1	4	3	0	11	20	Explicit on Patterns 1 and 2. On Pattern 3: recursive in constructing a near generalisation and applied a 'whole object, no adjustment' approach in constructing a far generalisation.
Daniel	1.1	10	1	0	10	4	Struggled to appreciate that patterns continue beyond the terms presented: "there is no 12th one". Demonstrated recursive thinking, and, also focused too heavily on whether the total number of components is odd or even. Made no constructive progress in establishing a general term.
Fiona	1.1	1	0	0	2	1	Demonstrated explicit thinking on Pattern 1, very uncertain on Pattern 2, and contributed very little to the discussion on Pattern 3.
Grace	1.1	2	0	0	7	23	Explicit on Patterns 1 and 2, struggled to construct far generalisation on Pattern 2. Recursive in extension on Pattern 3, and didn't succeed in constructing far generalisation correctly
Alex	1.2	4	1	0	11	5	Whole object on near generalisation on Pattern 1, ended up with 24 squares, including 4 shaded. Correct far

							generalisation, and described well. Tried to solve Pattern 2 with a recursive approach, but numbers overwhelmed him, recursive on 3rd
Arina	1.2	5	0	0	4	19	Explicit on all three
Cherry	1.2	1	0	0	1	4	Very quiet, didn't extend correctly or offer thinking for any element of discussions. More engaged with task 3 but couldn't explain thinking. Correctly added 6 posts onto 7th term to construct the near generalisation of the 9th term on Pattern 3.
Jay	1.2	3	0	0	0	5	Pattern 1: drew connection between term and position number, but failed to apply it to generate future terms. Did not engage well with Patterns 2 and 3.
Emily	1.5	2	3	0	14	23	On Pattern 1 used both Whole Object and Explicit approaches. On Patterns 2 and 3 explicit thinking dominated her approach.
Luigi	1.5	14	1	0	12	13	Began to make connection between the number of tiles on the bottom row and the term no on Pattern 1. On Patterns 2 and 3 Luigi demonstrated a tendency to use the term number, but did not construct a general term.
Orla	1.5	1	0	0	0	3	Very quiet, extended Patterns 1 incorrectly, but demonstrated some indications of explicit thinking on patterns 2 and 3.

							Disinclined to voice her thinking.
Wyatt	1.5	1	0	0	13	23	Explicit on all three patterns
Christopher	2.3	0	0	0	6	9	Explicit on Patterns 1 and 3, and succeeded in constructing generalisations. Recursive on Pattern 2 and did not construct a general term.
Danny	2.3	1	0	1	5	3	Did not connect term to position for any pattern, even though discussion in group prompted him to do so.
Jane	2.3	0	0	0	5	14	Initial descriptions of Pattern 1 and Pattern 2 tended to be recursive, but from near generalisation of Pattern 1 Jane also connected terms to their position and generalised for all patterns.
Lily Rose	2.3	0	0	0	4	9	Recursive in descriptions but explicit in near generalisation from Pattern 1 onwards. Constructed general terms for Patterns 2 and 3.

APPENDIX U: AN OVERVIEW OF THE CHILDREN'S COMMENTS CODED AS DEMONSTRATING FIGURAL OR NUMERICAL THINKING

	Fig	Num	Totals per pattern	
1 : School 1 Group 1	75	19		
Ciaran	38	12	38	12
3 : Pattern 1 Description	3	3		
4 : Pattern 1 Extension	7	1		
5 : Pattern 1 Far generalisation	5	0		
6 : Pattern 1 Near generalisation	1	0		
			16	4
7 : Pattern 2 Description	0	0		
8 : Pattern 2 Extension	7	1		
9 : Pattern 2 Far generalisation	10	7		
10 : Pattern 2 Near generalisation	1	0		
			18	8
11 : Pattern 3 Description	0	0		
12 : Pattern 3 Extension	1	0		

