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MSc Thesis

# Matched Asymptotics for a Generalisation of a Model Equation for Optical Tunnelling

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ABSTRACT

The fundamental purpose of this thesis is to consider the equation

$$y''(x)+(\lambda+\epsilon x^{n})y(x)=0 \quad , \quad x \in (0,\infty)$$

with $y'(0) + hy(0)=0$ , $h$ positive constant

and $y(x)$ has controlling behaviour $e^{ip(x)}$

as $x \rightarrow +\infty$ for some positive function $p(x)$.

The equation with $n = 1$ models the leakage of energy from the core of a bent fibre optic waveguide, the rate of leakage corresponding to $\mathrm{Im}(\lambda)$, which was show to be exponentially small like $O(\exp[-1/\epsilon])$ by Paris and Wood. The extension to $n = 2$ by Brazel, Lawless and Wood obtained $\mathrm{Im}(\lambda) = O(\exp[-1/\epsilon^{1/2}])$. Both these papers involve delicate analysis of the asymptotics of special functions near to Stokes' lines. When $n > 2$ no special functions are available and completely different methods must be employed to obtain the result $\mathrm{Im}(\lambda) = O(\exp[-1/\epsilon^{1/n}])$. In this thesis we obtain this result by matched asymptotics across the turning pointnearest to the positive real axis by WKB type approximate solution which includes the earlier result of $n = 1,2$.

In Chapter 1, we provide an overview of optical tunnelling literature and introduce the model of optical tunnelling by Kath and Kriegsmann and the work done by Paris and Wood and by Brazel Lawless and Wood for cases $n = 1,2$.

In Chapter 2, we introduce the theory of asymptotic matching and detailed WKB theory , Airy functions and Stokes' phenomenon..

In Chapter 3, we derive the solution to the generalised model optical tunnelling equation by asymptotic matching.

In Chapter 4,we use the asymptotic solution we obtain in Chapter 3 to obtain the general formula for the imaginary part of the eigenvalue $\lambda$, which turns out to be exponentially small as predicted.

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# CHAPTER 1

# REVIEW OF OPTICAL TUNNELLING LITERATURE

In this chapter, we first introduce the optical tunnelling problem briefly, derive the model problem and review the research work which has already been done by Paris and Wood and by Brazel.

## 1.1 The Optical Tunnelling Problem

Kath and Kriegsman [4] have recently considered radiation losses in bent fibre-optic waveguides where arbitrary deformations, including torsion, in three dimensions are allowed. Such waveguides do not trap light perfectly and energy slowly tunnels out of the core region into the cladding. The authors assume that the radius of curvature of the bent fibre is very large compared to the wavelength of light used. The fibre is also taken to be weakly guiding, so that the refractive index in the cladding deviates only slightly from that in the core. Thus the main direction of light propagation is along the axis of the fibre and a scalar wave equation is a good approximation to Maxwell's equations. With a choice of coordinate system following the fibre, suitably scaled in terms of the radius a of the inner core, the amplitude A of the transverse component of the electric field is described by the equation

$$2iA_{\sigma}+A_{\xi\xi}+A_{\eta\eta}+f(\xi,\eta)A+k^{-2}(2k_1\alpha A+A_{\sigma\sigma})=0 \qquad [1.1.1]$$

where $\alpha=\xi\cos\Theta - \eta\sin\Theta$. Here $(\xi,\eta)$ is a transverse coordinate system following the fibre, based on a dimensionless Frenet-Serret formula, and $\Theta$ is a rotation of the fibre which removes the torsion. The variable $\sigma$ measures a scaled length along the fibre, k is a dimensionless wave number (of order 10 to 20) of the light used and $k_1$ is a scaled curvature which is $o(1)$. The function $f(\xi,\eta)$ is the scaled difference in the index of refraction in the core.

To find the energy loss caused by this perturbation, Kath and Kriegsman set

$$A(\sigma,\xi,\eta)=y(\xi,\eta)e^{i\Lambda\sigma}$$

which gives

$$\nabla^2 y + f(\xi,\eta) + \lambda y + \epsilon\alpha y=0 \qquad [1.1.2]$$

on setting $\epsilon=2k_1/k^2$, $\lambda=2\Lambda-(\Lambda/k)^2$

The object of this exercise is to find the decay rate Im ($\Lambda$), which must be negative. An obvious solution to try is a perturbation expansion

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots$$

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \ldots$$

Substitution in [1.1.2] gives to leading order, the straight waveguide problem

$$\nabla^2 y_0 + f(\xi,\eta)y_0 + \lambda y_0 = 0 \qquad [1.1.3]$$

with symmetric boundary conditions, which is well known to have a real, negative eigenvalue if $f(\xi,\eta) > 0$. Comparison of terms of $O(\epsilon)$ gives the equation

$$\nabla^2 y_1 + f(\xi,\eta)y_1 + \lambda_0 y_1 + (\lambda_1 + \alpha)y_0 = 0$$

Multiplying by $y_0$, using [1.1.3], and integrating over $(\xi,\eta)$, we find

$$\lambda_1 \iint |y_0|^2 d\xi d\eta + \iint \alpha |y_0|^2 d\xi d\eta = 0 \qquad [1.1.4]$$

where we use the divergence theorem to eliminate cross terms such as $y_0\nabla^2 y_1$. We conclude from [1.1.4] that $\lambda_1$ is real and successive calculation of higher order terms produce the result

$$\mathrm{Im}(\lambda_n) = 0, \qquad n = 1,2,3,\ldots$$

Thus no information on the rate of decay is obtained by this analysis.

Kath and Kriegsman point out that this method fails because the perturbation is really singular. While this regular perturbation method gives a good approximation in the core, the term $\epsilon\alpha$ need not be small in the cladding. In the region $\alpha=O(1/\epsilon)$, the solution changes from evanescent to propagating. Instead they obtain an estimate of $\mathrm{Im}(\Lambda)$ by using a WKB expansion in a variational formula given by Bender and Wu [11]. The argument in [11] is essentially heuristic. To avoid the difficulties of treating subdominant terms inherent in this method we put forward alternative approaches to the problem.

