Hyperasymptotic solutions of differential equations with singularities of arbitrary integer rank

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I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Ph.D. is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Abstract

We develop hyperasymptotic expansions for two general classes of homogeneous differential equations with an irregular singularity of arbitrary integer rank at infinity. The first is the general second-order differential equation. The second extends this result to a restricted case of a higher order linear differential equation. In both of these cases the result is expressed in terms of certain hyperterminant integrals which are generalisations of those found in other papers in the case of a rank one singularity.

Contents

Chapter 1

Introduction

This thesis consists of two related works. The second chapter is concerned with the computation of *hyperasymptotics* for solutions of a general second- order linear ordinary differential equation with a high rank irregular singularity at infinity. In the third chapter we calculate hyperasymptotics for high order differential equations (greater order than two) which have some restrictions applied. For any differential equation with a higher order than two there is the possibility of having solutions of mixed exponential rank. Exponential rank is defined for our purposes as the order of the polynomial in the leading (exponential) behaviour of the solution as it tends towards the (irregular) singularity. The hyperasymptotic results in the third chapter are restricted to the case of single rank problems.

The word *hyperasymptotics* was first used by Berry and Howls [6] to indicate an improvement beyond *superasymptotics*. To understand these terms we need to understand the definition of Poincaré asymptotics [4, 19]. Poincaré's definition of the asymptotic relation $\sim$ (is asymptotic to) is that if we have a function $f(z)$ and a power series $\sum_{s=0}^{\infty} a_s z^{-s}$ (possibly divergent) where a_s are some complex numbers, then

$$\left[f(z) - \sum_{s=0}^{n} a_s z^{-s} \right] = \mathcal{O}(z^{-n-1}) \ , \text{ as } z \to \infty \ . \tag{1.1}$$

If we use the asymptotic series as an approximation to the original function then the error is of polynomial order. A convergent expansion satisfies the Poincaré definition of asymptotics but the properties of convergent series are well studied and for the purposes of asymptotics in general, not interesting. If we look at the properties of the error for a series which is divergent we can see that the asymptotic series first gets closer and closer to the value of the original function, the error falls to a minimum and then rises without bound. When the error is at a minimum it is of exponentially small size, this is superasymptotics. The value of n at the point at which we truncate the series to minimise the error depends in general on $|z|$. The truncation generally occurs after the term of the series which is smallest in modulus.

The beautiful property of the Poincaré definition is that these representations are unique. That is the coefficients in (1.1) can be determined uniquely from the analytic function f. This means if we have two representations satisfying the Poincaré definition then the values of the coefficients in the two series are the same.

The asymptotic relation (1.1) is not complete. These relations are only valid, in general, for limited sectors of the complex plane. In this thesis we attempt to give maximal sectors of validity in all cases. However the analytic solutions of the differential equation are usually valid for all arguments. This leads to the property of Stokes' Phenomenon where the analytic solution has different asymptotic representations in different sectors of the complex plane. These sectors of validity overlap. In the sector of common validity of two representations the functions must differ by an exponentially small factor. It is this factor, not apparent in the Poincaré definition that we study in hyperasymptotics.

In hyperasymptotics we re-expand the remainder term, in some manner, to reduce this error still further. In this way hyperasymptotics is a method of generating very accurate approximations to functions which in the case of this thesis are solutions of differential equations.

The original works by Berry [5] and Berry and Howls [6] were on the hyperasymptotics of ordinary differential equations. Olde Daalhuis and Olver developed the concepts for differential equations in [13, 14] and these two papers are the ones which develop the central technique we use in the thesis. In these two papers Olde Daalhuis and Olver develop their idea of using a Stieltjes' transform and the connection formulae for the solutions of the second order linear ordinary equation with an irregular singularity of rank one to generate the hyperasymptotic expansions.

The second chapter of the thesis is derived from these last two papers in conjunction with a paper on the calculation of Stokes' multipliers (also by Olde Daalhuis and Olver [15]).

Olde Daalhuis has used the techniques developed by Balser, Jurkat and Lutz [3] in the closely related area of summability to develop hyperasymptotic expansions for a high order equation with a singularity of unit rank [18].

In the third chapter of this thesis we use the same sources as the first but in addition we use some of the techniques of summability and resurgence. The results are of the same character as the first but are generated by using quite different methods.

The area of summability attempts to take divergent series and *sum* these series to generate an analytic function. This analytic function then has the asymptotic behaviour of the divergent series it is derived from. In general the divergent series are generated by looking for series which satisfy an equation formally. In our case we have formal solutions to a differential equation. An early work on summability is that of Turritin [24] in which he generates all formal solutions to a linear Ordinary Differential Equation in matrix form and then proceeds to show which of these he can sum. An important idea developed in this paper is the idea that any matrix form Ordinary Differential Equation can be transformed into a canonical form. Braaksma[7] uses Turritin's idea to write a very deep and general paper about the calculation of Stokes' multipliers for these types of equation. This paper of Braaksma's uses the more general theory of multisummability. A very well written and clear account of this generalisation of the summability theory is in Balser [2]. It also contains a good introduction to the more basic ideas of summability. Multisummability is needed to deal with the cases of mixed exponential rank.

Summability uses the techniques of the Laplace transform and its inverse, the Borel transform. The action of the appropriate (formal) Borel transform on a divergent series is to transform it into a convergent series which describes an analytic function out to its radius of convergence. If it is then possible to analytically continue this function to a function of less than exponential size then the Laplace transform can be taken. By a simple application of Watson's lemma the Laplace transform can be shown to be asymptotic to the original formal expansion in certain sectors of the complex plane. The extent of these sectors is related closely to the structure of the singularities of the analytically continued Borel transform.

To generate hyperasymptotics more information is needed than is given by the basic techniques of summability. The process of summation will generate many more solutions (in general) than the total number of linearly independent solutions possible for the particular differential equation studied. This leads to linear relationships between sets of solutions. The relationships between the many solutions of the differential equation are connection formulae. The basic idea of the general area of resurgence is to generate these connection formulae by looking at the behaviour of the Borel transformed solutions' singularity structure using the *alien calculus.* These techniques have not been presented for the problems in these chapters but this approach could have be used. A good introduction to the ideas (and a source of references) of resurgence is the paper by Delabaere [8]. I have also been led through some of the concepts by Chris Luke who has given papers and lectures to the staff at Dublin City University [10, 9].

Chapter 2

The Second Order Problem

2.1 Introduction

The general linear homogeneous differential equation of the second order is given by

$$\frac{d^2W}{dz^2} + f(z)\frac{dW}{dz} + g(z)W = 0 \ . \tag{2.1}$$

The problem we shall study is that of an irregular singularity of rank r at infinity. In this case the functions f and g can be expanded in power series about infinity of the form

$$f(z) = z^{r-1}\sum_{s=0}^{\infty}\frac{f_s}{z^s}, \qquad g(z) = z^{2r-2}\sum_{s=0}^{\infty}\frac{g_s}{z^s}$$

which converge in an open annulus $|z| > a$. At least one of the coefficients f_0, g_0, g_1 is non-zero otherwise the singularity would have lower rank.

This equation (2.1) is studied in detail for the case $r = 1$ in [14] where a method of rigorous re-expansion of the remainder terms in the asymptotic expansion of the solution is developed. The re-expansions are in terms of certain multiple integrals, the so called *hyperterminant integrals.* This equation is also studied in [15] for the case r arbitrary and a method for the calculation of Stokes' multipliers is derived. Using these results we have developed the *hyperasymptotic expansions* (see [12] and [14] for references) for the differential equation (2.1) for the general case of the second order linear differential equation of arbitrary rank r.

2.2 Setting up the problem

By making the transformation

$$w(z) = \exp\left(\frac{1}{2}\int^{z} f(t)dt\right) W(z) \tag{2.2}$$

the differential equation (2.1) is transformed to the equation

$$\frac{d^2w}{dz^2} = \phi(z)w \tag{2.3}$$

where

$$\phi(z) = \frac{1}{4}f^2(z) + \frac{1}{2}f'(z) - g(z) \ .$$

We may assume without loss of generality that $\frac{1}{4}f_0^2 - g_0$ is non zero* then the square root of ϕ can be expanded in the form

$$\{\phi(z)\}^{\frac{1}{2}} = z^{r-1}\sum_{s=0}^{\infty}\frac{\phi_s}{z^s} \ .$$

We now define sectors

$$S_k = \{z : \frac{(-k-\frac{1}{2}-\sigma)}{r}\pi \le \text{ph}z \le \frac{(-k+\frac{1}{2}-\sigma)}{r}\pi\}$$

where $\sigma = \text{ph}\phi_0$.

If we define $\hat{S}_k$ to be any closed sector properly interior to $S_{k-1}\cup S_k\cup S_{k+1}$ then the differential equation (2.3) has unique solutions $w_k(z)$ defined by

$$w_k(z) \sim e^{-\xi(z)}z^{\mu_1}\sum_{s=0}^{\infty}\frac{a_{s,1}}{z^s} \qquad z\to\infty \text{ in } \hat{S}_k \tag{2.4}$$

for k even and

$$w_k(z) \sim e^{\xi(z)}z^{\mu_2}\sum_{s=0}^{\infty}\frac{a_{s,2}}{z^s} \qquad z\to\infty \text{ in } \hat{S}_k \tag{2.5}$$

for k odd (see [15] and [22]). These sectors differ slightly from those in [15] and because of this so do the solutions w. The order r polynomial ξ is given by

$$\xi(z) = z^r\sum_{s=0}^{r-1}\frac{\phi_s}{(r-s)z^s}$$

*The case when $\frac{1}{4}f_0^2 - g_0$ is zero is dealt with by using the transformation of Fabry, see [22].

and the coefficients μ_1, μ_2, $a_{s,1}$ and $a_{s,2}$ can be calculated using a recursion relation derived by substituting the expressions (2.4) and (2.5) into the differential equation (2.3) (see [15]).

We would now like to define a new variable x such that

$$x^r = 2\xi(z)$$

so that the polynomial ξ in the exponent of the asymptotic form of the solutions (2.4), (2.5) becomes simply x^r. We can do this by writing

$$z = x\sum_{s=0}^{\infty}\frac{c_s}{x^s}$$

calculating the coefficients c_s by reversion of power series. Performing a full reversion of power series is laborious but if we now truncate this series and make the change of variables

$$z = x\sum_{s=0}^{r-1}\frac{c_s}{x^s}$$

we can then transform equation (2.1) to write down a new differential equation for W in the variable x

$$\frac{d^2W}{dx^2} + \hat{f}(x)\frac{dW}{dx} + \hat{g}(x)W = 0 \ .$$

We can apply the transform (2.2) to this differential equation and this yields an equation with solutions in the form (2.4), (2.5) with the polynomial ξ having the simple form $x^r/2$ in the new variable x. Without loss of generality we can now assume that equation (2.1) is in the correct form initially so that when we derive (2.3) there are two solutions which have the following behaviour

$$w_k(z) \sim e^{-z^r/2} z^{\mu_1}\sum_{s=0}^{\infty}\frac{a_{s,1}}{z^s} \qquad z\to\infty \text{ in } \hat{S}_k$$

for k even and

$$w_k(z) \sim e^{z^r/2} z^{\mu_2}\sum_{s=0}^{\infty}\frac{a_{s,2}}{z^s} \qquad z\to\infty \text{ in } \hat{S}_k$$

for k odd.

There are only two linearly independent solutions to the second order differential equation (2.3) so there must be a linear relationship between any three solutions; a *connection formula*. In particular we can write

$$w_{k+2}(z) = C_{k+1}w_{k+1}(z) + w_k(z) \ . \tag{2.6}$$

The coefficient of $w_k(z)$ is unity because $w_{k+2}(z)$ and $w_k(z)$ have the same dominant asymptotic form in their common sector of validity.

We now define functions

$$u_k(z) = e^{z^r/2} z^{-\mu_1} e^{\mu_1 k\pi\iota/r} w_k(ze^{-k\pi\iota/r})$$

for k even and

$$u_k(z) = e^{z^r/2} z^{-\mu_2} e^{\mu_2 k\pi\iota/r} w_k(ze^{-k\pi\iota/r})$$

for k odd. These functions have the asymptotic form

$$u_k(z) \sim \sum_{s=0}^{\infty} \frac{a_{s,1}}{(ze^{-k\pi\iota/r})^s} \qquad z \to \infty \text{ in } \hat{S}_0$$

for k even and

$$u_k(z) \sim \sum_{s=0}^{\infty} \frac{a_{s,2}}{(ze^{-k\pi\iota/r})^s} \qquad z \to \infty \text{ in } \hat{S}_0$$

for k odd. Using the connection formula (2.6) for w we can now define connection formulae for u

$$u_{k+2}(ze^{2\pi\iota/r}) = C_{k+1} e^{z^r} z^{\omega} e^{-k\pi\iota\omega/r} u_{k+1}(ze^{\pi\iota/r}) + u_k(z) \tag{2.7}$$

for k even

$$u_{k+2}(ze^{2\pi\iota/r}) = C_{k+1} e^{z^r} z^{-\omega} e^{k\pi\iota\omega/r} u_{k+1}(ze^{\pi\iota/r}) + u_k(z) \tag{2.8}$$

for k odd. The number $\omega = \mu_2 - \mu_1$. Note that $u_{k+2r}(z) = u_k(z)$.

In a similar manner to [15] we can now write down a Stieltjes integral representation for each of the functions u_k. The representation has a slightly different form depending on whether k is an even or odd integer.