	Ciaran		Daniel		Fiona		Grace		All	
	Fig	Num	Fig	Num	Fig	Num	Fig	Num	Fig	Num
Pattern 1	16	4	3	7	1	0	15	4	35	15
Pattern 2	18	8	18	4	6	0	16	8	58	20
Pattern 3	4	0	1	0	0	0	1	1	6	1
	38	12	22	11	7	0	32	13	99	36

		Ciaran		Daniel		Fiona		Grace	
		Fig	Num	Fig	Num	Fig	Num	Fig	Num
Pattern 1	Description	10	4	2	7	0	0	7	4
	Far generalisation	6	0	1	0	1	0	8	0
Pattern 2	Description	7	1	9	0	6	0	8	1
	Far generalisation	11	7	9	4	0	0	8	7

13 : Pattern 3 Far generalisation	2	0		
14 : Pattern 3 Near generalisation	1	0		
			4	0
Daniel	22	11	22	11
16 : Pattern 1 Description	0	7		
17 : Pattern 1 Extension	2	0		
18 : Pattern 1 Far generalisation	1	0		
19 : Pattern 1 Near generalisation	0	0		
			3	7
20 : Pattern 2 Description	4	0		
21 : Pattern 2 Extension	5	0		
22 : Pattern 2 Far generalisation	9	4		
23 : Pattern 2 Near generalisation	0	0		
			18	4
24 : Pattern 3 Description	0	0		
25 : Pattern 3 Extension	1	0		
26 : Pattern 3 Far generalisation	0	0		

Pattern 3	Description	1	0	1	0	0	0	0	0
	Far generalisation	3	0	0	0	0	0	1	1
		38	12	22	11	7	0	32	13

<u>Proportion of total comments that were figural</u>

	Ciaran	Daniel	Fiona	Grace	All
Pattern 1	0.2	0.7	0	0.21053	0.3
Pattern 2	0.3077	0.1818	0	0.33333	0.2564
Pattern 3	0	0	NC	0.5	0.1429
Total	0.24	0.3333	0	0.28889	0.2667

	Pattern 1	Pattern 2	Pattern 2	Total
Ciaran	0.2	0.3077	0	0.24

27 : Pattern 3 Near generalisation	0	0		
			1	0
Fiona	7	0	7	0
29 : Pattern 1 Description	0	0		
30 : Pattern 1 Extension	0	0		
31 : Pattern 1 Far generalisation	1	0		
32 : Pattern 1 Near generalisation	0	0		
			1	0
33 : Pattern 2 Description	4	0		
34 : Pattern 2 Extension	2	0		
35 : Pattern 2 Far generalisation	0	0		
36 : Pattern 2 Near generalisation	0	0		
			6	0
37 : Pattern 3 Description	0	0		
38 : Pattern 3 Extension	0	0		
39 : Pattern 3 Near generalisation	0	0		
Far generalisatin	0	0		
			0	0

Daniel	0.7	0.1818	0	0.33333
Fiona	0	0	NC	0
Grace	0.2105	0.3333	0.5	0.28889
Group 1	0.3	0.2564	0.143	0.26667

Grace	32	13	32	13
41 : Pattern 1 Description	1	3		
42 : Pattern 1 Extension	6	1		
43 : Pattern 1 Far generalisation	2	0		
44 : Pattern 1 Near generalisation	6	0		
			15	4
45 : Pattern 2 Description	4	1		
46 : Pattern 2 Extension	4	0		
47 : Pattern 2 Far generalisation	5	7		
48 : Pattern 2 Near generalisation	3	0		
			16	8
49 : Pattern 3 Description	0	0		
50 : Pattern 3 Extension	0	0		
51 : Pattern 3 Far generalisation	1	1		
52 : Pattern 3 Near generalisation	0	0		
			1	1