## 1.2 The Model Problem

In order to gain a better understanding of the problem of estimating the small, but crucial, imaginary part of the eigenvalue in [1.1.2], the following one dimensional problem has been proposed. Observe that in the original problem, for small $\epsilon$, We are in the cladding region where the perturbation $f(\xi,\eta)$ in the refractive index is zero. We are interested in the behaviour of solutions in the neighbourhood of a turning point which is situated well into the cladding region. We therefore feel justified in considering the model problem:

$$i\emptyset_t = -\emptyset_{xx} - \epsilon g(x)\emptyset \qquad [1.2.1]$$

with the general linear homogeneous boundary condition

$$\emptyset_x(0,t) + h\emptyset(0,t)=0 \qquad [1.2.1a]$$

where the positive constant h is essentially a matching parameter.

In what follows we shall take $g(x)=x^n$, where n is a positive integer.

Making the separation of variables

$$\emptyset(x,t)=e^{-i\lambda t}y(x) \quad \text{with } Im(\lambda) < 0$$

then equation [1.2.1] becomes

$$\lambda e^{-i\lambda t}y(x)=-e^{-i\lambda t}y''(x) - \epsilon g(x)e^{-i\lambda t}y(x)$$

Hence,

$$y''(x) + (\lambda + \epsilon x^{n})y(x)=0$$

with the boundary condition becoming

$$e^{-i\lambda t}y'(0) + e^{-i\lambda t}hy(0)=0$$

that is,

$$y'(0) + hy(0) = 0$$

The solution must be an outgoing wave beyond the turning point at $x=(-\lambda/\epsilon)^{1/n}$ which is chosen and fixed nearest to the positive real axis. The choice of outgoing wave depends on the time dependence sign in separation of variables. It will be seen later that this turning point is exponentially close to the real axis as $\epsilon \to 0^{+}$. We express this condition by constraining any solution y to have controlling behavior of the form $e^{ip(x)}$ where p(x) is a positive function in x as $x \to +\infty$ .

Thus our model problem is of the form

$$y''(x) + ( \lambda + \epsilon x^{n})y(x)=0 \qquad [1.2.2a]$$

$$y'(0)+hy(0)=0 \qquad [1.2.2b]$$

$y(x)$ has controlling behavior $e^{ip(x)}$ as $x \to +\infty$ [1.2.3]

where $p(x)$ is a positive function in $x$, $h$ is a positive constant and $\epsilon > 0$

The problem [1.2.2a] [1.2.2b] appears at first glance to be self-adjoint, in which case the spectrum would be real, but a careful analysis shows that the conditions for self-adjointness are broken at infinity. To see this let $Lu = u'' - \epsilon x^n u$ and denote by $\langle,\rangle$ the usual inner product in the Hilbert space $L^2(0,\infty)$. Then for any functions $u,v \in L^2(0,\infty)$ satisfying the boundary conditions, an integrations by parts shows that

$$\langle Lu,v\rangle = -[u'(x)\overline{v(x)}]\Big|_0^\infty + \int_0^\infty u'(x)\overline{v'(x)}dx - \epsilon\int_0^\infty x^n u(x)\overline{v(x)}dx$$

and

$$\langle u,Lv\rangle = -[u(x)\overline{v'(x)}]\Big|_0^\infty + \int_0^\infty u'(x)\overline{v'(x)}dx - \epsilon\int_0^\infty x^n u(x)\overline{v(x)}dx$$

The self-adjointness condition $\langle Lu,v\rangle=\langle u,Lv\rangle$ holds if and only if the integrated terms are equal. Since they are clearly equal at the origin, this condition is equivalent to

$$\lim_{x\to\infty} u'(x)\overline{v(x)} = \lim_{x\to\infty} u(x)\overline{v'(x)}$$

When we insert the proposed outgoing wave behavior $e^{ip(x)}$, we find that $u'(x)\overline{v(x)} \sim ip'(x)$, but $u(x)\overline{v'(x)} \sim -ip'(x)$ as $x \to +\infty$ The problem is thus non-adjoint and non-real eigenvalues may occur.

Straightforward perturbation expansion about $\epsilon=0$ shows that the eigenvalue of [1.2.2a] [1.2.2b] is given by asymptotic series

$$n=1: \quad \lambda= -h^2 - \epsilon/(2h) - \epsilon^2/(8h^4) - 5\epsilon^3/(32h^7) + O(\epsilon^4) \tag{1.2.3}$$

$$n=2: \quad \lambda= -h^2 - \epsilon/(2h^2) - 7\epsilon /(8h^6) - 121\epsilon /(16h^{10})+O(\epsilon^4) \tag{1.2.4}$$

$$n\geq 2: \quad \lambda= -h^2 - n!\epsilon/(2h)^n + O(\epsilon^2) \tag{1.2.5}$$

It is apparent that this method yields no information on $\mathrm{Im}\lambda$.

The fact that $\exp[-1/\epsilon^{1/k}] \ll \epsilon^k$, as $\epsilon \to 0^+$, for any $k \in N$, renders $\mathrm{Im}(\lambda)$ very difficult to compute by standard regular perturbation methods or numerical schemes. For Instance when $n = 2$ and $\epsilon = 0.0001$ , $\mathrm{Im}(\lambda) = O(10^{-69})$. When compared with a $\mathrm{Re}(\lambda)$ of $O(1)$, computation becomes very difficult. We are thus in the area of exponential asymptotics (Meyer), asymptotics beyond all orders (Segur and Kruskal) or hyperasymptotics (Berry).