Lemma 1 *For even k*

$$u_k(z) = -\frac{z}{2\pi\iota}\left[\sum_{j=0}^{r-1}\int_{\rho e^{(-2j+k-1)\pi\iota/r}}^{\rho e^{(-2j+k+1)\pi\iota/r}} \frac{u_{2j}(te^{(2j-k)\pi\iota/r})}{t(t-z)}dt + \right.$$
$$\left. e^{-k\pi\iota\omega/r}\sum_{j=0}^{r-1} C_{2j+1}\int_{\rho e^{-(2j-k+1)\pi\iota/r}}^{\infty e^{-(2j-k+1)\pi\iota/r}} \frac{e^{t^r} t^{\omega} u_{2j+1}(te^{(2j-k+1)\pi\iota/r})}{t(t-z)}dt\right] . \tag{2.9}$$

For odd k

$$u_k(z) = -\frac{z}{2\pi\iota}\left[\sum_{j=0}^{r-1}\int_{\rho e^{(-2j+k-2)\pi\iota/r}}^{\rho e^{(-2j+k)\pi\iota/r}} \frac{u_{2j+1}(te^{(2j-k+1)\pi\iota/r})}{t(t-z)}dt + \right.$$

$$\left. e^{k\pi\iota\omega/r}\sum_{j=0}^{r-1} C_{2j}\int_{\rho e^{-(2j-k)\pi\iota/r}}^{\infty e^{-(2j-k)\pi\iota/r}} \frac{e^{t^r}t^{-\omega}u_{2j}(te^{(2j-k)\pi\iota/r})}{t(t-z)}dt\right] . \quad (2.10)$$

We use these integral representations for u_k to derive an integral representation for the remainder after truncation of its asymptotic series. This is done by expanding the term $(t-z)^{-1}$ as a finite geometric series. The details of the proof are similar to [15] and are omitted. As a byproduct of this process we also find an integral representation for the coefficients of the asymptotic expansion for u_k. These can be used to develop asymptotic expansions for the coefficients (see [15, 14]).

Theorem 1

$$u_k(z) = \sum_{s=0}^{n-1}\frac{a_{s,1}}{(ze^{-k\pi\iota/r})^s} + R_k^0(z,n) \quad (2.11)$$

where the coefficients $a_{s,1}$ in the expansion are given by

$$a_{s,1} = \frac{1}{2\pi\iota}\left[\sum_{j=0}^{r-1} e^{-2js\pi\iota/r}\int_{\rho e^{-\pi\iota/r}}^{\rho e^{\pi\iota/r}} u_{2j}(t)t^{s-1}dt \right.$$

$$\left. + \sum_{j=0}^{r-1} C_{2j+1}e^{-(2j+1)(s+\omega)\pi\iota/r}\int_\rho^\infty e^{-t^r}t^{s+\omega-1}u_{2j+1}(t)dt\right] \quad (2.12)$$

and

$$R_k^0(z,n) = -\frac{1}{2\pi\iota z^{n-1}}\epsilon_k^0(z;\rho,n)$$

$$- \frac{1}{2\pi\iota z^{n-1}}e^{kn\pi\iota/r}\sum_{j=0}^{r-1} C_{2j+1}e^{-(2j+1)(n+\omega)\pi\iota/r}\int_\rho^\infty \frac{e^{-t^r}t^{n+\omega-1}u_{2j+1}(t)}{te^{-(2j-k+1)\pi\iota/r}-z}dt \quad (2.13)$$

for k even and

$$u_k(z) = \sum_{s=0}^{n-1}\frac{a_{s,2}}{(ze^{-k\pi\iota/r})^s} + R_k^0(z,n) \quad (2.14)$$

where

$$a_{s,2} = \frac{1}{2\pi\iota}\left[\sum_{j=0}^{r-1} e^{-(2j+1)s\pi\iota/r}\int_{\rho e^{-\pi\iota/r}}^{\rho e^{\pi\iota/r}} u_{2j+1}(t)t^{s-1}dt \right.$$

$$\left. + \sum_{j=0}^{r-1} C_{2j}e^{-2j(s-\omega)\pi\iota/r}\int_\rho^\infty e^{-t^r}t^{s-\omega-1}u_{2j}(t)dt\right] \quad (2.15)$$

and

$$R_k^0(z,n) = -\frac{1}{2\pi\iota z^{n-1}}\epsilon_k^0(z;\rho,n) - \frac{1}{2\pi\iota z^{n-1}} e^{kn\pi\iota/r}\sum_{j=0}^{r-1} C_{2j}e^{-2j(n-\omega)\pi\iota/r}\int_\rho^\infty \frac{e^{-t^r}t^{n-\omega-1}u_{2j}(t)}{te^{-(2j-k)\pi\iota/r}-z}dt \quad (2.16)$$

for k odd.

The definition of ϵ_k^0 is given by

$$\epsilon_k^0(z;\rho,n) = \sum_{j=0}^{r-1} e^{n(-2j+k)\pi\iota/r}\int_{\rho e^{-\pi\iota/r}}^{\rho e^{\pi\iota/r}} \frac{u_{2j}(t)t^{n-1}}{te^{(-2j+k)\pi\iota/r}-z}dt$$

for k even and by

$$\epsilon_k^0(z;\rho,n) = \sum_{j=0}^{r-1} e^{n(-2j+k-1)\pi\iota/r}\int_{\rho e^{-\pi\iota/r}}^{\rho e^{\pi\iota/r}} \frac{u_{2j+1}(t)t^{n-1}}{te^{(-2j+k-1)\pi\iota/r}-z}dt$$

for k odd.

In the integrals in (2.13) and (2.16) above z has been restricted to the phase range $|\mathrm{ph}z| \leq \pi/r-\delta$. We would like now to include the phases $\pm\pi/r$. We do this by analytically continuing the integrals. This is performed in the standard way, indenting the straight line contours from $t=\rho$ to $t=\infty$ in a semi-circle $||z|-t|=\delta$ where the indentation goes to the left (resp. to the right) of z when $0 \leq \mathrm{ph}z \leq \pi/r$ (resp. $-\pi/r \leq \mathrm{ph}z \leq 0$) (see figures 2.1,2.2). We shall call this contour $\mathcal{P}$. With this extension and the continuation formulae we can get a representation for any solution of the differential equation for any z.

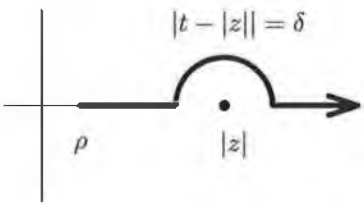

Figure 2.1: $\mathcal{P}$ for $0 \leq \mathrm{ph}z \leq \pi/r$

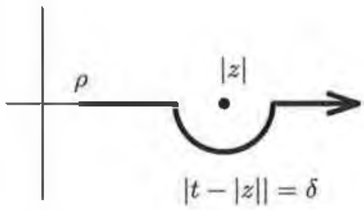

Figure 2.2: $\mathcal{P}$ for $-\pi/r \leq \mathrm{ph}z \leq 0$

2.3 Optimal expansion at level zero (Poincaré asymptotics)

We now wish to minimise the remainder in (2.11) and (2.14) to determine the optimal number of terms of these expansions to use with the single standard Poincaré asymptotic series. We will consider in detail the case for even k; the calculations are similar for odd k.

Let $n = N_0$ in (2.11). The remainder term is given in (2.13). We deal with the two terms separately. The first term is estimated by

$$\frac{1}{2\pi\iota z^{N_0-1}}\epsilon_k^0(z;\rho;N_0) = \mathcal{O}(\rho^{N_0}z^{-N_0}) \ . \tag{2.17}$$

In the second term of (2.13) we can see that the dominant contribution to the bound occurs when $-\pi/r \leq \text{ph}z \leq 0$ and arises from the integral for which $2j-k=0$ †. The path along which we integrate is indented at $|z|$ to pass to the right of $|z|$. To derive sharp error bounds we need to perform the analytic continuation of the previous section in a different way.

Starting with the dominant integral (assume $0 \geq \text{ph}z > -\pi/r$)

$$\int_\rho^\infty \frac{e^{-t^r}t^{N_0+\omega-1}u_{k+1}(t)}{te^{-\pi\iota/r}-z}dt$$

we replace z by $z\exp(-\pi\iota/r)$ and make the substitution $t = v^{1/r}$ (taking the principal branch) to give

$$\frac{e^{\pi\iota/r}}{r}\int_{\rho^r}^\infty \frac{e^{-v}v^{(N_0+\omega)/r-1}u_{k+1}(v^{\frac{1}{r}})}{v^{1/r}-z}dv \ .$$

Now we perform the analytic continuation allowing z to be real and indenting the contour in a semi-circle centred on and to the right of z of radius δ_1 in the v plane.

We bound this integral in two parts. For the integrals

$$\frac{e^{\pi\iota/r}}{r}\left\{\int_{\rho^r}^{|z|^r-\delta_1} + \int_{|z|^r+\delta_1}^\infty\right\}\frac{e^{-v}v^{(N_0+\omega)/r-1}u_{k+1}(v^{\frac{1}{r}})}{v^{1/r}-z}dv$$

we can say that $|v^{1/r}-z| = \mathcal{O}(1)$, u_k is $\mathcal{O}(1)$ uniformly on the region $|z| \geq \rho$ so that the sum of the integrals is

$$\mathcal{O}(1)\,\Gamma((N_0+\Re\omega)/r). \tag{2.18}$$

†or symmetrically when $0 \leq \text{ph}z \leq \pi/r$ and $2j-k=-2$.

For the part of the integral around the semicircle we have $|v - |z|^r| = \delta_1$ so that

$$\frac{e^{\pi\iota/r}}{r}\int_{|v-|z|^r|=\delta_1}\frac{e^{-v}v^{(N_0+\omega)/r-1}u_{k+1}(v^{\frac{1}{r}})}{v^{1/r}-z}dv = e^{-|z|^r}||z|^r+\delta_1|^{(N_0+\Re\omega)/r-1}\,\mathcal{O}(1) \qquad (2.19)$$

uniformly in the region of validity for z.

Now we assume that

$$N_0 = \beta_0|z|^r + \alpha_0 \qquad (2.20)$$

where $\beta_0 > 0$ is $\mathcal{O}(1)$ and α_0 is bounded. Now using this form for N_0 and comparing (2.19) to (2.18) and using Stirling's asymptotic estimate for the gamma function we have

$$\frac{e^{-|z|^r}||z|^r+\delta_1|^{(N_0+\Re\omega)/r-1}}{\Gamma((N_0+\Re\omega)/r)} = \mathcal{O}(1)e^{-|z|^r}|z|^{\beta_0|z|^r+\alpha_0+\Re\omega-r}e^{(\beta_0|z|^r+\alpha_0+\Re\omega)/r}\times$$

$$[(\beta_0|z|^r+\alpha_0+\Re\omega)/r]^{-(\beta_0|z|^r+\alpha_0+\Re\omega)/r+\frac{1}{2}}$$

$$= \mathcal{O}(1)e^{-|z|^r}|z|^{\beta_0|z|^r+\alpha_0+\Re\omega-r}e^{\beta_0|z|^r/r}(\beta_0/r)^{-(\beta_0/r)|z|^r}|z|^{-(\beta_0|z|^r+\alpha_0+\Re\omega)+\frac{r}{2}}$$

$$= \mathcal{O}(1)e^{-|z|^r}|z|^{-r/2}\left(e^{\beta_0/r}(\beta_0/r)^{-\beta_0/r}\right)^{|z|^r} = \mathcal{O}(|z|^{-r/2}) \; .$$

The last step is due to the fact that $e^{\beta_0/r}(\beta_0/r)^{-\beta_0/r}$ has its maximum at $\beta_0 = r$. This estimate for the semi-circular indentation of the integral can therefore be absorbed in the estimate for the straight line part of the integral (2.18) and this is our final estimate for the integral. The remainder term in (2.13) is then estimated by

$$\mathcal{O}(\Gamma((N_0+\Re\omega)/r)z^{-N_0+1}) \; . \qquad (2.21)$$

Using the value of N_0 in (2.20) and Stirling's formula we can minimise (2.21) with respect to β_0. We find that

$$z^{-N_0+1}\Gamma((N_0+\Re\omega)/r) = \mathcal{O}(1)\left[e^{-(\beta_0/r)}(\beta_0/r)^{(\beta_0/r)}\right]^{|z|^r}|z|^{1+\Re\omega-r/2}$$

and $\beta_0 = r$ for the remainder to be minimal. Substituting this value of β_0 in (2.21) we find that the minimal remainder estimate for k even is

$$\mathcal{O}(z^{1-r/2+\Re\omega}e^{-|z|^r}) \; .$$

The corresponding estimate for k odd may be shown in a similar manner to be

$$\mathcal{O}(z^{1-r/2-\Re\omega}e^{-|z|^r}) \; .$$

2.4 Optimal expansion at level one

To construct the first level of hyperasymptotic expansions we re-expand the remainder terms in (2.13) and (2.16). The calculations for even and odd k are similar so only the even k calculations will be shown. Substituting the expressions (2.14) into (2.13) we find that

$$R_k^0(z,N_0) = -\frac{z^{1-N_0}}{2\pi\iota} e^{kN_0\pi\iota/r} \sum_{s=0}^{N_1-1} a_{s,2} \sum_{j=0}^{r-1} C_{2j+1} e^{-(2j+1)(N_0+\omega-s)\pi\iota/r} \times$$
$$G_{2j-k+1}^{1,\omega}(z;N_0-s) + R_k^1(z;N_0,N_1)$$

where

$$G_k^{1,\omega}(z,N_0) = \int_0^\infty \frac{e^{-t^r} t^{N_0+\omega-1}}{te^{-k\pi\iota/r}-z} dt \tag{2.22}$$

and

$$\begin{aligned} R_k^1(z;N_0,N_1) = &-\frac{z^{1-N_0}}{2\pi\iota}\epsilon_k^0(z;\rho;N_0) \\ &+\frac{z^{1-N_0}}{2\pi\iota} e^{kN_0\pi\iota/r} \sum_{s=0}^{N_1-1} a_{s,2} \sum_{j=0}^{r-1} C_{2j+1} e^{-(2j+1)(N_0+\omega-s)\pi\iota/r} \int_0^\rho \frac{e^{-t^r} t^{N_0+\omega-1-s}}{te^{-(2j-k+1)\pi\iota/r}-z} dt \\ &-\frac{z^{1-N_0}}{2\pi\iota} e^{kN_0\pi\iota/r} \sum_{j=0}^{r-1} C_{2j+1} \times \\ &e^{-(2j+1)(N_0+\omega)\pi\iota/r} \int_{\mathcal{P}} \frac{e^{-t^r} t^{N_0+\omega-1} R_{2j+1}^0(t,N_1)}{te^{-(2j-k+1)\pi\iota/r}-z} dt \ . \end{aligned} \tag{2.23}$$

Now we estimate the remainder as in the previous section and then proceed to minimise it. The first term on the right hand side of (2.23) is estimated as before in (2.17) to be $\mathcal{O}(z^{-N_0}\rho^{N_0})$.