APPENDIX V: TRANSCRIPT OF GROUP 2 IN SCHOOL 1'S DISCUSSION OF PATTERN 2

AT	So take a few minutes and just look at term one, at term two, at term three and at term four and tell each other everything that you spot
Jay	Guys, guys, guys, the first one is not the x, that one's an x, but that one's kind of and that one's an x
Arina	But I think it's… it's like the… it's like the shaded one
Alex	It's like the shaded one as well
Arina	Yeah I think it's like the shaded one, you see like…
AT	OK. Don't talk about the previous pattern. Try to forget about that because that might confuse you, so just try to forget about the previous one.
Arina	OK. It's like when it's term one it's like it's in the middle square it's like the middle square is kind of like the odd one out and when it's term one there's a square here and in term two there's squares over here and in term three it's three here.
Alex	They keep adding two on instead of one this time. There's only, there's only, there's three there but then they add on two, then they add on another two, then they add on another two.
Jay	That's what I'm talking about. That's what I'm talking about. Hey, guess what….[22:31]
Arina	So it's like, just like adding two but like up here. If you take…
Arina	So in the middle, like you see that square in the middle.
Jay	That should be shaded.
Alex	[some reference to wiping memory]
Arina	When it's term four, like on term four…
Cherry	Yeah go on
Arina	When it's term 4, it's like, up here, Like 1, 2, 3, 4.
AT	Right, would you like to make term five? And again you can make one each or you can make it together, whichever you'd like.
Alex	I think I know how much that one [23:08]
Alex	1, 2, 3, 4, 5, something, 8, 9.
Alex	Wait, I know, term five would have eleven. If the fifth term [23:17].
AT	OK, That's interesting Alex
Arina	If the fifth term…if you take term 4.
Alex	If you add two 5s you add half 2 on to that. That would make 11.
Alex	Let's make a big, big, big…
Arina	If we're making term five [23:38]. Like only two up here and three up here.
Cherry	Yeah.
Arina	Yeah 1, 2, 3, 4…
Arina	OK now we need to do the x. So ehm, think the
Cherry	Oh yeah.
Alex	So that's half 11 on. We all add two on to one, we add two on to that it's 11. So half 9?
Cherry	You guys are copying off us?
Jay	No we're not.
AT	The point is that you share your ideas. So copying is perfectly allowed.
Jay	Will we be leaving when it's lunchtime?
Arina	I think we're done. Oh wait no, we're not. We need to add one more.
Alex	1, 2, 3, 4, 5, 6, 7, 8, 9. Wait, 10.
AT	Now, is this the, is this? Are you finished?
Arina	We're finished.

Alex	We're finished
Jay	Yeah.
Alex	How interesting that an adult can learn from children [27:41]
AT	Now is everybody happy with those as the fifth terms? [Children reply "yeah"]
AT	OK. Now for the sixth term would you like to draw them or make another one?
Arina	Make another one.
Jay	Draw them, draw them.
AT	It might be… it's up to you now. If you're going to draw them what I'd suggest you do is use circles instead of squares. Circles are much quicker and easier to draw. So on the face of each square use a circle. If you want to draw a square, draw a square but if you'd like to draw a circle you're very welcome to do so. Are you going to make one, Alex or are you going draw it?
Alex	I'll draw… I'll make one.
AT	OK. And, Arina, would you like to make it or draw?
Arina	Erm… make
AT	OK. Now off you go. This is your sixth term. And if you're drawing it make sure you put your name on the page.[28:31]
Arina	You're done?
Alex	Yeah I just needed two more for that. So just put, just make an x
Arina	I just put 11.
Arina	Ow, don't mess my hairstyle
Alex	It's lovely [and more about hair]
Arina	So 1, 2, 3. 1, 2, 3.
Alex	I'm done.
Jay	Stop [Alex is waving to the cameras]
Jay	There's 2 on [gesturing to cameras]
Jay	It doesn't matter, let's just ignore the cameras
Arina	I;m done mine as well
Arina	1, 2, 3. It's 1, 2, 3. 6. So… wait, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13.
Jay	OK, I think…
Alex	Because if you add two on the 12, because 12 was
Arina	It's the number
Jay	Two people draw it and then two people…
Arina	Two people what?
Alex	I used a lot of them for mine
Arina	OK.
Jay	I'm still drawing, so.
Arina	I'm done. Are you?
Cherry	Yeah, but I think….
Arina	[waves her finger in gesture indicating 'No']
Jay	I think it's actually done
Alex	Oh yeah I like the x but I think they're actually I'm doing a lot of squares.
Arina	Try to draw circles.
Alex	They think it's funny.
Alex	You have to make circles remember
Jay	Circles?
Alex	Yes, she said you have to do circles. It makes your drawing go faster.
Jay	Oh my God
Alex	If you had an x make out of circles, it would have to be 13 circles.