## 1.3. Introduction of Research of This Model Equation

The inspiration of this thesis lies in the work carried out by R. Paris and A. Wood (see reference [3]) who concern themselves with the model problem given below with g(x)=x:

$$i\phi_t = - \phi_{xx} - \epsilon g(x)\phi \qquad [1.3.1]$$

with the general linear homogeneous boundary condition

$$\phi_x(0,t) + h\phi(0,t) = 0 \qquad [1.3.1a]$$

and for physical reasons, any solution $\phi$ is constrained to be an outgoing wave beyond the turning point. They in turn were motivated by the I.M.A lecture entitled "Mathematics in Industry and the Prevalence of the free boundary problems" given by Dr. Ockendon who expressed the need for a rigorous proof of results W. Kath and G.Kriegsmann in a forthcoming paper (see reference [4]). In this paper, the authors attempt to estimate the energy loss in a fibre optic waveguide due to curvature in the fibre. This requires estimating the imaginary part of an eigenvalue which is extremely small. R. Paris and A. Wood successfully solved the model equation [1.3.1] with g(x)=x by using Airy functions and dealing with Stokes' phenomenon.

The imaginary part of eigenvalue is found to be

$$\mathrm{Im}(\lambda) \sim \left( - 2h^2/e\right) \exp\left[ - 4h^3/(3\epsilon) \right] \quad \epsilon \rightarrow 0^+ \qquad [1.3.2]$$

Brazel,Lawless and Wood solved the case $g(x)=x^2$ by using Weber's solution of parabolic cylinder equation. In [10] they find that the exponentially small behaviour of imaginary part of $\lambda$ persists, namely

$$\mathrm{Im}(\lambda) \sim - 2h^2 \exp\left[ - \pi h^2/(\epsilon^{1/2}) \right], \; \epsilon \rightarrow 0^+ \qquad [1.3.3]$$

In the case n=1,2 the appearance of these eigenvalues with exponentially small imaginary part is intimately connected to the smoothing of Stokes' discontinuities (Berry[9]) which is equivalent to the exponentially-improved asymptotics of Olver[3]. The methods employed for n=1,2 depend on a knowledge of explicit solutions in terms of special functions, Airy and Hankel functions in [4] and parabolic cylinder functions in [10]. The appearance of exponentially small terms arises from careful consideration of the full asymptotic expansion in the neighbourhood of Stokes' lines of these special functions using the result of [2] and [3].

Case n=1 is used by Paris and Wood as idealised model equation for optical tunnelling. We are interested in mathematically generalising this model equation to $n \geq 1$ and finding whether imaginary part of $\lambda$ persists in exponentially small form.

Comparison of relations [1.3.2] and [1.3.3] leads us to speculate that for general case $g(x)=x^n$ in the model problem,

$$\mathrm{Im}(\lambda) \sim - A \exp\left[- B/ \epsilon^{1/n}\right] \quad , \epsilon \to 0^+ \quad n = 1,2,3,...$$

where A and B are positive real constants. The establishment of this result for general positive integer n is the main purpose of the current thesis. We will obtain the leading term of the asymptotic expansion of $\mathrm{Im}(\lambda)$ for $n \in N$ and show that it agrees with the results for $n = 1,2$.

For $n \geq 3$ there are no special function solutions available and must rely on totally different methods, the concept of matched asymptotic expansions given in Bender and Orszag[2]. The method proceeds as follows. We identify the WKB approximate solution for large x which satisfies the "outgoing wave" condition [1.2.3] and match this to the Airy function approximate solution valid in the neighbourhood of the turning point $x=(-\lambda/\epsilon)^{1/n}$ nearest to positive real axis. This turning point will be exponentially close to the axis and tend to it as $\epsilon \to 0^+$. We then match this combination of

Airy functions to the WKB solution valid to the left of the turning point and substitute this into the boundary condition[1.2.2a]. This leads to the eigenvalue relation from which we obtain a general formula of Im($\lambda$) which includes the earlier result of n = 1,2.

# CHAPTER 2

## BACKGROUND OF METHOD OF MATCHED ASYMPTOTICS

### 2.1 Asymptotic Matching

The purpose of this section is to introduce the notation of matched asymptotic expansions. Asymptotic matching is an important perturbative method which is used in both boundary-layer theory and WKB theory to determine analytically the approximate global properties of the solution to a differential equation. Asymptotic matching is usually used to determine a uniform approximation to the solution of a differential equation and to find other global entities of differential equations such as eigenvalues. Asymptotic matching may also be used to develop approximation to integrals.

The principle of asymptotic matching is simple. The interval on which a boundary-value problem is posed is broken into a sequence of two or more overlapping subintervals. Then, on each subinterval perturbation theory is used to obtain an asymptotic approximation to the solution of the differential equation valid on that interval. Finally, the matching is done by requiring that the asymptotic approximation have the same functional form on the overlap of every pair of intervals. This gives a sequence of asymptotic approximations to the solution of the differential equation; by construction, each approximation satisfies all the boundary conditions given at various points on the interval. Thus the end result is an approximate solution to a boundary-value problem valid over the entire interval.

On the overlap region, the asymptotic approximation solutions to the differential equations are being matched rather than exact solution being matched. Matching is done by comparing functions over an interval whose length approaches $\infty$ as $\epsilon$, the perturbing parameter , approaches 0 from above.

## 2.2. WKB Method

WKB theory is a powerful tool for obtaining a global approximation of a linear differential equation whose highest derivative is multiplied by a small parameter $\epsilon$; it contains boundary-layer theory as a special case.

The WKB approximation to a solution of a differential equation has a simple structure. The exact solution may be some unknown function of overwhelming complexity, yet, the WKB approximation, order by order in powers of $\epsilon$, consists of exponentials of elementary integral of algebraic functions, and well-known special functions, such as Airy functions (for linearisation about a simple turning point) or parabolic cylinder functions (double turning point). WKB approximation is suitable for linear differential equations of any order, for intial-value and boundary-value problems, and for eigenvalue problems. It may also be used to evaluate integrals of the solution of a differential equation. The limitation of conventional WKB techniques is that they are only useful for linear equations.

For a differential equation that exhibits dissipative and dispersive phenomena, it is natural to seek an approximate solution of the form

$$y(x) \sim A(x)e^{S(x)/\delta} \quad , \delta \to 0 \qquad [2.2.1]$$

The phase S(x) is assumed nonconstant and slowly varying in a breakdown region. When S is real, there is a boundary layer of thickness; when S is imaginary, there is a region of rapid oscillation characterized by waves having wavelength of order $\delta$. When S(x) is constant, the behaviour of y(x), which is characteristic of an outer solution in boundary-layer theory, is expressed by the slowly varying amplitude function A(x).