In the second term in (2.23) we have that $|te^{-(2j-k+1)\pi\iota/r} - z| \geq |z| - \rho$ in the worst case so that

$$\int_0^\rho \frac{e^{-t^r} t^{N_0+\omega-1-s}}{te^{-(2j-k+1)\pi\iota/r}-z} dt = \mathcal{O}(z^{-1}\gamma((N_0+\Re\omega-s)/r,\rho^r)) = \mathcal{O}(z^{-1}\rho^{N_0}/N_0) \ .$$

From [15] we have that

$$a_{s,2} = \mathcal{O}(\Gamma((s-\Re\omega)/r)) \text{ as } s \to \infty$$

so that the whole second term is estimated by

$$\mathcal{O}(\Gamma((N_1-\omega)/r) z^{-N_0}\rho^{N_0}/N_0) \ .$$

In the third term taking the expression for $R^0_{2j+1}(t, N_1)$ from (2.16) and replacing all occurrences of ρ by $\rho - \delta$ (to ensure convergence of the integrals), we have that

$$\int_{\mathcal{P}} \frac{e^{-t^r} t^{N_0+\omega-1} R^0_{2j+1}(t, N_1)}{te^{-(2j-k+1)\pi\iota/r} - z} dt$$
$$= \frac{1}{2\pi\iota} \int_{\mathcal{P}} \frac{e^{-t^r} t^{N_0-N_1+\omega}}{te^{-(2j-k+1)\pi\iota/r} - z} dt \Bigg[\epsilon^0_{2j+1}(t; \rho - \delta; N_1) +$$
$$+ e^{(2j+1)N_1\pi\iota/r} \sum_{l=0}^{r-1} C_{2l} e^{-(2l)(N_1-\omega)\pi\iota/r} \int_{\rho-\delta}^{\infty} \frac{e^{-t_1^r} t_1^{N_1-\omega-1} u_{2l}(t_1)}{t_1 e^{-(2l-2j-1)\pi\iota/r} - t} dt_1 \Bigg] .$$

The estimate for the integral of the ϵ term is given by (2.17) so we have that

$$\int_{\mathcal{P}} \frac{e^{-t^r} t^{N_0-N_1+\omega} \epsilon^0_{2j+1}(t; \rho - \delta; N_1)}{te^{-(2j-k+1)\pi\iota/r} - z} dt = \mathcal{O}((\rho - \delta)^{N_1} \Gamma((N_0 - N_1 + \Re\omega)/r) .$$

The double integral written out in full is

$$\int_{\mathcal{P}} \int_{\rho-\delta}^{\infty} \frac{e^{-t^r} t^{N_0-N_1+\omega} e^{-t_1^r} t_1^{N_1-\omega-1} u_{2l}(t_1)}{(te^{-(2j-k+1)\pi\iota/r} - z)(t_1 e^{-(2l-2j-1)\pi\iota/r} - t)} dt_1 dt . \tag{2.24}$$

We estimate this by noting that $|te^{-(2j-k+1)\pi\iota/r} - z| \geq \delta$ and $|t_1 e^{-(2l-2j-1)\pi\iota/r} - t| \geq |t|\mathcal{O}(1)$. Then (2.24) splits into the product of two single integrals and is estimated by

$$= \mathcal{O}(\Gamma((N_0 - N_1 + \Re\omega)/r)\Gamma((N_1 - \Re\omega)/r)) .$$

All of the other terms can be absorbed into this estimate so that we have our final estimate that

$$R^1_k(z; N_0, N_1) = \mathcal{O}(z^{-N_0+1} \Gamma((N_0 - N_1 + \Re\omega)/r)\Gamma((N_1 - \Re\omega)/r)) . \tag{2.25}$$

Following (2.20) we now assume the standard form for N_0 and N_1

$$N_0 = \beta_0 |z|^r + \alpha_0 ,$$

$$N_1 = \beta_1 |z|^r + \alpha_1 .$$

$\beta_0 > \beta_1 > 0$ are $\mathcal{O}(1)$ and α_0, α_1 are bounded.

Using Stirling's formula to give an asymptotic estimate for the gamma functions in (2.25) we find that

$$R^1_k(z; N_0, N_1) = \mathcal{O}(z^{-\beta_0|z|^r - \alpha_0 + 1}) |z|^{(\beta_0-\beta_1)|z|^r + \alpha_0 - \alpha_1 + \Re\omega - r/2} \times$$
$$\left(e^{-\frac{\beta_0-\beta_1}{r}} \left[\frac{\beta_0 - \beta_1}{r}\right]^{\frac{\beta_0-\beta_1}{r}} \right)^{|z|^r} |z|^{\beta_1|z|^r + \alpha_1 - \Re\omega - r/2} \left(e^{-\frac{\beta_1}{r}} \left[\frac{\beta_1}{r}\right]^{\frac{\beta_1}{r}} \right)^{|z|^r} . \tag{2.26}$$

We find that the estimate (2.26) is minimised when $\beta_0 - \beta_1 = r$ and $\beta_1 = r$ so that $\beta_0 = 2r$ and the optimal estimate for the remainder at level 1 is

$$R_k^1(z; N_0, N_1) = z^{1-r} e^{-2|z|^r} \ .$$

2.5 General Levels

The complete expansion for $u_k(z)$ can now be written down and proved by induction. The number ν_k in Theorem 2 is 1 when k is even and 0 when k is odd.

Theorem 2 $-\pi/r \leq \mathrm{ph} z \leq \pi/r$. *For k even*

$$\begin{aligned} u_k(z) = \sum_{s=0}^{N_0-1} \frac{a_{s,1}}{(ze^{-k\pi\iota/r})^s} + z^{1-N_0} e^{kN_0\pi\iota/r} & \\ \times \sum_{n=1}^{p} (-)^n \sum_{s=0}^{N_n-1} a_{s,2-\nu_n} & \\ \times \prod_{l=0}^{n-2} \left[\sum_{j_l=0}^{r-1} \frac{C_{2j_l+\nu_l}}{2\pi\iota} e^{-(2j_l+\nu_l)(N_l-N_{l+1}+(-)^l\omega)\pi\iota/r} \right] & \\ \times \sum_{j_{n-1}=0}^{r-1} \frac{C_{2j_{n-1}+1-\nu_n}}{2\pi\iota} e^{-(2j_{n-1}+1-\nu_n)(N_{n-1}-s-(-)^n\omega)\pi\iota/r} & \\ \times G^{n,\omega}_{2j_0-k+1, 2j_1-2j_0-1, \ldots, 2(j_{n-1}-j_{n-2})-(-1)^n}(z; N_0-N_1, \ldots, N_{n-2}-N_{n-1}, N_{n-1}-s) & \\ + R_k^p(z; N_0, N_1, \ldots, N_p) & \end{aligned}$$

where the remainder is estimated by

$$\begin{aligned} R_k^p(z; N_0, N_1, \ldots, N_p) = \mathcal{O}(z^{-N_0+1} \Gamma((N_0 - N_1 + \Re\omega)/r) \Gamma((N_1 - N_2 - \Re\omega)/r) \times \ldots & \\ \times \Gamma((N_{p-1} - N_p + (-)^{p-1}\Re\omega)/r) \Gamma((N_p + (-)^p \Re\omega)/r)) \ . & \end{aligned}$$

For k odd

$$u_k(z) = \sum_{s=0}^{N_0-1} \frac{a_{s,2}}{(ze^{-k\pi\iota/r})^s} + z^{1-N_0} e^{kN_0\pi\iota/r}$$
$$\times \sum_{n=1}^{p} (-)^n \sum_{s=0}^{N_n-1} a_{s,1+\nu_n}$$
$$\times \prod_{l=0}^{n-2} \left[\sum_{j_l=0}^{r-1} \frac{C_{2j_l+1-\nu_l}}{2\pi\iota} e^{-(2j_l+1-\nu_l)(N_l-N_{l+1}-(-)^l\omega)\pi\iota/r} \right]$$
$$\times \sum_{j_{n-1}=0}^{r-1} \frac{C_{2j_{n-1}+\nu_n}}{2\pi\iota} e^{-(2j_{n-1}+\nu_n)(N_{n-1}-s+(-)^n\omega)\pi\iota/r}$$
$$\times G^{n,-\omega}_{2j_0-k,2j_1-2j_0+1,\ldots,2(j_{n-1}-j_{n-2})+(-1)^n}(z; N_0-N_1,\ldots,N_{n-2}-N_{n-1},N_{n-1}-s)$$
$$+ R_k^p(z; N_0, N_1, \ldots, N_p)$$

where the remainder is estimated by

$$R_k^p(z; N_0, N_1, \ldots, N_p) = \mathcal{O}(z^{-N_0+1}\Gamma((N_0-N_1-\Re\omega)/r)\Gamma((N_1-N_2+\Re\omega)/r) \times \ldots$$
$$\times \Gamma((N_{p-1}-N_p+(-)^{p-2}\Re\omega)/r)\Gamma((N_p+(-)^{p-1}\Re\omega)/r)) \ .$$

In the case where the expansions are optimally truncated after n series the number of terms in the final re-expansion is $N_p = r|z|^r + \alpha_p$ then in each previous expansion the number of terms increases approximately by this amount, i.e. $N_{p-i} = (i+1)r|z|^r + \alpha_{p-i}$, etc. The optimal error term in this case is $R_k^p = \mathcal{O}(z^{1-p\frac{r}{2}}e^{-p|z|^r})$.

The general integral G appearing in the expansions above is given by

$$G^{n+1,\omega}_{k,k_1,\ldots,k_n}(z; M_0, M_1, \ldots, M_n) =$$
$$\int_0^\infty \cdots \int_0^\infty \frac{e^{-t^r-t_1^r\cdots-t_n^r} t^{M_0+\omega} t_1^{M_1-\omega} \cdots t_{n-1}^{M_{n-1}+(-)^{n-1}\omega} t_n^{M_n+(-)^n\omega-1}}{(te^{-k\pi\iota/r}-z)(t_1e^{-k_1\pi\iota/r}-t)\ldots(t_ne^{-k_n\pi\iota/r}-t_{n-1})} dt_n \ldots dt_1 dt \quad (2.27)$$

for $n \geq 1$ and for $n = 0$ it is given by (2.22).

2.6 On calculation of the integrals G appearing in expansions

The integrals (2.27) and (2.22) (often called terminant integrals) can be calculated numerically by writing them in terms of the integrals in [14]. We can then use some results in [12] which express these simpler integrals as a convergent infinite series of confluent hypergeometric functions.

To recast our integrals in terms of those in [12] we first substitute $t = v^{1/r}$, $t_j = v_j^{1/r}$, $j = 1, \ldots, n$ into (2.27) taking the principal branch in all cases. We then use the results

$$\frac{1}{(ve^{-k\pi\iota})^{1/r} - z} = \frac{1}{ve^{-k\pi\iota} - z^r} \sum_{j_0=0}^{r-1} (ve^{-k\pi\iota})^{j_0/r} z^{r-1-j_0}$$

and

$$\frac{1}{(v_n e^{-k_n\pi\iota})^{1/r} - v_{n-1}^{1/r}} = \frac{1}{v_n e^{-k_n\pi\iota} - v_{n-1}} \sum_{j_n=0}^{r-1} (v_n^{-k_n\pi\iota})^{j_n/r} v_{n-1}^{(r-1-j_n)/r}$$

to show that (2.27) can be expressed as a certain sum of integrals F defined in [12] (in the case where $k_1, \ldots, k_n$ are all odd integers):

$$G^{n+1,\omega}_{k,k_1,\ldots,k_n}(z; M_0, M_1, \ldots, M_n) =$$
$$\frac{1}{(-r)^{n+1}} \sum_{j_0=0}^{r-1} z^{r-1-j_0} \sum_{j_1=0}^{r-1} \cdots \sum_{j_n=0}^{r-1} e^{-(kj_0+k_1j_1+\cdots+k_nj_n)\pi\iota/r} F^{n+1}(z^r; (M_0 + j_0 - j_1 + \omega)/r + 1,$$
$$(M_1 + j_1 - j_2 - \omega)/r + 1; \ldots, (M_n + j_n + (-1)^n\omega)/r)\ .$$

To calculate with Theorem 2 we then use the results of [12]. These calculations for various examples confirm numerically the theoretical error estimates of Theorem 2.

2.7 Conclusions

In Theorem 2 we have obtained in general form a hyperasymptotic expansion at all levels for solutions of a second order homogeneous linear ordinary differential equation which has an irregular singularity at infinity of arbitrary rank r.

As in [14] this expansion for sufficiently large level p is numerically unstable. This instability is due to the fact that in general we sum a divergent series past the point where the last term added is of order one, leading to severe cancellation. This may be dealt with in the same manner as in [14]. The optimal numerically stable scheme will use less terms than the corresponding optimal series from Theorem 2 but will not be as accurate for the same level p.

We may also extend the region of validity of the exponentially improved expansions in a similar manner to [14] section 10 with a corresponding weakening of the error estimates in the expanded sectors. This is really only of theoretical interest, however; in practice the high accuracy results in this paper can be used to generate approximations to any solution anywhere in the complex plane by direct use of the connection formulae (2.7) and (2.8).