Jay	Good boy
Alex	Just made the sixth term.
Arina	Remember the middle is like… the middle one was not there.
Alex	Are we allowed to go on or what do we have to do?
Arina	Lesson 3
Alex	Tenth term! It's not the tenth term.
Jay	So we're on lesson 3, draw a sixth term, draw a sixth term.
Jay	75 term
Alex	What do you have?
Cherry	Look at number 7.
Alex	What?
Cherry	Look at number 7
Arina	5, 10, 15, 20.
Jay	Don't look at the camera
Arina	…20… I know question four as well.
Jay	Just be … and look around
Arina	I know
Jay	If you wanted to look at the camera, you just look around like this.
Arina	OK. 5 plus 5, 5 plus 5 will be 20, 21, yeah. I know question 4 [31:50].
AT	OK, what is question 4 then? We'll take a look at this. How many tiles will be needed? OK. So you do the sixth term. Everybody happy with the sixth term?
Jay	Yeah.
AT	And now we're going to look at the tenth term. So what I want you to do first of all is describe it, think about it, what would it look like? And then tell me about how many tiles would you need for the tenth term.
Jay	Erm…
Arina	21
AT	Now Arina why? Why 21 for the tenth term?
Arina	Because for the tenth term, like, I… for term four there's like, two and then like the middle square is kind of like I forget the square is not there and then I add these. So it's like…
Jay	I think it might be 23.
Arina	I added 5 and 5 [29:57] so 20 and then I added the middle square so it'll be 21.
AT	21, OK, lovely. Anybody else have any other ideas?
Alex	I think there might be 23.
Alex	I agree with Arina.
Alex	But I'm not sure so I agree with Arina.
AT	OK. What about the 75^{th} term? [30:20]
Jay	So 75 plus 75.
Alex	Oh yeah.
AT	Why 75 plus 75, Jay?
Jay	So you can add it on and see what's [30:40].
AT	And why would you add 75 plus 75?
Jay	To get a bigger answer?
Alex	Well we're all doing it.
Jay	Oh no, I'm just doing [30:56].
Alex	40.
AT	OK, Alex, why? Look back at the terms at the top of the page and tell me why… what will you need to do to work out how many tiles are needed with seventy fifth term?