The exponential approximation in [2.2.1] is conventionally known as a WKB approximation, named after Wentzel, Kramers, and Brillouin who independently but simultaneously popularized the theory. However, credit should also be given to many others including Rayleigh and Jeffreys who contributed to its early development.

The exponential approximation in [2.2.1] is not in a form most suitable for deriving asymptotic approximations because the amplitude and phase functions A(x) and S(x) depend implicitly on $\delta$. It is best to represent the dependence of these functions on $\delta$ explicitly by expanding A(x) and S(x) as series in powers of $\delta$. We can then combine these two series in a single exponential series of the form

$$y(x) \sim \exp\left[1/\delta \sum_{n=0}^{\infty} \delta^n S_n(x)\right], \quad \delta \to 0 \qquad [2.2.2]$$

This expression is the starting formula from which all WKB approximations are derived.

## 2.3. Conditions For Validity of the WKB Approximation

WKB theory is a singular perturbation theory because it is used to solve a differential equation whose highest derivative is multiplied by a small parameter (when the small parameter vanishs, the order of the differential equation changes abruptly). the singular nature of WKB theory is clearly evident in the $1/\delta$ term in the exponential approximation [2.2.2]

$$y(x) \sim \exp\left[1/\delta \sum_{n=0}^{\infty} \delta^n S_n(x)\right], \quad \delta \to 0 \qquad [2.3.1]$$

Unless $S_o(x)=0$, the approximation ceases to exist when $\delta=0$. The singular nature of this approximation also surfaces in a more subtle way - the WKB series $\Sigma\ \delta^n S_n$ usually diverges. ( The series converges if it truncates, but this is rare. This is why we use $\sim$ rather than $=$. Nevertheless, even though the WKB series diverges, it can give an extremely accurate numerical approximation to $y(x)$.

This section develops criteria for predicting when the WKB approximation will be useful. These criteria are quantitative; i.e, they specify how small $\delta$ must be for the WKB series in [2.3.1] to approximate y(x) to some prescribed relative error.

In order that the WKB approximation [2.3.1] be valid on an interval, it is necessary that the series $\Sigma\ \delta^{n-1}S_n(x)$ be an asymptotic series in $\delta$ as $\delta \to 0$ uniformly for all x on the interval. This requires that the asymptotic relations

$$\begin{aligned}
&\delta S_1(x) \ll S_0(x)\ , \quad \delta \to 0 \\
&\delta S_2(x) \ll S_1(x), \quad \delta \to 0 \qquad\qquad [2.3.2] \\
&\cdots \\
&\delta^n S_{n+1}(x) \ll \delta^{n-1} S_n(x), \quad \delta \to 0
\end{aligned}$$

hold uniformly in x. These conditions are equivalent to the requirement that each of the functions $S_{n+1}(x)/S_n(x)$ (n=0,1,2,3,...) be bounded functions of x on the interval (although these bounds may be arbitrary functions of n). If the series $\Sigma\ \delta^{n-1}S_n(x)$ is uniformly asymptotic in x as $\delta \to 0$, the optimal truncation rule suggests that truncating the series before the smallest term $\delta^N S_{N+1}(x)$ gives an approximation to ln y with uniformly small relative error throughout the x interval.

However, because the WKB series appears in the exponent [2.3.1], the asymptotic conditions in [2.3.2] are not sufficient to ensure that $\exp[\Sigma\delta^{n-1}S_n(x)]$ will be a good approximation to y(x). For the WKB series truncated at the term $\delta^{N-1}S_N(x)$ to be a good approximation to y(x), the next term must be small compared with 1 for all x in the interval of approximation, that is,

$$\delta^N S_{N+1}(x) \ll 1, \quad \delta \to 0 \qquad [2.3.3]$$

If this relation holds, then

$$\exp\left[\delta^N S_{N+1}(x)\right] = 1 + 0\left[\delta^N S_{N+1}(x)\right], \quad \delta \to 0$$

Thus the relative error between y(x) and WKB approximation is small:

$$\left\{y(x) - \exp\left[1/\delta \sum_{n=0}^{N} \delta^n S_n(x)\right]\right\} / y(x) \sim \delta^N S_{N+1}(x), \quad \delta \to 0$$

Both conditions [2.2.2] and [2.2.3] must be satisfied for WKB to be useful. i.e.

$$\delta S_1(x) \ll S_0(x), \quad \delta \to 0$$

$$\delta S_2(x) \ll S_1(x), \quad \delta \to 0 \qquad [2.3.4]$$

$$\cdots$$

$$\delta^n S_{n+1}(x) \ll \delta^{n-1} S_n(x), \quad \delta \to 0$$

$$\delta^N S_{N+1}(x) \ll 1, \quad \delta \to 0 . \qquad [2.3.5]$$

## 2.4. Definition of Geometrical and Physical Optics

If we retain only the first term in the WKB series, we are making the approximation of geometrical optics:

$$\exp\left[S_0(x)/\delta\right]$$

The first two terms in the WKB series constitute the approximation of physical optics:

$$y(x) \sim \exp\left[S_0(x)/\delta + S_1(x)\right], \quad \delta \to 0 \qquad [2.4.1]$$

## 2.5 Dominant Balance Method

Dominant balance method is one of the approach to solve differential equations approximately. It is used to identify those terms in an equation that may be neglected in an asymptotic limit. The technique of dominant balance consists of three steps:

1. We drop all terms that appear small and replace the exact equation by an asymptotic relation.
2. We replace the asymptotic relation with an equation by exchanging the sign ~ for an = sign and solve the resulting equation exactly (the solution to this equation automatically satisfies the asymptotic relation although it is certainly not the only function that does so).