Chapter 3

The High Order Problem

3.1 Introduction

In this chapter we study solutions of the linear differential equation

$$D\, w(z) = 0 \tag{3.1}$$

in the complex z plane. The operator D is defined by

$$D = \sum_{n=0}^{\nu} z^{-n(r-1)} a_n(z) \frac{d^n}{dz^n} \tag{3.2}$$

where the numbers ν and r are non-negative integers and the coefficient functions $a_n(z)$ have the form

$$a_n(z) = \sum_{m=0}^{\infty} a_{nm} z^{-m} \ .$$

These sums converge in some common annulus $|z| > \rho$. The order of the differential equation is then ν and the *rank* of the essential singularity at infinity is r. We wish, first, to study the asymptotic behaviour of the solutions of this equation at infinity. We must impose some conditions on the coefficient functions a_n to ensure that the type of behaviour of the solutions is restricted to one exponential *level* r*. We therefore assume that the coefficient $a_{\nu 0}$ is non-zero.

*i.e. the form of the leading asymptotic term in each solution is $e^{\lambda_n z^r}$ with λ_n non-zero in each case.

There are two main results in this chapter. First we will find solutions in sectors (with maximum region of validity) which have the formal solutions of (3.1) as their asymptotic behaviour. The approach taken is to obtain a Laplace transform representation of the analytic solutions. Then secondly we will obtain hyperasymptotic expansions for these solutions using methods similar to those in chapter 2.

3.2 Formal Solutions

For our present purposes we have found it convenient to transform the differential operator (3.2) which has rank r to another operator which has rank one. To this end we make the transform $z_r = z^r$. This gives us a second differential operator

$$D_r = \sum_{n=0}^{\nu} b_n(z_r) \frac{d^n}{dz_r^n} \ . \tag{3.3}$$

The coefficient functions b_n are now power series in $z_r^{-1/r}$ which converge in an annulus $|z_r| > \rho^r$.

$$b_n(z_r) = \sum_{m=0}^{\infty} b_{nm} z_r^{-m/r} \ . \tag{3.4}$$

Now we solve the equation

$$D_r W = 0 \ . \tag{3.5}$$

Obviously the solutions $w(z)$ and $W(z_r)$ are equal.

The characteristic polynomial of (3.3) is

$$P(\lambda) = \sum_{n=0}^{\nu} b_{n0} \lambda^n \ .$$

Due to the conditions on the coefficients of the functions a_n both $b_{\nu 0}$ and b_{00} are non zero. This means that $P(\lambda)$ has ν zeros. Additionally, at this point we make the assumption that the zeros of the polynomial P are simple. In general the admission of non-simple roots leads to an equation with many ranks (or exponential levels) of solutions or logarithmic-type solutions neither of which we wish to discuss in this chapter. This condition implies that $P'(\lambda_n)$ is not zero for all $n = 1, \ldots, \nu$.

We wish to show now that the unique formal solutions to (3.5) are of the form

$$\hat{W}_i(z_r) = \exp(\lambda_i z_r + \sum_{s=1}^{r-1} \lambda_{is} z_r^{1-\frac{s}{r}}) z_r^{\tau_i} \hat{\phi}_i(z_r) \ , \tag{3.6}$$

$$\hat{\phi}_i(z_r) = 1 + \sum_{n=1}^{\infty} c_{in} z_r^{-n/r} \ .$$

Throughout this chapter the hat on any symbol means that the symbol is to be interpreted in a formal sense, i.e. any infinite series appearing in the definition of the symbol may not converge.

As a by-product of this demonstration we will also find equations which each of the $\hat{\phi}_i(z_r)$ satisfies. These equations will then be used to calculate the analytic solutions through the process of Borel summation. We proceed by showing that if we substitute the ansatz for the formal solutions into the differential equation each of the coefficients can be calculated uniquely. When the Borel sum has been calculated it is a simple matter to use Watson's lemma to show that the analytic solution (in the form of a Laplace transform) is asymptotic to the formal expansion from which it has been derived and find the maximal sectors of validity.

3.2.1 The derivation of the formal solution and as a by-product differential equations for $\hat{\phi}_i$

The first step in the process of demonstrating that the functions $\hat{W}_i(z_r)$ defined in (3.6) are formal solutions of the differential equation (3.5) is to split the coefficient functions b_n into two components, the constant and variable parts. To this end we define the functions

$$b_n^*(z_r) = b_n(z_r) - b_{n0}$$

derived from the coefficient functions b_n defined in (3.4).

At this stage we also define the operator

$$\partial = \frac{d}{dz_r}$$

to simplify the notation somewhat.

We also define functions $b_n^k(z_r)$ which appear during the following calculation which are defined for $k = 1, \ldots, r$. The functions b_n^k are convergent power series in the variable $z_r^{1/r}$,

$$b_n^k(z_r) = \sum_{m=0}^{\infty} b_{nm}^k z_r^{-m/r} \ .$$

Following the definition b_{n0}^k is the constant term in the power series and thus we define

$$b_n^{k*}(z_r) = b_n^k(z_r) - b_{n0}^k \ .$$

The exact values of the coefficients are derived from those of the functions $b_n(z_r)$ in the following manner:

$$b_{s0}^1 = \sum_{n=s}^{\nu} b_{n0} \binom{n}{s} \lambda_i^{n-s} \ , \tag{3.7}$$

$$b_s^{1*}(z_r) = \sum_{n=s}^{\nu} b_n^*(z_r) \binom{n}{s} \lambda_i^{n-s}$$

and

$$b_n^{k+1}(z_r) = e^{-\lambda_{ik} z_r^{1-k/r}} \sum_{s=n}^{\nu} b_s^k(z_r) \binom{s}{n} \partial^{s-n} e^{\lambda_{ik} z_r^{1-k/r}} \ .$$

If we study these equations we can see that despite complex behaviour in the higher order terms the constant coefficients remain the same, i.e. that

$$b_{n0}^k = b_{n0}^1$$

for $k = 2, \ldots, r$. Note that

$$b_{00}^k = 0 \ .$$

To calculate the coefficients in the formal expansion we need to peel off the highest order terms in sequence, to do this we define the functions

$$\hat{W}_i(z_r) =: e^{\lambda_i z_r} \hat{W}_{i1}(z_r) \tag{3.8}$$

$$\hat{W}_{i,k-1}(z_r) =: e^{\lambda_{i,k-1} z_r^{1-(k-1)/r}} \hat{W}_{ik}(z_r) \text{ for } k = 2, \ldots, r \ . \tag{3.9}$$

Now we write the formal solution in the form defined in 3.8, substitute this in the differential equation (3.5) and multiply by the factor $e^{-\lambda_i z_r}$. This gives us the sequence of equations

$$\begin{aligned} 0 &= e^{-\lambda_i z_r} D_r \hat{W}_i(z_r) \\ &= \sum_{n=0}^{\nu} b_{n0} \sum_{s=0}^{n} \binom{n}{s} \lambda_i^{n-s} \hat{W}_{i1}^{(s)} + \sum_{n=0}^{\nu} b_n^*(z_r) \sum_{s=0}^{n} \binom{n}{s} \lambda_i^{n-s} \hat{W}_{i1}^{(s)} \\ &= \sum_{n=0}^{\nu} b_{n0} \lambda_i^n \hat{W}_{i1} + \sum_{n=1}^{\nu} b_{n0} \sum_{s=1}^{n} \binom{n}{s} \lambda_i^{n-s} \hat{W}_{i1}^{(s)} + \sum_{n=0}^{\nu} b_n^*(z_r) \sum_{s=0}^{n} \binom{n}{s} \lambda_i^{n-s} \hat{W}_{i1}^{(s)} \ . \end{aligned}$$

The first sum in the last equation is the characteristic polynomial $P(\lambda)$ times $\hat{W}_{i1}$. As this is the highest order term in z^r and the only order one term we set this to zero to give the characteristic values λ_i. We note here that the derivatives of $\hat{W}_{ik}$ are all of order at least $z_r^{-1/r}$ smaller than $\hat{W}_{ik}$, $k = 1, \ldots, r$. We now reverse the order of the sums to give the equation

$$\sum_{s=1}^{\nu} b_{s0}^1 \hat{W}_{i1}^{(s)} + \sum_{s=0}^{\nu} b_s^{1*}(z_r) \hat{W}_{i1}^{(s)} = 0 \ . \tag{3.10}$$

The functions $b_s^1(z_r)$ we have defined earlier (3.7).

We remark that we have from (3.7) that

$$b_{10}^1 = \sum_{n=1}^{\nu} n b_{n0} \lambda_i^{n-1} = P'(\lambda_i) \neq 0 \ .$$

Because the reciprocal of this term will occur when the remaining coefficients in the formal solution are calculated the assumption that there are no repeated roots of the characteristic polynomial is essential for formal solutions of this type. Also we note here that $b_s^{1*} = \mathcal{O}(z_r^{-1/r})$, for $s = 1, \ldots, \nu$.

The calculations involving the product of derivatives of the exponential and a function are too complicated to work with so in subsequent calculations we will use the fact that

$$\partial^m e^{K z_r^{1-k/r}} = (K^m (1 - k/r)^m z_r^{-mk/r} + \mathcal{O}(z_r^{-(m-1)k/r-1})) e^{K z_r^{1-k/r}}$$

because we only require the leading order term. The lower orders are absorbed by the coefficient functions b. This statement is true for $k = 1, \ldots, r-1$.

Now we substitute the next term of the desired formal solution into (3.10) i.e. we write the equation in terms of $\hat{W}_{i2}(z_r)$ and this gives us on substitution and pre-multiplication by

$\exp(-\lambda_{i1} z_r^{1-1/r})$

$$\begin{aligned} 0 &= \mathrm{e}^{-\lambda_{i1} z_r^{1-1/r}} \sum_{s=1}^{\nu} b_{s0}^1 \sum_{n=0}^{s} \binom{s}{n} \left(\partial^{s-n} \mathrm{e}^{\lambda_{i1} z_r^{1-1/r}} \right) \hat{W}_{i2}^{(n)} \\ &\quad + \mathrm{e}^{-\lambda_{i1} z_r^{1-1/r}} \sum_{s=0}^{\nu} b_s^{1*}(z_r) \sum_{n=0}^{s} \binom{s}{n} \left(\partial^{s-n} \mathrm{e}^{\lambda_{i1} z_r^{1-1/r}} \right) \hat{W}_{i2}^{(n)} \\ &= \sum_{n=1}^{\nu} b_{n0}^2 \hat{W}_{i2}^{(n)} + \sum_{n=0}^{\nu} b_n^{2*}(z_r) \hat{W}_{i2}^{(n)} \ . \end{aligned}$$

The $\mathcal{O}(z_r^{-1/r})$ terms are

$$(b_{01}^1 + (1 - 1/r) b_{10}^1 \lambda_{i1}) z_r^{-1/r}$$

which we set to zero to give

$$\lambda_{i1} = -\frac{b_{01}^1}{(1 - 1/r) b_{10}^1} \ .$$

The principle is now clear and the calculations in practice are straightforward to carry out. At the kth stage we will have the differential equation:

$$\sum_{n=1}^{\nu} b_{n0}^k \hat{W}_{ik}^{(n)} + \sum_{n=0}^{\nu} b_n^{k*}(z_r) \hat{W}_{ik}^{(n)} = 0 \ .$$

We calculate the coefficients λ_{ik} in (3.9) successively by setting the highest order term in the equation to zero. Because of this procedure we can see that $b_0^{k*} = \mathcal{O}(z_r^{-k/r})$ and at each stage we get a similar formula for the λ_{ik}.

$$\lambda_{ik} = -\frac{b_{0k}^k}{(1 - k/r) b_{10}^1}$$

for $k = 2, \ldots, r-1$.

We now consider the rth stage. We will have reached the equation

$$\sum_{n=1}^{\nu} b_{n0}^r \hat{W}_{ir}^{(n)} + \sum_{n=0}^{\nu} b_n^{r*}(z_r) \hat{W}_{ir}^{(n)} = 0 \ .$$

We can see that the coefficient functions have the following behaviour

$$\begin{aligned} b_0^{r*}(z_r) &= \mathcal{O}(z_r^{-1}) \ , \\ b_{n0}^r &= b_{n0}^1 \ . \end{aligned}$$

Now if we substitute into this equation the functions

$$\hat{W}_{ir} = z_r^{\tau_i} \hat{\phi}_i(z_r)$$

we find that

$$\tau_i = -\frac{b^r_{0r}}{b^1_{10}} \ .$$

Since we can assume the first coefficient of the power series $\hat{\phi}_i(z_r)$ is unity it is useful for our purposes to write the differential equation for $\hat{\phi}_i(z_r)$ in terms of the function $\hat{\psi}_i(z_r) = \hat{\phi}_i(z_r) - 1$. We can write the differential equation for $\hat{\psi}_i(z_r)$ as

$$\sum_{s=0}^{\nu} d_s(z_r)\hat{\psi}^{(s)} + d_0(z_r) = 0 \tag{3.11}$$

with

$$d_s(z_r) = z^{-\tau_i} \sum_{n=s}^{\nu} \binom{n}{s} (z_r^{\tau_i})^{(n-s)} b^r_n(z_r) \text{for } 0 \leq s \leq \nu \ .$$

Writing the system in this way and then substituting the series for all the functions into the differential equation we can easily develop a sequence of equations for the coefficients of $\hat{\psi}_i(z_r)$. We can then find all of the coefficients uniquely in terms of the coefficients of the functions $d_s(z_r)$.

With this the functions $\hat{W}_i(z_r)$ have been shown to be formal solutions of (3.5). During this process we have also developed a differential equation which the formal power series $\hat{\psi}_i(z_r)$ satisfies. This will be used in the next section to prove the existence of analytic solutions to (3.5) and to show that their asymptotic form in certain sectors is the corresponding formal solution.