Alex	You could add the smaller… you could add numbers that are smaller than 75 but bigger than the term ten or term 6 [34:30].
AT	OK. And have you any ideas what numbers they would be? Can you explain to me, Alex, how the tenth term is made up? What would the tenth term look like? And why?
Alex	I think it would be, like, 21 or 23.
AT	No, I don't want to hear a total number, I want you to tell me what it would look like and why?
Alex	I've taken… two added on to term nine and four added on to term eight and twelve added on to term seven. And sixteen added on to term six.
AT	OK. Very interesting. Right for the… now I'm not too worried about the total number for the 75th term, I'm more interested in how you're thinking about it. Arina, tell me how you thought about the 75th term? What it would look like?
Arina	It would look like a big x.
Jay	Very big
Arina	It would have, like, 75 on each side I think and…
AT	Now 75 legs on each diamond is it? Sorry 75 diamonds on each leg?
Arina	Like 75 here and 75 here. Like 1 middle square here and 75 here and here and here and here.
AT	OK. And Cherry, do you think… would you agree with that?
Jay	Yes.
AT	That if term four has 2 here that term 75 will have 75 here?
Cherry	Yeah.
AT	What do you think, Alex?
Alex	Erm… yes, I think I would agree with her
AT	So if term four has 2 on this leg do you think that term 75 will have 75 on that leg?
Alex	Hmm…
AT	No, why not?
Alex	Because that's term four and there isn't 4 on that term.
AT	OK. And what is there rather than four?
Alex	I think there's…
AT	OK. But I'm not asking for the total, again, we're just looking at the number on the legs. [34:01]
Alex	There's two. In term four there's 2 on each leg. On term three there was 1 here and 2 down here. Term two there's 1 on each leg. So for term six can you tell me again how many, Cherry would you tell me how many again, how many diamonds would be on each leg of term six? [37:15]
Cherry	13.
AT	OK, 13 total but I'm asking for each leg. How many diamonds would be on each leg for term six? I'm just asking you to think back over the earlier ones again to help you to think about the bigger one.
Cherry	1, 3 and then that one, 1?
AT	OK. So the legs would all be different lengths, would they? For term six? And Jay, for term five? What would be on each leg? Term five. The fifth term, the next one after term four.
Jay	This?
AT	That was your term six, I think. Term five. Alex?
Alex	I think… on each leg?
AT	Yes.

Alex	So I think there are 2 on two legs but 3 on the [35:27].

AT	OK. Now, Alex can you use that same thinking to tell me about the 75th term? What would the 75th term look like?

Alex	It would be 2 added onto the 74th term.

AT	OK. [Laughter] But what would the 74th term look like?

Jay	I don't know.

Alex	Ehmm…

AT	That's OK, they're very new, different kind of questions to what you usually come across, aren't they? OK, Arina, what do you think?

Arina	Well I made a mistake with the 75, because I got with the other pattern, I still remember that other pattern so I'm just thinking what's half of 75 so I will know what's half of 75 so to make 75, so what's half of 75

AT	I can help you that then, I can tell you that half of 75 is 37 and a half. Hopefully that helps you to think about it.

Arina	[gestures in the air using arms to indicate 2 sides]

Arina	Are we doing 75 or 74?

AT	Term 75.

Arina	Oh well I have 74.

AT	OK, so describe term 74 to me.

Arina	So like 30? I think 37, two 37s, yeah two 37s… Oh wait.

Alex	I think er;re not going to be able to fit everything on the page.

AT	Well to sketch, you can just sketch it like this. So this is whatever number and this is whatever number.

Alex	Should we still do circles?

AT	Well you can, or squares, it doesn't really matter. And just put in whatever number you think. OK…

Alex

AT	Well Arina, you know, the number, the answer doesn't really matter, I'm just interested in how you are thinking about it. So you are adding 37 four times, why is that?

Arina	Well because first I thought, so like if 37 plus 37 would be like 74 but I thought on each side but it would make 148 so I made… I needed to make something to make 74 so something at that side and that side it will be something and on another side and another side and a square in the middle makes 74 but I don't really know the number.

Alex	I don't think anybody knows this

AT	OK. Any more ideas? Will we take a look at the last two questions? Right, finish this sentence: to fit into this pattern a term must, what?

Arina	I think it must have a middle square.

AT	OK, very interesting to start with, must have a middle square.

Alex	Yeah but the…

AT	And does that ever change?

Jay	No.

Alex	Yeah if you take away the middle square and cover up the hole, it'll just be the same[39:19]

AT	Oh I know what you mean.

Alex	And then it would be the middle [inaudible]

AT	But not if you made the tiles.

Alex	Oh yeah, OK.