3. We check that the solution we have obtained is consistent with the approximation made in (1). If it is consistent, we must still show that the equation for the function obtained by factoring off the dominant balance solution from the exact solution itself has a solution that varies less rapidly than the dominant balance solution. When this happens, we conclude that the controlling factor (and not the leading behaviour) obtained from the dominant balance relation is the same as that of the exact solution.

## 2.6. WKB Solution to a Schrödinger Equation

A second-order homogeneous linear differential equation is in Schrödinger form if the $y'$ term is absent and the leading coefficient is constant. The approximate solution to the Schrödinger equation

$$\epsilon^2 y'' = Q(x)y, \quad Q(x) \neq 0 \qquad [2.6.1]$$

are easy to find using WKB analysis when $\epsilon$ is small. We merely substitute [2.3.1] into [2.6.1]. Differentiating [2.3.1] twice gives

$$y' \sim \left[ 1/\delta \sum_{n=0}^{\infty} \delta^n S'_n(x) \right] \exp\left[ 1/\delta \sum_{n=0}^{\infty} \delta^n S_n(x) \right]$$

[2.6.2a]

$$y' \sim \left[ \delta^{-2}\left(\sum_{n=0}^{\infty} \delta^{n} S_n'(x)\right)^2 + \sum_{n=0}^{\infty} \delta^{n} S_n''(x) \right] \exp\left(\sum_{n=0}^{\infty} \delta^{n} S_n'(x)\right)$$

$$\delta \to 0. \qquad [2.6.2b]$$

Next we substitute [2.6.2a] , [2.6.2b] and [2.3.1] into [2.6.1] and divide off the exponential factors:

$$\delta^{-2}\epsilon^{2} S_0'^{2} + 2\delta^{-1}\epsilon^{-1} S_0' S_1' + \delta^{-1}\epsilon^{2} S_0'' = Q(x) \qquad [2.6.3]$$

The largest term on the left side of [2.6.3] is $\delta^{-2}\epsilon^{2} S_0'$. By dominant balance this term must have the same order of magnitude as Q(x) on the right side. (Here we have used the assumption that Q(x) $\neq$ 0 for x in the interval under consideration). Thus, $\delta$ is proportional to $\epsilon$ and for simplicity we choose $\delta = \epsilon$. Setting $\delta = \epsilon$ in [2.6.3] and comparing powers of $\epsilon$ gives a sequence of equations which determines $S_0$, $S_1$, $S_2$,...:

$$S_0'^{2} = Q(x) \qquad [2.6.4]$$

$$2S_0'S_1' + S_0'' = 0 \qquad [2.6.5]$$

$$2S_0'S_n' + S'' + S_j'S_{n-j}' = 0 \quad , n \geq 2 \qquad [2.6.6]$$

The equation for $S_0$ [2.6.4] is called eikonal equation; its solution is

$$S_0(x) = \pm \int^{x} [Q(t)]^{1/2} dt \quad . \qquad [2.6.7]$$

The equation for $S_1$ [2.5.5] is called thetransport equation. Apart from an additive constant, the solution is

$$S_1(x) = - \frac{1}{4} \ln Q(x) \quad . \qquad [2.6.8]$$

Combining [2.6.7] and [2.6.8] gives a pair of approximate solutions to the Schrödinger equation [2.6.1] one for each sign of $S_0$. The general solution is a linear combination of the two:

$$y(x) \sim C_1 Q(x)^{-1/4} \exp\left[ 1/\delta \int_a^x [Q(t)]^{1/2} dt \right]$$

$$+ C_2 Q(x)^{-1/4} \exp\left[ - 1/\delta \int_a^x [Q(t)]^{1/2} dt \right], \quad \epsilon \to 0 \qquad [2.6.9]$$

where $C_1, C_2$ are constants to be determined from initial or boundary conditions and a is an arbitrary but fixed integral point chosen to suit our convenience. This expression is the leading-order WKB approximation to the solution of [2.6.1]; it differs from the exact solution by terms of order $\epsilon$ in regions where $Q(x) \neq 0$.

A more accurate approximation to y(x) may be constructed from the higher terms in the WKB series. The next four terms, as computed from [2.5.6] by repeated differentiation, are

$$S_2 = \pm\int^x \left[\frac{Q''}{8Q^{3/2}} - \frac{5(Q')^2}{32Q^{5/2}}\right] dt \qquad [2.6.10]$$

$$S_3 = -\frac{Q''}{16Q^2} + \frac{5Q'^2}{64Q^3} \qquad [2.6.11]$$

$$S_4 = \pm\int^x \left[\frac{d^4Q/dx^4}{32Q^{5/2}} - \frac{7Q'Q'''}{32Q^{7/2}} - \frac{19(Q'')^2}{128Q^{7/2}} + \frac{221Q''(Q')^2}{256Q^{9/2}} - \frac{1,105(Q')^4}{2,048Q^{11/2}}\right] dt \qquad [2.6.12]$$

$$S_5 = -\frac{d^4Q/dx^4}{64Q^3} + \frac{7Q'Q'''}{64Q^4} + \frac{5(Q'')^2}{64Q^4} - \frac{113(Q')^2Q''}{256Q^5} + \frac{565(Q')^4}{2,048Q^6} \qquad [2.6.13]$$

## 2.7 Stokes phenomenon

We first introduce two symbols which express the relative behaviour of two functions. Let $f(Z)$ and $g(Z)$ be two complex fucntions. The notation $f(Z) << g(Z)$, $Z \rightarrow Z_0$ ,which is read "$f(Z)$ is much smaller than $g(Z)$ as $Z$ tends to $Z_0$," means

$$\lim_{Z\rightarrow Z_0} f(Z)/g(Z) = 0$$

We say that f is asymptotic to g, and write

$f(Z) \sim g(Z)$ as $Z \rightarrow Z_0$ in a sector of complex plane, say,centered on $Z_0$.

if $f(Z) - g(Z) << g(Z)$ in S as $Z \rightarrow Z_0$ .

Observe that $f(Z)$ may be written as

$$f(Z) = \{ f(Z) - g(Z)\} + g(Z).$$

Then what we are saying by writing $f(Z) \sim g(Z)$ as $Z\rightarrow Z_0$ in S is that $f(Z) - g(Z)$ is small(or subdominant) in S as compared with $g(Z)$ (which is dominant).