3.2.2 Analytic Solutions

When we use the Borel-Laplace method three things must be shown. The first is that the formal Borel transform (which is an analytic germ) can be analytically continued outside its circle of convergence at the origin. In the present case we can show that analytic continuation of the germ to infinity is possible. The second is to show that the analytically continued function is of exponential size to ensure convergence of its Laplace transform. The third is to show that the computed Laplace transform representation solves the original differential equation.

First we will specify the particular form of Borel transform used in this chapter. There are two types of Borel transform in this chapter the analytic and the formal. The analytic is equivalent to the classical definition of the inverse Laplace transform. The formal Borel transform has the same effect as the analytic on analytic functions but is also allowed to operate on formal power series term by term.

If $f(z)$ is an analytic function with no singularities in $|z| > \rho$ then the analytic Borel transform of f is defined as

$$\mathcal{B}(f(z))(\xi) = \frac{1}{2\pi i} \int_C f(z) e^{z\xi} dz \tag{3.12}$$

where the contour C begins at $-\infty - 0i$, proceeds in a counter-clockwise loop around the singularities and finishes at $-\infty + 0i$. In particular

$$\mathcal{B}(z^{-\alpha})(\xi) = \frac{\xi^{\alpha-1}}{\Gamma(\alpha)} \ .$$

This is basically the definition of the formal Borel transform. To be more precise if we take a formal (possibly divergent) power series with no constant term

$$\hat{\phi}(z) = \sum_{k=1}^{\infty} c_k z^{-k/r}$$

then

$$\hat{\mathcal{B}}(\hat{\phi}(z))(\xi) = \sum_{k=1}^{\infty} c_k \frac{\xi^{k/r-1}}{\Gamma(k/r)}$$

with $r \in \mathbf{N}^+$. If the form of the divergence in the formal power series is mild (the coefficients grow like a geometric series times a gamma function) then the process of taking the formal Borel transform results in a convergent analytic power series in some neighbourhood of the origin. If this power series can be analytically continued to infinity and can be bounded by an exponential function then we can take the inverse transform. This defines an analytic function which is asymptotic to the formal series it is derived from: This is the process of Borel summation. We note here that

$$\mathcal{B}(\partial f(z_r))(\xi) = -\xi \mathcal{B}(f(z_r))(\xi) \ .$$

The Borel transform of the product of two analytic functions is the convolution of their corresponding Borel transforms. If $f(\xi)$ and $g(\xi)$ are analytic functions then their convolution is defined as

$$f * g = \int_0^{\xi} f(\xi - u) g(u) du \ .$$

For simplicity during calculations we choose the contour to be a straight line. This is general enough for our purposes and makes the estimation of convolutions much simpler.

If $f(\xi)$ is an analytic function in ξ the Laplace transform that we use is

$$\mathcal{L}(f(\xi))(z) = \int_0^{\infty e^{i\theta}} f(\xi)\mathrm{e}^{-\xi z} d\xi$$

where $\theta \in \mathbf{R}$ less some set of points Ω. The contour is a straight line from the origin to infinity in the direction θ. We may only calculate this integral where $f(z)$ is non-singular, hence the set of permissible angles θ is restricted to those without any singularities of the function we are transforming. The set Ω is therefore the set of directions for which $f(z)$ has a singularity.

The Borel transform specified in (3.12) is not strictly the inverse of this Laplace transform. The exact inverse is the Borel transform with the contour rotated to a direction $-\bar{\alpha}$ where the direction of the Laplace transform is in a direction α, for some complex number α. θ is the phase of α

Before we make a statement of the main theorem of the first section of the chapter some definitions are necessary.

If λ_i, $i = 1, \ldots, \nu$ are the roots of the characteristic equation $P(\lambda) = 0$ for the differential operator (3.5) then we define numbers

$$\theta_{ij} = \mathrm{ph}(\lambda_i - \lambda_j) \ , \ i \neq j \tag{3.13}$$

on the half open interval $(0, 2\pi]$. We order these numbers θ_{ij} for each i on this interval and we call these ordered numbers

$$\sigma_1^i, \ldots, \sigma_{\nu-1}^i \text{ for } i = 1, \ldots, \nu \ .$$

In particular $\sigma_1^i = \min_j \theta_{ij}$. We have consequently the relationship that $\sigma_1^i \leq \sigma_2^i \leq \cdots \leq \sigma_{\nu-1}^i$. We then define these values for all integer subscripts $\sigma^i_{n(\nu-1)+j} = \sigma^i_j + 2n\pi$ for $j = 1, \ldots, \nu - 1$.

We define extensions of the numbers θ_{ij},

$$\theta_{ij}^q = \theta_{ij} + 2\pi i q$$

for all integer q.

Now we define sectors

$$S_k^i = \{z : -\frac{\pi}{2} - \sigma_k^i < \text{ph}z < \frac{\pi}{2} - \sigma_{k-1}^i\} \ .$$

Lemma 2 *For integer k there exist analytic solutions of the differential equation (3.5)*

$$W_k^i(z_r) \sim \exp(\lambda_i z_r + \sum_{s=1}^{r-1} \lambda_{is} z_r^{1-\frac{s}{r}}) z_r^{\tau_i} \hat{\phi}_i(z_r)$$

as $|z| \to \infty$ in any closed sub-sector of S_k^i.

Note If equality holds for any two or more successive σ_k^i then the corresponding W_k^i will be identical. This happens when λ_i and two or more of the λ_j are co-linear.

Lemma 2 is proved as stated above by showing that the Borel transform of the series part of the solution $\hat{\phi}_i(z_r)$ is summable. For this we have to prove the following theorem regarding the Borel transform of the analytic solutions $W_k^i(z_r)$.

Theorem 3 *The Borel transform of $\hat{\phi}_i(z_r)$ is analytic in $\xi^{1/r}$ and is analytically continuable to infinity with only a finite number of singularities at the points $\lambda_i - \lambda_j$, $j \neq i$. The Borel transform of $\hat{\phi}_i(z_r)$ is of exponential size in sectors bounded away from and not containing singularities.*

During the proof of Theorem 3 we will need the following bound on the Borel transform of an analytic function.

Lemma 3 *Let $f(z_r)$ be an analytic function of $z_r^{1/r}$ which can be represented by a convergent power series in a neighbourhood $|z_r| > \rho^r$ of infinity*

$$f(z_r) = \sum_{k=n}^{\infty} f_k z_r^{-k/r} \quad n > 0 \ .$$

There exists an α such that the Borel transform of f, $\underline{f}(\xi)$, is entire in the transform variable $\xi^{1/r}$ and

$$|\underline{f}(\xi)| \leq C \frac{|\xi|^{n/r-1}}{\Gamma(n/r)} e^{\Re \alpha \xi}$$

where $\bar{\alpha}$, the complex conjugate of α, is on the centre line of the sector

$$P = \{\xi : \theta_1 < \text{ph}\xi < \theta_2\} \ ,, \ \theta_2 - \theta_1 < \pi \ .$$

Lemma 4 *For $\alpha, \beta \geq 1/r$, r a positive integer, the Beta function, $B(\alpha, \beta)$, is bounded above by a positive constant B.*

Proof

We prove this using three cases.

- **Case 1** for $\alpha, \beta \geq 1$.

  An integral definition of the beta function is

  $$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt \ .$$

  Taking absolute values this is clearly bounded above by 1 when $\alpha, \beta \geq 1$. In this case, therefore,

  $$B(\alpha, \beta) \leq 1 \ .$$

- **Case 2** for $\alpha > 1, \beta < 1$.

  Using the same integral definition of the Beta function as in Case 1 we can see the first term $(1-t)^{\alpha-1}$ is bounded above by 1. On integration we can see that the bound for the integral becomes $1/\beta$ which is maximised for $\beta = 1/r$. So that we have the bound that

  $$B(\alpha, \beta) \leq r \ .$$

  Since the Beta function is symmetric in its arguments this bound holds also for $\alpha < 1, \beta > 1$. Finally we have

- **Case 3** for $\alpha < 1, \beta < 1$. The Beta function is decreasing in both α and β for increasing α and β so the largest value of the Beta function is for $\alpha, \beta = 1/r$. Using the fact that the Beta function can also be written in terms of a ratio of Gamma functions we can show that

  $$B(1/r, 1/r) = 2B(1/r + 1, 1/r) \ .$$

  According to case 2 the last Beta function is bounded above by r and so

  $$B(\alpha, \beta) \leq 2r.$$

Now if we choose $B = 2r$ then we have the result. •

Proof of Lemma 3.

The Borel transform of f, from our definition of the analytic Borel transform in section 3.2.2, is

$$\underline{f}(\xi) := \mathcal{B}(f(z_r))(\xi) = \sum_{k=n}^{\infty} \frac{f_k \xi^{k/r-1}}{\Gamma(k/r)}$$

Using the ratio test and knowing that the original analytic functions' power series expansion converged for $|z_r| > \rho^r$ we can show that the radius of convergence of the Borel transform $\underline{f}(\xi)$ is infinite and so the function represented by it is entire.

We rewrite this expression in the form

$$\underline{f}(\xi) = \frac{\xi^{n/r-1}}{\Gamma(n/r)} \sum_{k=0}^{\infty} \frac{f_{k+n}\xi^{k/r}}{\Gamma((k+n)/r)/\Gamma(n/r)}$$

and taking the modulus

$$|\underline{f}(\xi)| \leq \frac{|\xi|^{n/r-1}}{\Gamma(n/r)} \sum_{k=0}^{\infty} \frac{|f_{k+n}||\xi|^{k/r}}{\Gamma((k+n)/r)/\Gamma(n/r)}$$

Now we need to show that the sum on the right hand side is less than an exponential function.

We use the fact that since the original series expansion in the z_r plane was convergent we can bound the absolute value of the coefficients by a geometric series. Each term $f_n \leq DK^n$ where D, K are some finite positive constants, $K > \rho$.

We also use the lemma above showing that the Beta function is bounded above by a finite constant B.

Let $\tilde{\gamma} = \min \Gamma(t), \;\; t > 0$.

Rewriting the sum

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{|f_{k+n}||\xi|^{k/r}}{\Gamma((k+n)/r)/\Gamma(n/r)} &= \sum_{k=0}^{\infty}\sum_{q=0}^{r-1} \frac{|f_{rk+n+q}||\xi|^{(rk+q)/r}}{\Gamma((rk+q+n)/r)/\Gamma(n/r)} \\
&= \sum_{k=1}^{\infty}\sum_{q=0}^{r-1} \frac{|f_{rk+n+q}||\xi|^{(rk+q)/r}}{\Gamma((rk+q+n)/r)/\Gamma(n/r)} + \sum_{q=0}^{r-1} \frac{|f_{n+q}||\xi|^{q/r}}{\Gamma((q+n)/r)/\Gamma(n/r)} \\
&\leq B/\tilde{\gamma} \left(\sum_{k=1}^{\infty}\sum_{q=0}^{r-1} \frac{DK^{rk+n+q}|\xi|^{(rk+q)/r}}{\Gamma(k)} + \sum_{q=0}^{r-1} DK^{n+q}|\xi|^{q/r} \right) \\
&\leq \frac{BDK^n}{\tilde{\gamma}} \left(\sum_{k=1}^{\infty} \frac{K^{rk}|\xi|^k}{\Gamma(k)} \sum_{q=0}^{r-1} K^q|\xi|^{q/r} + \sum_{q=0}^{r-1} K^q|\xi|^{q/r} \right) .
\end{aligned}$$

We must bound this sum for $|\xi|$ greater than and less than unity. For $|\xi| \le 1$

$$\sum_{q=0}^{r-1} K^q |\xi|^{q/r} \le \sum_{q=0}^{r-1} K^q =: M$$

and when $|\xi| > 1$

$$\sum_{q=0}^{r-1} K^q |\xi|^{q/r} \le |\xi| \sum_{q=0}^{r-1} K^q =: M|\xi| \ .$$

We can now bound the sum by an exponential $C\exp(Q|\xi|)$ where C and Q are positive real numbers. $Q = 1 + 2K^r$ and $C = 2DBMK^n/\bar{\gamma}$. Once we choose $|\bar{\alpha}|$ large enough this is equivalent to the sum being bounded by $C\exp(\Re\alpha\xi)$ in a sector P whose centre-line is any direction $|\bar{\alpha}|$. •

3.2.3 Proof of Theorem 3

Consider (3.11) and the definitions following it. We have that

$$d_s(\infty) = b^1_{s0} =: d_{s0} \ . \tag{3.14}$$

We define

$$d^*_s(z_r) := d_s(z_r) - d_{s0}$$

and write (3.11) in the form

$$\sum_{s=0}^{\nu} d_{s0}\hat{\psi}^{(s)}_i + \sum_{s=0}^{\nu} d^*_s(z_r)\hat{\psi}^{(s)}_i = -d_0(z_r) \ . \tag{3.15}$$

Note here that

$$d_0(z_r) = \mathcal{O}(z_r^{-1-1/r}) \ , \tag{3.16}$$

$$d_s(z_r) - d_{s0} = \mathcal{O}(z_r^{-1/r})), \ 1 \le s \le \nu \ . \tag{3.17}$$

Considering the differential equation (3.11) we write

$$\sum_{s=0}^{\nu} d_{s0}\hat{\psi}^{(s)}_i(z_r) + \epsilon \sum_{s=0}^{\nu} d^*_s(z_r)\hat{\psi}^{(s)}_i(z_r) = -d_0(z_r) \tag{3.18}$$

introducing the parameter ϵ to flag the subdominant terms in the differential equation and write

$$\hat{\psi}_i(z_r) = \sum_{j=0}^{\infty} \epsilon^j \hat{\psi}^j_i(z_r) \ . \tag{3.19}$$