AT	If you take away the middle tile it'd still be there. But if you did it on the page, if I just took out that little box when I was printing this you wouldn't

	notice. Because the sides of the other boxes would make it look like there was still a box there. OK. Anything else other than just the middle square? Anything else? Or the middle diamond?
Alex	Oh I think I know. [39:56] this square right here because if you take away this square then this and this would just be [inaudible]
AT	OK. But Alex, tell me something else about to be a term in this pattern what else would a term have to have or have to be?
Alex	It would have to be a shape?
AT	OK. What kind of a shape?
Alex	symmetrical.
AT	Always symmetrical? Which way is it symmetrical?
Alex	Like the side or some side or [40:39]
AT	And is there always? OK so you're saying that there's a line of symmetry here, so this will follow here, is there always a line of symmetry there?
Alex	No.
AT	OK. Arina, anything else you wanted to say?
Arina	I think on term three and in term one it's like you're making a building and you're making kitchen floor tiles. So if you put, like, on term three it's like put one square up there and one square like that and then two squares up here so and then in term one it's like two squares at that side and one square at that side. So it won't really cover it up [44:23].
Alex	I think you should ask Cherry because she didn't talk in ages
Facilitator	Yes, she has. She told me all about the sixth term and this… Cherry any ideas what a term would have to have to fit into this pattern? You can think about it for another few minutes, I don't want to put you under pressure because there's no need for it, we don't have anything we need to have done, I just want to hear what you think. So can you talk to me about a connection between the term number and the term? Now Jay keep looking here, keep looking at the terms. A new file, because everybody might find a different connection, everybody might see a different thing. A connection between the number and the term, and the number and the term, and the number and the term. Would you like your own copy, Jay, to look at? You look in with Alex.
Alex	I can only see one connection between them and I think you won't like it.
Arina	
AT	Any connection? Cherry, can you see there?

[Conversation peters out and I ask children to move onto the next pattern].

APPENDIX W: AN EXCERPT FROM FIELD-NOTES TAKEN DURING THE SECOND PILOT STUDY

1. 2nd group far less chatty than first, far less co-construction, more input needed from me to encourage children to work through tasks. Looked back through footage to see how I had introduced the work. In the video of the first group, I emphasised that they would be working together far more and I used my introduction from the first with other groups.

2. Individual worksheets encouraged children to work individually, seemed to reduce interaction between children and placed emphasis on filling in rather than talking.

3. Needed multiple copy of group worksheet, all children wanted to read the questions for themselves.

4. In Radford diamonds – put even numbered near generalisation to avoid confusion with 11th term having a total of 11 diamonds and odd numbered far term.

    a. Changed both near and far terms to even numbered – there is no significance of odd-numbered terms with this pattern. Tried to avoid multiples, i.e. did not choose 10 because children have 5th in front of them and could be led to count and multiply. Chose 12 because 14 seems to be getting too far from the terms that the children have constructed. Even though it is a multiple of 6 and the 6th is in front of them, they may be less likely to quantify the 6th than the 5th because it is larger. Also, if a child builds recursively from the 5th to create the 6th, they may need to work from the 6th to construct the 12th, rather than building all answers on the 5th.

5. Some groups don't talk to each other as much. Need 1 sheet and 1 pencil?
6. Shauna: "I'm just going with my answer" – are there tips in research to encourage kids to self-correct, improve their own thinking/work?
7. Read Rivera to see how he asked them to describe the general term what words did he use:
   a. He didn't ask children to describe a general term in the interview schedules he shared with me. Check papers. 2011: He writes – "find a direct formula" – not appropriate for 4th class in an Irish primary school because they haven't encountered formulae yet.
8. Warren (2008) asked children to "write the general rule" but again Irish children would probably have no meaning for 'the general rule'. In analysis, Warren looked for when children connected term to number. 2 new questions added to group worksheet:
9. Finish this sentence: To fit into this pattern, a term must……
10. Can you describe a connection between the term number and each term?
11. Read Gaye Williams to see how to encourage children to always improve, constantly self-improve own thinking. Also Moss and Beatty and Goos and Galbraith. See Group interactions doc.
12. Descriptions are very limited and tend to be "it adds one every time" or derivatives thereof. Change the question to include all kinds of information spotting.
13. Need to maximise interactions among the group.
14. Read Webb (1991) "task-related verbal interaction and mathematics learning in small groups"