On the boundary of D, both $f(Z) - g(Z)$ and $g(Z)$ are of equal magnitude and as we cross this line, the part of $f(Z) - g(Z)$ becomes dominant while $g(Z)$ becomes subdominant. This occurrence is known as Stokes' phenomenon.

We define Stokes' lines to be those curves in the complex plane upon which the difference between the dominant and subdominant terms is greatest in magnitude.

Similarily, we define anti-Stokes lines to be those asymptotics in the complex plane upon which the "dominant" and "subdominant" terms are of equal magnitude.

If the controlling behaviour of solution to a second order differential equation are given by $\exp[S_1(x)]$ and $\exp[S_2(x)]$ as $Z \rightarrow Z_0$, then the Stokes lines are the asymptotes as $Z \rightarrow Z_0$ of the curves

$$\mathrm{Im}\ [\ S_1(x)\ -\ S_2(x)\ ] = 0 \qquad [2.7.1]$$

while the anti-Stokes lines are the asymptotes as $Z \rightarrow Z_0$ of the curves

$$\mathrm{Re}\ [\ S_1(x)\ -\ S_2(x)\ ] = 0\ . \qquad [2.7.2]$$

The reason for using an asymptotic approximation is to replace a complicated transcendental function like Ai(z) by simpler expression involving elementary functions like exponentials and powers of z. From a practical point of view, much is gained by such approximations. However, one pays for these advantages by having to deal with the complexities of the Stokes' phenomenon. The Stokes' phenomenon is not an intrinsic property of a function like Ai(z), but rather it is a property of the functions that are used to approximate it.

## 2.8 Stokes phenomenon and Ai(Z)

The Airy equation is given by

$$y''(x) = x\, y(x) \quad . \qquad [2.8.1]$$

When x is real, solutions of [2.8.1] are linear combinations of the Airy functions Ai(x) and Bi(x), while if x is complex they are combinations of the Airy functions with complex argumentss, Ai(x), $Ai(xe^{\pm 2\pi i/3})$, Bi(x), $Bi(xe^{\pm 2\pi i/3})$.

The general asymptotic expansion of the Ai function is

$$Ai(z) \sim C_1 z^{-1/4} \exp\left( 2/3\ z^{3/2}\right) \sum_{n=0}^{\infty} a_n z^{-3n/2}$$

$$+ C_2 z^{-1/4} \exp\left( -\ 2/3\ z^{3/2}\right) \sum_{n=0}^{\infty} (-1)^n a_n z^{-3n/2}. \qquad [2.8.2]$$

Here $C_1$ and $C_2$ are Stokes multipliers which depend on the region x is on the complex plane ( see figure 2). The two series in [2.8.2] are obtained by repeated application of the WKB approximation and a pair of linearly independent approximate solutions of [2.8.1]. The constants $C_1$ and $C_2$ are chosen so that [2.8.2] matches to the Taylor series solution for Ai(x) about x = 0. (See[2] Bender and Orszag page 130-131).

From Bender and Orszag we have

On sector $S_{-1}$: $-2\pi/3 < \arg(z) < 0$

$$C_1 = 0 \qquad C_2 = 1/2\ \pi^{-1/2}\ ,$$

On sector $S_{-2}$: $-4\pi/3 < \arg(z) < -2\pi/3$

$$C_1 = -i/2\pi^{-1/2} \qquad C_2 = 1/2\ \pi^{-1/2}\ .$$

In Chapter 3 we need to know the Stokes' multipliers for $\arg(z)=-4\pi/3$, so we need to calculate the Stokes multipliers $C_1$, $C_2$ on Sector $S_3$: $-2\pi < \arg(z) < -4\pi/3$ by using the relation from Bender and Orszag (3.8.21a).

$$Ai(Z) = -\omega\ Ai(\omega Z) - \omega^2 Ai(\omega^2 Z) \qquad [2.8.3]$$

where $\omega = e^{-2\pi i/3}$ is a cube root of unity.

Let $Z = \omega^{-2} z$, we have

$$Ai(z) = -\omega^{-1} Ai(\omega^{-1} z) - \omega^{-2} Ai(\omega^{-2} z) \qquad [2.8.3a]$$

If $z \in S_{-3}$ then $\omega^{-1} z \in S_{-2}$: $-4\pi/3 < \arg(z) < -2\pi/3$

$$Ai(\omega^{-1}x) \sim\ 1/2\ \pi^{-1/2}\omega^{1/4}z^{-1/4}\exp\left(2/3\ z^{3/2}\right)\sum_{n=0}^{\infty} a_n z^{-3n/2}$$

$$-\ i/2\pi^{-1/2}\omega^{1/4}z^{-1/4}\exp\left(-\ 2/3 z^{3/2}\right)\sum_{n=0}^{\infty}(-1)^n a_n z^{-3n/2}$$

Also $\omega^{-2} z \in S_{-1}$: $-2\pi/3 < \arg(x) < 0$ implies that

$$Ai(\omega^{-2}z) \sim\ 1/2\ \pi^{-1/2}\omega^{1/2}z^{-1/4}\exp\left(-\ 2/3\ z^{3/2}\right)\sum_{n=0}^{\infty}(-1)^n\ a_n z^{-3n/2}.$$

By using [2.8.3a] we have, when $z \in S_{-3}$

$$Ai(z) \sim - 1/2\ \pi^{-1/2}\omega^{-3/4}z^{-1/4}\exp\left(2/3\ z^{3/2}\right)\sum_{n=0}^{\infty} a_n z^{-3n/2}$$