Later we set ϵ to one to give our result. Substituting this formula into (3.18) we get differential equations for each of the functions $\hat{\psi}_i^j(z_r)$:

$$\begin{aligned} \sum_{s=0}^{\nu} d_{s0}\hat{\psi}_i^{0(s)}(z_r) &= -d_0(z_r) \ , \\ \sum_{s=0}^{\nu} d_{s0}\hat{\psi}_i^{k(s)}(z_r) &= -\sum_{s=0}^{\nu} d_s^*(z_r)\hat{\psi}_i^{k-1(s)}(z_r) \ , \ k \geq 1 \ . \end{aligned}$$

At this stage we take the Borel transform of this set of differential equations and derive a set of convolution equations for $\underline{\psi}_i^k$, $k = 0, \ldots, \infty$ the Borel transform of the functions $\hat{\psi}_i^k$.

$$\begin{aligned} \sum_{s=0}^{\nu} d_{s0}(-\xi)^s \underline{\psi}_i^0 &= -\underline{d}_0(\xi) \ , \\ \sum_{s=0}^{\nu} d_{s0}(-\xi)^s \underline{\psi}_i^k &= -\sum_{s=0}^{\nu} \underline{d}_s^*(\xi) * [(-\xi)^s \underline{\psi}_i^{k-1}] \ , \ k \geq 1 \ . \end{aligned}$$

From (3.14) we have that

$$d_{s0} = b_{s0}^1 = \sum_{n=s}^{\nu} b_{n0} \binom{n}{s} \lambda_i^{n-s}$$

and this means that

$$\begin{aligned} \sum_{s=0}^{\nu} d_{s0}(-\xi)^s &= \sum_{s=0}^{\nu}(-\xi)^s \sum_{n=s}^{\nu} b_{n0}\binom{n}{s}\lambda_i^{n-s} \\ &= \sum_{n=0}^{\nu} b_{n0}(\lambda_i - \xi)^n \\ &= P(\lambda_i - \xi) \end{aligned}$$

so that the equations for the $\underline{\psi}_i^k(\xi)$ can be written

$$P(\lambda_i - \xi)\underline{\psi}_i^0 = \underline{d}_0(\xi) \tag{3.20}$$

$$P(\lambda_i - \xi)\underline{\psi}_i^k = -\sum_{s=0}^{\nu} \underline{d}_s^*(\xi) * [(-\xi)^s \underline{\psi}_i^{k-1}] \ , \ k \geq 1 \ . \tag{3.21}$$

Looking at equation (3.20) we can see that $\underline{\psi}_i^0(z_r)$ is analytic on the Riemann surface of $z_r^{1/r}$ bar the singularities which may occur at the points $\xi = \lambda_i - \lambda_j$ on all of the sheets.

The convolution of an entire function with one with singularities will produce a function with singularities at the same points. The functions $d_s^*(z_r)$ are analytic and the functions $(-\xi)^s\underline{\psi}_i^0$ have singularities at the points $\xi = \lambda_i - \lambda_j$ so the convolution has singularities at the same points. Consequently $\underline{\psi}_i^1$ and all the higher iterates $\underline{\psi}_i^k$ are analytic apart from the points $\xi = \lambda_i - \lambda_j$.

Now we must show that the sum defined in (3.19) converges when ϵ is set to unity. In fact we prove that the sum of the Borel transforms of these functions converges. We then show that the inverse transform exists and solves the differential equation (3.15).

The functions $\underline{d}_0(\xi)$, $\underline{d}_s^*(\xi)$, $s = 0 \dots \nu$ are Borel transforms of functions satisfying the conditions of Lemma 3 and consequently from (3.17) we have the following bounds

$$\begin{aligned} |\underline{d}_0(\xi)| &\leq C\frac{|\xi|^{1/r}}{\Gamma(1/r+1)}e^{\Re\alpha\xi} , \\ |\underline{d}_0^*(\xi)| &\leq C\frac{|\xi|^{1/r}}{\Gamma(1/r+1)}e^{\Re\alpha\xi} , \\ |\underline{d}_s^*(\xi)| &\leq C\frac{|\xi|^{1/r-1}}{\Gamma(1/r)}e^{\Re\alpha\xi} , \ 1 \leq s \leq \nu \end{aligned} \tag{3.22}$$

where C is a suitably chosen real constant and the complex number α is chosen so that $|\alpha|$ is sufficiently large. $\bar{\alpha}$ is the centre line of some sector $S = \xi : \theta_1 < \mathrm{ph}\xi < \theta_2,\ \theta_2 - \theta_1 < \pi$. In addition we choose θ_1 and θ_2 so that $\sigma_{k-1}^i \leq \theta_1 < \theta_2 \leq \sigma_k^i$ for any k. If we take any closed sub-sector of S, Q, then the closest distance from Q to the nearest singularity of the functions $\underline{\psi}$ is bounded below. We bound in a different way for $|\xi|$ less than and greater than 1. For $|\xi| \geq 1$ we have

$$\left|\frac{\lambda_i - \lambda_j}{\xi} - 1\right| < \delta_2 .$$

For $|\xi| < 1$ we have

$$|\lambda_i - \lambda_j - \xi| < \delta_1 .$$

If $\xi \in Q$ then

$$\frac{1}{|P(\lambda_i - \xi)|} \leq \begin{cases} K/|\xi| \text{ for } |\xi| < 1 \\ K/|\xi|^\nu \text{ for } |\xi| \geq 1 \end{cases} . \tag{3.23}$$

Using this information and the equation (3.20) we can bound $\underline{\psi}_i^k$ in a sector Q. We take the cases where $|\xi| < 1$ and $|\xi| \geq 1$ separately.

- Case: $|\xi| < 1$

  From equation (3.20) we can see that

$$|\underline{\psi}_i^0| \leq KC\frac{|\xi|^{1/r-1}}{\Gamma(1/r+1)}e^{\Re\alpha\xi} . \tag{3.24}$$

Using this bound and substituting it into the expression for $|\underline{\psi}_i^1|$, equation (3.21), we can develop a bound for $|\underline{\psi}_i^1|$.

From equation (3.21) we have that

$$\underline{\psi}_i^1 \leq \frac{K}{|\xi|}\sum_{s=0}^{\nu}\left|\underline{d}_s^*(\xi) * [(-\xi)^s \underline{\psi}_i^0(\xi)]\right| \ , \tag{3.25}$$

$\underline{d}_s^*(\xi)$ has different bounds for for $s=0$ and for $s=1,\ldots,\nu$. Therefore the terms in the sum have identical bounds for $s=1,\ldots,\nu$ and for $s=0$. The calculations however are similar. For $s=0$ we have

$$\left|\underline{d}_0^*(\xi) * \underline{\psi}_i^0(\xi)\right| = \left|\int_0^{\xi} \underline{d}_0^*(\xi-u)\underline{\psi}_i^0(u)du\right| \ .$$

In the convolution integrals we simplify the calculations by integrating only along straight-line paths. We can do this because all functions inside the integrals are analytic inside the sector in which we integrate. Substituting in the bounds given in equations (3.22) and (3.24) and taking the absolute value inside the integral means that the $s=0$ term is bounded by

$$C^2K\int_0^{|\xi|}\frac{(||\xi|-u|)^{1/r}}{\Gamma(1/r+1)}\mathrm{e}^{\Re\alpha(\xi-u)}\frac{u^{1/r-1}}{\Gamma(1/r+1)}\mathrm{e}^{\Re\alpha u}du \ .$$

Finally we integrate by parts to transform this term to a form similar to the terms we have for $s=1,\ldots,\nu$. Then using the definition of the Beta function we get the following bound for the $s=0$ term

$$rC^2K\mathrm{e}^{\Re\alpha\xi}\frac{|\xi|^{2/r}}{\Gamma(2/r+1)} \ .$$

This bound differs from the bound we derive for the other terms in the sum in equation 3.25 only by a factor of r so our final bound for the sum is

$$|\underline{\psi}_i^1(\xi)| \leq (r+\nu)C^2K^2\mathrm{e}^{\Re\alpha\xi}\frac{|\xi|^{2/r-1}}{\Gamma(2/r+1)} \ .$$

Substituting this bound to calculate a bound for the next $\underline{\psi}$ we begin to see the forming pattern. The general result we arrive at, which we may prove by induction, is that

$$|\underline{\psi}_i^k(\xi)| \leq (r+\nu)^k(CK)^{k+1}|\xi|^{-1}\frac{|\xi|^{\frac{k+1}{r}}}{\Gamma(\frac{k+1}{r}+1)}\mathrm{e}^{\Re\alpha\xi} \ . \tag{3.26}$$

- Case: $|\xi| \geq 1$

From equation (3.20) and (3.23) we can see that

$$|\underline{\psi}_i^0| \leq KC\frac{|\xi|^{1/r} - \nu}{\Gamma(1/r+1)}e^{\Re\alpha\xi} \ . \tag{3.27}$$

Again we substitute this bound into the expression for $|\underline{\psi}_i^1|$, equation (3.21) and get that

$$\underline{\psi}_i^1 \leq \frac{K}{|\xi|^\nu}\sum_{s=0}^{\nu}\left|\underline{d}_s^*(\xi) * [(-\xi)^s \underline{\psi}_i^0(\xi)]\right| \ . \tag{3.28}$$

For $s = 1, \ldots, \nu$ the bounds for the terms in the sum are again identical. We will again deal with the special case for $s = 0$ to illustrate our method.

For $s = 0$ we have

$$\left|\underline{d}_0^*(\xi) * \underline{\psi}_i^0(\xi)\right| \leq \int_0^{|\xi|}\left|\underline{d}_0^*(\xi - u)\underline{\psi}_i^0(u)du\right| \ .$$

Obviously now we have to separate the integral into the region from 0 to 1 and from 1 to $|\xi|$. Substituting in the bounds given in equations (3.22) and (3.27) we get the bound

$$\frac{C^2K}{\Gamma(1/r+1)\Gamma(1/r+1)}e^{\Re\alpha\xi}\left\{\int_0^1(||\xi| - u|)^{1/r}u^{1/r-1}du + \int_1^{|\xi|}(||\xi| - u|)^{1/r}u^{1/r-\nu}du\right\} \ .$$

We can now, however, recombine the two integrals by noting that in the second $u \geq 1$ and so we can rewrite $u^{1/r-\nu}$ as $u^{1/r-1}$. Finally again for the special case we integrate by parts to transform this term to a form similar to the terms we have for $s = 1, \ldots, \nu$. Then using the definition of the Beta function we get the following bound for the $s = 0$ term

$$rC^2Ke^{\Re\alpha\xi}\frac{|\xi|^{2/r}}{\Gamma(2/r+1)} \ .$$

Then when we combine the terms in the sum in (3.28)

$$|\underline{\psi}_i^1(\xi)| \leq (r+\nu)C^2K^2e^{\Re\alpha\xi}\frac{|\xi|^{2/r-\nu}}{\Gamma(2/r+1)}$$

and the general result is

$$|\underline{\psi}_i^k(\xi)| \leq (r+\nu)^k(CK)^{k+1}|\xi|^{-\nu}\frac{|\xi|^{\frac{k+1}{r}}}{\Gamma(\frac{k+1}{r}+1)}e^{\Re\alpha\xi} \ . \tag{3.29}$$

We have reached the stage where we have a bound on the Borel transform of the $\hat{\psi}_i^j$ functions which appearing on the right hand side of equation (3.19). We need to find $\hat{\psi}_i$. First we need to sum these Borel transforms to give the Borel transform of $\hat{\psi}_i$. Then we take the inverse transform to get $\hat{\psi}_i$. For the inverse transform to converge we need to prove the sum is less than an exponential function.

We need to sum the Borel transforms over all $k = 0, \ldots, \infty$. The part which varies with k is the same in both cases The ratio test can be used to show that the sum of these terms converges for all ξ. Now we show that the sum is smaller than an exponential function. First let us take a generic series

$$\begin{aligned}\sum_{k=0}^{\infty} M^k \frac{|\xi|^{(k+1)/r}}{\Gamma((k+1)/r+1)} &= |\xi|^{1/r} \sum_{k=0}^{\infty}\sum_{q=0}^{r-1} M^{rk+q} \frac{|\xi|^{k+q/r}}{\Gamma(k+(q+1)/r+1)} \\ &= |\xi|^{1/r} \sum_{k=0}^{\infty} M^k \frac{|\xi|^k}{k!} \sum_{q=0}^{r-1} M^q \frac{|\xi|^{q/r}}{\Gamma(k+(q+1)/r+1)/k!} .\end{aligned}$$

The last sum over q is smaller than a constant if $|\xi| < 1$. If $|\xi| \geq 1$ then it is smaller than a constant, L, times $|\xi|$. Our bound for the sum is then given by

$$L|\xi|^{1/r} e^{(1+M)|\xi|} . \tag{3.30}$$

In the case where $|\xi| \geq 1$, $|\xi|^{-\nu} \leq |\xi|^{-1}$, so that the bound (3.29) is bounded by (3.26) and they can be merged.

Let $M = (r+\nu)CK$. Then the bound becomes

$$|\underline{\psi}_i^k(\xi)| \leq CK e^{\Re\alpha\xi} |\xi|^{-1} M^k \frac{|\xi|^{\frac{k+1}{r}}}{\Gamma(\frac{k+1}{r}+1)}$$

for all values of ξ. Using (3.30) we arrive at the final bound

$$|\underline{\psi}_i(\xi)| \leq Q|\xi|^{1/r-1} e^{\Re\alpha\xi}$$

where $Q = CK(1+L)$ and α has been adjusted suitably.

Using this result we may now prove Lemma 2.

3.2.4 Proof of Lemma 2

Proof Since $\underline{\psi}_i$ is of less than exponential size we can take its Laplace transform along some straight line $[0, \infty e^{i\theta}]$,

$$\int_0^{\infty e^{i\theta}} \underline{\psi}_i e^{-z_r \xi} d\xi \tag{3.31}$$

provided that direction does not contain one of the singularities of $\underline{\psi}_i$. The singularities of $\underline{\psi}_i$ occur at the set of points θ_{ij} defined in (3.13).