Thus the Stokes' multipliers of Ai(x) in $S_{-3}$ are:

$$C_1 = - 1/2\ \pi^{-1/2}\omega^{-3/4} = -i/2\ \pi^{-1/2} \quad ,$$

$$C_2 = 0 \quad .$$

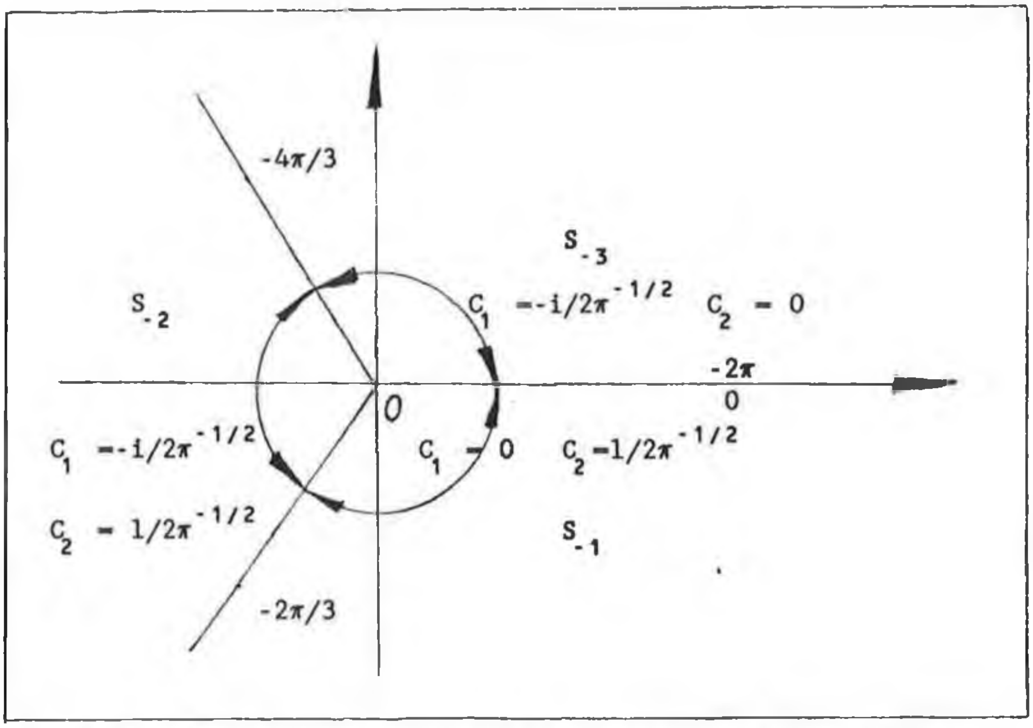


*Figure 2* Stokes' lines for Ai(z) with values of multipliers

According to Berry's paper[9], across a Stokes' line, where one exponential in an asymptotic expansion maximally dominates another, the multiplier of the small exponential changes rapidly. If the expansion is truncated near its least term the change is not discontinuous but smooth and moreover universal in form.

According Berry's notation in [9],

$$\text{In sector } S_{-2}: \qquad S(\sigma) = C_2/(iC_1) = 1$$

$$\text{In sector } S_{-3}: \qquad S(\sigma) = C_2/(iC_1) = 0$$

Berry's resultshows that on Stokes'line $\arg(z)=-4\pi/3$ we have,

$S = 1/2$ ,or in our notation $C_2 = 1/4\ \pi^{-1/2}$

Hence on Stokes line $\arg(z) = -4\pi/3$ we have

$$Ai(z) \sim -\ i/2\ \pi^{-1/2} z^{-1/4} \exp\left[2/3\ z^{3/2}\right]$$

$$+\ 1/4\ \pi^{-1/2} z^{-1/4} \exp\left[-\ 2/3\ z^{3/2}\right] \qquad [2.8.4]$$

$$\text{as} \quad |z| \to \infty \quad \text{in} \quad \arg(z) = -4\pi/3$$

# CHAPTER 3

## Matched Asymptotic for Generalised Optical Tunnelling Model Equation

In this chapter, we shall derive the solution to the model optical tunnelling equation by asymptotic matching. We first transform the model equation by changing the variable x to the variable w which will be convenient for asymptotic matching. We then divide real w-axis into three regions. In region I and region III we will have WKB type asymptotic solutions and in region II we will have Airy function solutions. Finally we will match them asymptotically on the overlap regions of validity. Thus we obtain the asymptotic solution to the transformed model equation.

## 3.1 Transforming the Model Problem

We restate the model problem here for clarity:

$$y''(x) + ( \lambda + \epsilon x^n )y(x) = 0 \text{ on } (0,\infty) \qquad [3.1.1]$$

$$y'(0)+hy(0) = 0 \qquad [3.1.2]$$

$y(x)$ has controlling behavior $e^{ip(x)}$ as $x \to \infty$ [3.1.3]

where $p(x)$ is a positive function in $x$, $h$ is a positive constant.

If we let $w = \delta(x-x_0)$ [3.1.4]

with $\delta = \epsilon^{1/n}$, $x_0 = \left(-\lambda/\epsilon\right)^{1/n}$ chosen and fixed to be the turning point closest to the positive real-axis.

then $dy/dx = \delta dy/dw$

$$dy^2/dx^2 = \delta^2 d^2y/dw^2$$

Substituting them into equations [3.1.1] and [3.1.2] we find [3.1.1] changes into

$$\delta^2 y'' = - \left[(w+\delta x_0)^n + \lambda\right] y$$

The boundary condition at the origin [3.1.2] becomes

$$\delta y'(-\delta x_0) + hy(-\delta x_0) = 0$$

Thus the transformed problem (where dash now denotes differentiation with respect to w) becomes

$$\delta^2 y'' = - \left[(w+\delta x_0)^n + \lambda\right] y \qquad [3.1.5]$$

$$\delta y'(-\delta x_0) + hy(-\delta x_0) = 0 \qquad [3.1.6]$$

y(w) has controlling behaviour exp(ip(w)) as $w \to +\infty$

p(w) is positive function [3.1.7]

## 3.2. Solution of the Transformed Problem

In terms of variable w, we divide the w-axis into three regions:

Region I: $w \gg \delta^{2/3}$, $\delta \to 0^+$

Region II: $|w| \ll 1$, $\delta \to 0^+$

Region III: Re $w < 0$, $-w \gg \delta^{2/3}$ $\delta \to 0^+$ .