As is well known for any particular valid direction of integration θ the Laplace transform (3.31) is well defined for $|z_r|$ large enough and for $-\pi/2 - \theta < \mathrm{ph} z < \pi/2 - \theta$. If we have a sector in the ξ plane with $\theta_1 \le \theta \le \theta_2$ and this sector contains no singularities of $\underline{\psi}_{i0}$ then by continuously varying θ between the two boundaries we define an analytic continuation of the Laplace transform valid in the sector $-\pi/2 - \theta_2 + \delta \le \mathrm{ph} z_r \le \pi/2 - \theta_1 - \delta$ where $\delta > 0$ is some small number. In particular we may choose $\sigma^i_{k-1} < \theta_1 < \theta_2 < \sigma^i_k$. The sector defined above for this choice of θ_1 and θ_2 defines the possible closed subsectors of S^i_k. Now using Watson's lemma we can show that the Laplace transform of $\underline{\psi}_{i0}$, (3.31), is asymptotic to $\hat{\phi}_i(z_r) - 1$ in any of these closed subsectors.

Finally we need to prove this Laplace transform satisfies the equation (3.11). This proof is straightforward and will be omitted. It uses the definitions of the Borel and Laplace transforms and the definition of convolution. •

Corollary 1 *The functions $u^i_k(z)$ are uniformly $\mathcal{O}(1)$ in closed sub-sectors of S^i_k.*

Proof By definition

$$u^i_k(z) = 1 + \int_0^{\infty e^{i\theta}} \underline{\psi}_{i0} e^{-z_r \xi} d\xi \ .$$

Taking absolute values and using the bound for $\underline{\psi}_{i0}$ obtained in the proof of Theorem 3 we can show the integral is $\mathcal{O}(1/z_r)$ and the result follows. •

3.3 Hyperasymptotics

Using the results derived in the first section, specifically lemma 2, which give the asymptotic behaviour of solutions of the differential equation (3.5), we now proceed to derive hyperasymptotic expansions for the same solutions. We do this in a similar manner to [11] and [14] by proceeding from a connection formula for the solutions to an integral transform. This integral transform is then rewritten as the first n terms of the asymptotic expansion plus a remainder. The remainder contains references to the other solutions and the iteration of this expression gives the hyperasymptotics.

3.3.1 Connection Formulae

The first stage in this process is the derivation of connection formulae for the solutions $W_k^i(z)$ to the differential equation (3.5).

In general there will exist a connection formula of the form

$$W_k^i(z) = \sum_{j=1}^{\nu} C^j_{l^j_{ik}} W^j_{l^j_{ik}}(z)$$

on a common line of validity of the functions W. The integer $l^j_{i,k-1}$ is defined as the l which gives the least σ^j_l for $\sigma^j_l \geq \sigma^i_{k-1}$.

However on observation of the relative asymptotic behaviour of the solutions this simplifies greatly to

$$W_k^i(z) = W^i_{k-1}(z) + C^j_{l^j_{ik}} W^j_{l^j_{ik}}(z)$$

where the value of j is the value for which $\theta_{ij} = \sigma^i_{k-1}$. If there is more than one value of j satisfying this statement then the connection formula becomes a sum over these values and $W_k^i(z)$ becomes $W^i_{k+q-1}(z)$ where q is the number of different values of j for which θ_{ij} satisfies the equality. This is not dealt with explicitly in the text because it leads to the same final representation as in the general case of no repeated values. This case corresponds to three or more co-linear values of λ_j.

Instead of directly using the solutions $W_k^i(z)$ we now strip off the leading asymptotic behaviour and analyse the solutions $u_k^i(z)$ where these are defined as

$$u_k^i(z) = \exp[-\lambda_i z - \sum_{s=1}^{r-1} \lambda_{is} z^{1-s/r}] z^{-\tau_i} W_k^i(z) \ .$$

The functions $u_k^i(z^r)$ are analytic in z and $\mathcal{O}(1)$ in S_k^i.

If we now define functions

$$D_{ij}(z) = z^{\tau_j - \tau_i} \exp[(\lambda_j - \lambda_i)z + \sum_{s=1}^{r-1} (\lambda_{js} - \lambda_{is}) z^{1-s/r}]$$

then the connection formulae for the functions $u_k^i(z)$ are

$$u_k^i(z) = u_{k-1}^i(z) + C^j_{l^j_{ik}} D_{ij}(z) u^j_{l^j_{ik}}(z) \ .$$

3.3.2 An integral transform of Stieltjes type

Because $u_k^i(z^r)$ is analytic in z for large enough z and assuming $\sigma_{k-1}^i < \sigma_k^i$ we can use the Cauchy integral formula to write it in the form

$$u_k^i(z^r) = \frac{-z}{2\pi i}\int_{C_k^i} \frac{u_k^i(\xi^r)}{\xi(\xi - z)} d\xi$$

where the contour C_k^i starts at $\rho e^{-\sigma_{k-1}^i i/r}$ goes along a straight line to $R e^{-\sigma_{k-1}^i i/r}$, along a circle centred at the origin to $R e^{-\sigma_k^i i/r}$, along a straight line to $\rho e^{-\sigma_k^i i/r}$ and back to the starting point along the circular path centred at the origin. This integral is valid for z inside the region bounded by the contour C_k^i. For z outside this region the integral around the contour is zero.

We know that these functions $u_k^i(z^r)$ are uniformly $\mathcal{O}(1)$ inside the sector S_i^k. The contour C_k^i lies entirely in this sector. If we now take the limit as $R \to \infty$ then the part of the integral along the larger circular arc, radius R tends to zero and we are left with the other three parts of the contour to integrate over.

In one of the straight line integrals we replace $u_k^i(\xi^r)$ using the connection formula and then transfer the integration contour to the other side of the sector. We repeat this process $r(\nu - 1)$ times which brings us through 2π when we use the fact that $u_k^i(z\exp(2\pi ir)) = u_{k+r(\nu-1)}^i(z)$ and $C_{k+r(\nu-1)}^i = C_k^i \exp[2\pi ir(\tau_j - \tau_i)]$. This process produces the following Stieltjes integral transform of $u_k^i(z^r)$. For $-\sigma_k^i < \mathrm{ph} z < -\sigma_{k-1}^i$

$$u_k^i(z^r) = \sum_{q=1}^{r}\sum_{p=1}^{\nu-1}\left(P_{qp+k-1}^i(z) + Q_{qp+k-1}^i(z)\right)$$

where

$$P_k^i(z) = -\frac{z}{2\pi i}\int_{\rho e^{-\sigma_{k-1}^i i/r}}^{\rho e^{-\sigma_k^i i/r}} \frac{u_k^i(\xi^r)}{\xi(\xi - z)} d\xi$$

and

$$Q_k^i(z) = -\frac{z}{2\pi i} C^i_{l^j_{i,(k+1)}} \int_{\rho e^{-\sigma_k^i i/r}}^{\infty e^{-\sigma_k^i i/r}} \frac{D_{ij}(z^r) u^j_{l^j_{i,(k+1)}}(\xi^r)}{\xi(\xi - z)} d\xi \ .$$

There is an implicit dependence here of j on i and k since $\sigma_k^i = \theta_{ij}$.

In all of these integrals we use the partial geometric series expansion

$$\frac{1}{\xi - z} = -\frac{1}{z}\sum_{s=0}^{n-1}\left(\frac{\xi}{z}\right)^s + \frac{\xi^n}{z^n(\xi - z)} \ .$$

Now we define

$$P_k^{is} = \frac{1}{2\pi i}\int_{\rho e^{-\sigma_{k-1}^i i/r}}^{\rho e^{-\sigma_k^i i/r}} u_k^i(\xi^r)\xi^{s-1}d\xi \ ,$$

$$Q_k^{is} = \frac{1}{2\pi i}C^i_{l^j_{i,(k+1)}}\int_{\rho e^{-\sigma_k^i i/r}}^{\infty e^{-\sigma_k^i i/r}} D_{ij}(\xi^r)u^j_{l^j_{i,(k+1)}}(\xi^r)\xi^{s-1}d\xi \ ,$$

$$\epsilon_k^{in}(z) = -\frac{z^{1-n}}{2\pi i}\int_{\rho e^{-\sigma_{k-1}^i i/r}}^{\rho e^{-\sigma_k^i i/r}} \frac{u_k^i(\xi^r)\xi^{n-1}}{\xi - z}d\xi \ ,$$

$$R_k^i(z) = \epsilon_k^{in}(z) - \frac{z^{1-n}}{2\pi i}C^i_{l^j_{i,(k+1)}}\int_{\rho e^{-\sigma_k^i i/r}}^{\infty e^{-\sigma_k^i i/r}} D_{ij}(z^r)u^j_{l^j_{i,(k+1)}}(\xi^r)\frac{\xi^{n-1}}{\xi - z}d\xi \ .$$

With these definitions we can write our representation for the functions $u_k^i(z^r)$ as

$$u_k^i(z^r) = \sum_{s=0}^{n-1} z^{-s}\sum_{q=1}^{r}\sum_{p=1}^{\nu-1}\left(P_{(qp)}^{is}(z) + Q_{(qp)}^{is}(z)\right) + \sum_{q=1}^{r}\sum_{p=1}^{\nu-1} R_{(qp)}^{in}(z) \ .$$

The dependence of the right hand side on k is gone. The sector of validity determines the value of k. We therefore define

$$u^i(z^r) := \sum_{s=0}^{n-1} z^{-s}\sum_{q=1}^{r}\sum_{p=1}^{\nu-1}\left(P_{(qp)}^{is}(z) + Q_{(qp)}^{is}(z)\right) + \sum_{q=1}^{r}\sum_{p=1}^{\nu-1} R_{(qp)}^{in}(z) \ .$$

In the remainder term and elsewhere we can remove the lower index on any u because the sector of the argument to the functions is always well defined by the integration range. This allows us to change all of the σ angles back to the original θ. By indenting the contour on the left (respectively right) in the straight line integrals if necessary we may analytically continue our representation so that the representation for $u_k^i(z^r)$ is valid for $-\sigma_k^i \leq \mathrm{ph} z \leq -\sigma_{k-1}^i$ and we have thus established the following theorem.

Theorem 4 *We may represent solutions $u^i(z)$ of the differential equation (3.5) in the form*

$$u^i(z^r) = \sum_{s=0}^{n-1} z^{-s}c_s^i + R_n^i(z)$$

in the sector $-\sigma_k^i \leq \mathrm{ph} z \leq -\sigma_{k-1}^i$ *for any integer* k. *The remainder term* $R_n^i(z)$ *on truncation of the asymptotic expansion for* $u^i(z^r)$ *is given by*

$$R_n^i(z) = \sum_{q=0}^{\nu-1}\sum_{p=1}^{r-1} \epsilon_{qp}^{in}(z) - \frac{z^{1-n}}{2\pi i}\sum_{j\neq i}\sum_{q=0}^{r-1} K_{jq}^i \int_{[\theta_{ij}^q]} \frac{D_{ij}(\xi^r)u^j(\xi^r)\xi^{n-1}}{\xi - z} d\xi \ . \tag{3.32}$$

The symbol $[\theta_{ij}^q]$ *stands for the contour from* $\rho\exp(-\theta_{ij}^q/r)$ *to* $\infty\exp(-\theta_{ij}^q/r)$.

As a by–product of the proof of this theorem we can find a representation for the coefficients in the asymptotic expansion (using the fact that a Poincaré expansion is unique) in the form of a sum of integrals.

3.3.3 Superasymptotics

In this section we estimate the remainder in Theorem 4 and then proceed to minimise the estimate to obtain superasymptotics for $u^i(z)$, i.e. the order of the remainder term is exponentially small. In the following we will assume that the number of terms in each series at optimal truncation is proportional to $|z|^r$ plus a bounded number. This means that in this section the number of terms in the first or zeroth level expansion is

$$N_0^i = \beta_0^i |z|^r + \alpha_0^i \ .$$

There are two parts to the remainder term (3.32): we estimate these separately. The first part involves the functions $\epsilon_k^{iN_0^i}(z)$. This is easily estimated for z inside the contour $\mathcal{C}_k^i$ to be

$$\epsilon_k^{iN_0^i}(z) = \mathcal{O}(|z|^{-N_0^i}\rho^{N_0^i}) \ .$$

This term is small compared to the next term and is absorbed during the estimation process into the second term estimate. The second term of the remainder consists of a double sum of integrals but only one of these is relevant to our estimate.

Taking an individual term from the double sum and by finding its asymptotic behaviour we can show, using Laplace's Method, that

$$z^{1-N_0^i}\int_{[\theta_{ij}^q]} \frac{D_{ij}(\xi^r)u^j(\xi^r)\xi^{N_0^i-1}}{\xi - z} d\xi = \mathcal{O}(1)\exp\left[-P_{ijq}\left(\left(\frac{\beta_0^i}{\mu_{ij}r}\right)^{1/r}|z|\right)\right]\left(\frac{\beta_0^i}{\mu_{ij}r}\right)^{\frac{\beta_0^i}{r}|z|^r} |z|^{\omega_{ij}-r/2}$$

where $\mu_{ij} = |\lambda_i - \lambda_j|$, $\omega_{ij} = \Re\tau_j - \tau_i$ and

$$P_{ijq}(\xi) = \mu_{ij}\xi^r - \sum_{s=1}^{r-1}(\lambda_{is} - \lambda_{js})(\xi e^{(\theta_{ij}+2\pi i q)/r})^{r-s} \ .$$

If we analytically continue the function by indenting the contour around the singularity this estimate can be shown to be true (under certain technical restrictions which do not limit us in practice) also for z on the line of integration . The details of this estimation are long and will be omitted.

For this estimate to be minimal we find that

$$\beta_0^i = r\mu_{ij} \ .$$

With this minimal estimate we can see that in the limit as $|z| \to \infty$ the dominant integral in the second sum in (3.32) is the one which has the smallest value of $\mu_{ij}, j \neq i$ as long as the multiplier, K_{jq}^i, is non-zero.

3.3.4 Level one

To obtain the level one expansion we substitute the series expansions with remainder term for the solutions $u^j(z)$ which occur in the remainder term in (3.32). When we do this we obtain the expansion

$$R_{N_0^i}^i(z) = \sum_{q=0}^{\nu-1}\sum_{p=1}^{r-1}\epsilon_{qp}^{iN_0^i}(z) - \frac{z^{1-N_0^i}}{2\pi i}\sum_{j\neq i}\sum_{q=1}^{r}K_{jq}^i\sum_{s=0}^{N_1^j-1}c_s^j\int_{[\theta_{ij}^q]}\frac{D_{ij}(\xi^r)\xi^{N_0^i-s-1}}{\xi - z}d\xi + R_{N_0^i;(N_1^j,j\neq i)}^i$$

where the second level remainder term

$$R_{N_0^i;(N_1^j,j\neq i)}^i(z) = -\frac{z^{1-N_0^i}}{2\pi i}\sum_{j\neq i}\sum_{q=1}^{r}K_{jq}^i\int_{[\theta_{ij}^q]}\frac{D_{ij}(\xi^r)u^j(\xi^r)\xi^{N_0^i-1}R_{N_1^j}^j(\xi)}{\xi - z}d\xi \ .$$

The first term is estimated as before. We now need to estimate the double integrals which occur in the second term. To estimate the integrals successfully we must first substitute $\rho - \delta$ for ρ in the integration limits in the inner integrals. $|z| - \rho > \delta > 0$ is a small number. Once we do this the estimate for integrals of ϵ terms can be absorbed in the second sum estimate. The estimate for the double integrals in the second sum decouples when the denominator of the integrand is bounded below by δ and becomes essentially equivalent to a product of the two integrals in the level zero case. The final estimate which we arrive at is that

$$R^i_{N^i_0;(N^j_1,j\neq i)}(z) = \mathcal{O}(1)\exp(-P_{ijq}((\frac{\beta^i_0-\beta^j_1}{\mu_{ij}r})^{1/r}|z|))\left(\frac{\beta^i_0-\beta^j_1}{\mu_{ij}r}\right)^{\frac{\beta^i_0-\beta^j_1}{r}|z|^r}\times$$
$$\exp(-P_{jj_1q_1}((\frac{\beta^j_1}{\mu_{jj_1}r})^{1/r}|z|))\left(\frac{\beta^j_1}{\mu_{jj_1}r}\right)^{\frac{\beta^j_1}{r}|z|^r}|z|^{\omega_{ij}+\omega_{jj_1}-r}\ .$$

If the two contours in successive integrations are in the same direction then the outer must avoid the inner on the right to give the correct analytic continuation. If we need to do this then the estimate is weakened by a factor of $|z|$ every time.

Minimising this estimate we find that

$$\begin{aligned}\beta^i_0 &= r(\mu_{ij}+\mu_{jj_1})\ ,\\ \beta^j_1 &= r\mu_{jj_1}\ .\end{aligned}$$

The largest of these minima determines the order of the remainder term and this is given by the term with the smallest value of the sum $\mu_{ij}+\mu_{jj_1}$ over all possible values of j and j_1 excluding those paths which have a Stokes' multiplier which is zero.

3.3.5 Some Integrals

To simplify the statement of the theorem we define the following integrals:

$$G\binom{ijq}{N}(z) = \int_{[\theta^q_{ij}]}\frac{D_{ij}(\xi^r)\xi^{N-1}}{\xi-z}d\xi$$

for the first order integrals and for the orders greater than one:

$$G\binom{ijq}{N_0};\binom{jj_1q_1}{N_1};\ldots;\binom{j_{n-1}j_nq_n}{N_n}(z) =$$
$$\int_{[\theta^q_{ij}]}\int_{[\theta_{jj_1},q_1]}\cdots\int_{[\theta_{j_{n-1}j_n},q_n]}\frac{D_{ij}(\xi^r)D_{jj_1}(\xi^r_1)\ldots D_{j_{n-1}j_n}(\xi^r_n)\xi^{N_0-1}\xi^{N_1-1}\ldots\xi^{N_n-1}}{(\xi-z)(\xi_1-\xi)\ldots(\xi_n-\xi_{n-1})}d\xi_n\ldots d\xi_1 d\xi\ .$$

3.3.6 The Hyperasymptotic Theorem

With the first few levels calculated we can see the emerging pattern. The full hyperasymptotic expansion for each of the functions u^i is expressed in the following theorem (which can be proved by induction).

Theorem 5

$$
\begin{aligned}
u^i(z^r) \;=\;& \sum_{s=0}^{N_0^i-1} z^{-s} c_s^i + z^{1-N_0^i} \sum_{j\neq i} \sum_{q=0}^{r-1} \frac{-K_{jq}^i}{2\pi i} \left(\sum_{s=0}^{N_1^{ij}} c_s^j G\begin{pmatrix} ijq \\ N_0^i - s \end{pmatrix}(z) \right. \\
&+ \sum_{j_1\neq j} \sum_{q_1=0}^{r-1} \frac{-K_{j_1 q_1}^j}{2\pi i} \left(\sum_{s=0}^{N_2^{jj_1}} c_s^{j_1} G\begin{pmatrix} ijq & ; & jj_1q_1 \\ N_0^i - N_1^j + 1 & ; & N_1^j - s \end{pmatrix}(z) \right. \\
&+ \sum_{j_2\neq j_1} \sum_{q_2=0}^{r-1} \frac{-K_{j_2 q_2}^{j_1}}{2\pi i} \left(\sum_{s=0}^{N_3^{jj_1j_2}} c_s^{j_2} G\begin{pmatrix} ijq & ; & jj_1q_1 & ; & j_1j_2q_2 \\ N_0 - N_1 + 1 & ; & N_1 - N_2 + 1 & ; & N_2 - s \end{pmatrix}(z) \right. \\
&+ \cdots \\
&+ \sum_{j_p\neq j_{p-1}} \sum_{q_p=0}^{r-1} \frac{-K_{j_p q_p}^{j_{p-1}}}{2\pi i} \left(\sum_{s=0}^{N_p^{jj_1\cdots j_p}} c_s^{j_p} G\begin{pmatrix} ijq & ; & jj_1q_1 & ; \cdots ; & j_{p-1}j_pq_p \\ N_0 - N_1 + 1 & ; & N_1 - N_2 + 1 & ; \cdots ; & N_p - s \end{pmatrix}(z) \right. \\
& \left.\left.\left.\left. \right) \cdots \right) \right) \right) \\
&+ R_p^i(z) \; .
\end{aligned}
$$

Our estimate for the remainder appearing in the theorem is

$$
\begin{aligned}
R^i_{N_0^i;(N_1^j, j\neq i);\ldots;(N_p^{j\cdots j_p}, j\neq j_{p-1})}(z) = \mathcal{O}(1) \sum_{j\neq i} \sum_{q=0}^{r-1} \frac{-K_{jq}^i}{2\pi i} \sum_{j_1\neq j} \sum_{q_1=0}^{r-1} \frac{-K_{j_1q_1}^j}{2\pi i} \cdots \\
\sum_{j_p\neq j_{p-1}} \sum_{q_p=0}^{r-1} \frac{-K_{j_pq_p}^{j_{p-1}}}{2\pi i} \exp(-P_{ijq}((\frac{\beta_0^i - \beta_1^j}{\mu_{ij} r})^{1/r}|z|)) \left(\frac{\beta_0^i - \beta_1^j}{\mu_{ij} r} \right)^{\frac{\beta_0^i - \beta_1^j}{r}|z|^r} \times \\
\exp(-P_{jj_1q_1}((\frac{\beta_1^j - \beta_2^{j_1}}{\mu_{jj_1} r})^{1/r}|z|)) \left(\frac{\beta_1^j - \beta_2^{j_1}}{\mu_{jj_1} r} \right)^{\frac{\beta_1^j - \beta_2^{j_1}}{r}|z|^r} \times \cdots \times \\
\exp(-P_{j_{p-1}j_pq_p}((\frac{\beta_p^{j_p}}{\mu_{j_{p-1}j_p} r})^{1/r}|z|)) \left(\frac{\beta_p^{j_p}}{\mu_{j_{p-1}j_p} r} \right)^{\frac{\beta_p^{j_p}}{r}|z|^r} |z|^{\omega_{ij} + \cdots + \omega_{j_{p-1}j_p} - r} \; .
\end{aligned}
$$

This estimate is minimised when

$$
\begin{aligned}
\beta_0^i &= r(\mu_{ij} + \mu_{jj_1}) \ , \\
\beta_1^j &= r(\mu_{jj_1} + \mu_{j_1j_2}) \ , \\
&\dots \\
\beta_p^{j_p} &= r\mu_{j_{p-1}j_p} \ .
\end{aligned}
$$

The largest of these minima determines the order of the remainder term and this is given by the term with the smallest value of the sum over all μ excluding those paths which have a Stokes' multiplier which is zero.

3.3.7 Conclusions

In this chapter we have shown the existence of the hyperasymptotic expansion for the class of linear ordinary differential equations of high order with a large (but non infinite) integer rank irregular singularity at infinity with a single type of dominant exponential behaviour at infinity. This gives results which are similar in nature to those of Olde Daalhuis [18] for rank one irregular singularities but using a different method and extending them to high rank irregular singularities. In the rank one case our result reduces to that of Olde Daalhuis.

The obvious and difficult generalisation of this work is to treat the case where the exponential behaviours of the solutions at infinity is not restricted to one level. This case of mixed exponential rank is very complicated and introduces very difficult treatments when a solution of lower exponential rank can not be seen at any hyperasymptotic level of a higher one. It is not clear whether for a general problem any general hyperasymptotic theorem can be found.

Bibliography

[1] M. Abramowitz and I.A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, volume 55 of *Nat. Bur. Standards Applied Math. Series.* U.S. Govt. Printing Office, Washington, DC, 1964.

[2] W. Balser. *From Divergent Power Series to Analytic Functions, Theory and Application of Multisummable Power Series.* Springer Verlag, 1994.

[3] W. Balser, W.B. Jurkat, and D.A. Lutz. On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities, I. *SIAM J. Math. Anal.*, 12(5), 1981.

[4] C.M. Bender and S.A. Orzag. *Advanced Mathematical Methods for Scientists and Engineers.* Mathematics Series. McGraw-Hill, Washington, DC, 1978.

[5] M. V. Berry. Asymptotics, superasymptotics, hyperasymptotics... In H. Segur, S. Tanveer, and H. Levine, editors, *Asymptotics beyond All Orders*, volume 284 of *NATO ASI Series, Series B:*, pages 1–14, 1991.

[6] M.V. Berry and C.J. Howls. Hyperasymptotics. *Proc. Roy. Soc. London Ser. A*, 434:657–675, 1991.

[7] B.L.J. Braaksma. Multisummability and Stokes' multipliers of linear meromorphic differential equations. *Journal of Differential Equations*, 92:45–75, 1991.

[8] E. Delabaere. Introduction to the Ecalle theory. In E. Tournier, editor, *Computer Algebra and Differential Equations*, volume 193 of *London Mathematical Society Lecture Notes Series.* Cambridge University Press, 1994.

[9] C.J. Luke. Examples of resurgence. *Series of Lectures in Dublin City University*, 1996.

[10] C.J. Luke. Summability and asymptotic representations of resonance poles. *In Personal Correspondence*, 1997.

[11] B.T. Murphy and A.D. Wood. Hyperasymptotic solutions of second order ordinary differential equations with a singularity of arbitrary integer rank. *Methods and Applications of Analysis*, 4(3):250–260, 1997.

[12] A.B. Olde Daalhuis. Hyperterminants I. *J. Comput. Appl. Math.*, 76(1-2):255–264, 1996.

[13] A.B. Olde Daalhuis and F.W.J. Olver. Exponentially-improved asymptotic solutions of ordinary differential equations II: irregular singularities of rank one. *Proc. Roy. Soc. London Ser. A*, 445:39–56, 1994.

[14] A.B. Olde Daalhuis and F.W.J. Olver. Hyperasymptotic solutions of second order linear differential equations I. *Methods and Applications of Analysis*, 2(2):173–197, 1995.

[15] A.B. Olde Daalhuis and F.W.J. Olver. On the calculation of Stokes' multipliers for linear differential equations of the second order. *Methods and Applications of Analysis*, 2(3):348–367, 1995.

[16] A.B. Olde Daalhuis and F.W.J. Olver. On the asymptotic and numerical solution of linear ordinary differential equations. *Submitted to SIAM review*, 1997.

[17] Adri Olde Daalhuis. Hyperterminants II. *Preprint*, 1997.

[18] Adri Olde Daalhuis. Hyperasymptotic solutions of higher order differential equations with a singularity of rank one. *Proc. Roy. Soc. London Ser A*, 1998.

[19] F. Olver. *Asymptotics and Special Functions.* Academic Press, New York, 1974.

[20] F. Olver. On Stokes' phenomenon and converging factors. In R. Wong, editor, *Asymptotic and Computational Analysis*, volume 124 of *Lecture Notes in Pure and Applies Mathematics*, pages 329–355. Marcel Dekker, New York, 1990.

[21] F. Olver. Asymptotic solutions of linear ordinary differential equations at an irregular singularity of rank unity. *Submitted to Meth. Appl. Anal.*, 1997.

[22] F.W.J. Olver and F. Stenger. Error bounds for asymptotic solutions of second order differential equations having an irregular singularity of arbitrary rank. *SIAM J. Numer. Anal.*, 2:244–249, 1965.

[23] N. Temme. Contour integrals and uniform expansions. In *From the introductory course in asymptotic analysis given at the Newton Institute Cambridge*, 1995.

[24] H.L. Turritin. Convergent solutions of ordinary linear homogenous differential equations in the neighborhood of an irregular singular point. *Acta Math.*, 93:27–66, 1955.

[25] R. Wong. *Asymptotic Approximation of Integrals.* Academic Press, New York, 1989.