In region I, the physical-optics WKB solution of the transformed equation is

$$y_I(w) = C_1\left[ -(w+\delta x_0)^n-\lambda \right]^{-1/4} \exp\left[ -1/\delta \int_0^w [-(t+\delta x_0)^n -\lambda]^{1/2} dt \right]$$

$$+ \quad C_2\left[ -(w+\delta x_0)^n-\lambda \right]^{-1/4} \exp\left[ 1/\delta \int_0^w [-(t+\delta x_0)^n -\lambda]^{1/2} dt \right].$$

[3.2.1]

The outgoing wave condition that y(w) has controlling behaviour exp(ip(w)) as $w \to +\infty$ (p(w) is positive function) excludes the term behaving like $\exp\left[- i/\delta \; p_1(w)\right]$ ($p_1(w)$ is positive function), thus we require $C_1 = 0$

Hence we have

$$y_I(w) = C\left[ -(w+\delta x_0)^n-\lambda \right]^{-1/4} \exp\left[ 1/\delta \int_0^w [-(t+\delta x_0)^n -\lambda]^{1/2} dt \right]. \quad [3.2.2]$$

Now we determine the precise region of validity of the approximation [3.2.2]. According to section 2.3 , two criteria for the validity of this approximation are

$$S_0/\delta \gg S_1 \gg \delta S_2 \quad , \; \delta \to 0^+ \qquad [3.2.3]$$

$$\delta S_2 \ll 1 \; , \qquad \delta \to 0^+ \; . \qquad [3.2.4]$$

If we set $$Q(w) = -(w+\delta x_0)^n - \lambda \qquad [3.2.5]$$

when w is small

$$Q(w) \simeq -n\left[-\lambda\right]^{(n-1)/n} w$$

We let $a = -n\left[-\lambda\right]^{(n-1)/n}$ where we also take the root which is closest to the positive real axis.

It follows that when w is small

$$Q(w) \sim aw$$

$$S_o(w) \sim \pm \frac{2}{3}a^{1/2}w^{3/2}, \qquad (w \to 0^+)$$

$$S_1(w) \sim -\frac{1}{4}\ln w, \qquad (w \to 0^+)$$

$$S_2(w) \sim \pm \frac{5}{48}a^{-1/2}w^{-3/2}, \qquad (w \to 0^+).$$

Thus, the criteria for validity of the WKB physical-optics approximation are satisfied if

$$w \gg \delta^{2/3}, \quad \delta \to 0^+. \qquad [3.2.7]$$

In Region II, we have the approximate differential equation

$$\delta^2 y'' = awy. \qquad [3.2.8]$$

This is effectively a linear approximation to the potential function Q(w) valid in the neighbourhood of the turning point.

By setting $t = \delta^{-2/3} a^{1/3} w$,

[3.2.8] is changed into the Airy equation

$$y_{II}'' = t\, y_{II} \quad [3.2.9]$$

The general solution of [3.2.9] is

$$y_{II}(w) = A_1 Ai(\delta^{-2/3} a^{1/3} w) + B_1 Bi(\delta^{-2/3} a^{1/3} w) \quad [3.2.10]$$

where $A_1, B_1$ are constants to be determined by asymptotic matching with $y_I(w)$. The approximation of $y_{II}(w)$ is valid so long as

$$w \ll 1, \quad \delta \rightarrow 0^{+}. \quad [3.2.11]$$

Combining [3.2.7] and [3.2.11], $y_I(w)$ in [3.2.2] and $y_{II}(w)$ in [3.2.10] have common region of validity

$$\delta^{2/3} \ll w \ll 1, \quad \delta \rightarrow 0^{+}. \quad [3.2.12]$$

Inside this region $y_I(w)$ and $y_{II}(w)$ are both approximate solutions to the same differential equations and therefore they must match asymptotically.

Consider $y_I(w)$ in the overlap region [3.2.12]. Here w is so small that Q(w) is approximately aw, therefore

$$\left[Q(w)\right]^{-1/4} \sim a^{-1/4} w^{-1/4} \qquad ( w \to 0^+)$$

$$\int_0^w [Q(t)]^{1/2} dt \sim 2a^{1/2} w^{2/3}/3 \qquad ( w \to 0^+ )$$

i.e $\quad y_I(w) \sim Ca^{-1/4} w^{-1/4} \exp\left( - 2a^{1/2} w^{3/2}/3\delta \right) \quad ( w \to 0^+ ) \quad [3.2.13]$

To find the precise region of validity of [3.2.13] we need further careful estimation. However, the upper edge of validity of [3.2.13] depends on Q(w).

Suppose $\qquad Q(w) - aw \sim bw^2 \qquad ( w \to 0^+ )$

then
$$\int_0^w [Q(t)]^{1/2} dt \sim \int_0^w [at+bt^2]^{1/2} dt$$

$$\sim \int_0^w (at)^{1/2} (1+bt/2a) dt$$

$$\sim \frac{2}{3} aw + bw^{5/2}/5a^{1/2} \qquad ( w \to 0^+ ).$$

To obtain [3.2.13], it is necessary to assume that w is sufficiently small so that $\exp(bw^{5/2}/5a^{1/2}\delta) \sim 1 \quad (\delta \to 0^+)$, hence we arrive at the condition that $w \ll \delta^{2/5} \quad (\delta \to 0^+)$.

Thus [3.2.13] is valid in the region

$$\delta^{2/3} \ll w \ll \delta^{2/5} \qquad (\delta \to 0^+). \qquad [3.2.14]$$

Now consider $y_{II}(w)$:

Matching to the functional form of $y_I$ on the overlap region requires that $B_1 = 0$ , thus

$$y_{II}(w) = A_1 Ai(\delta^{-2/3} a^{1/3} w) \qquad [3.2.15]$$

On the overlap region, we can approximate the Airy function's leading asymptotic behaviour for large positive argument.

Hence,

$$y_{II}(w) \sim \frac{1}{2} A_1 \pi^{-1/2} \delta^{1/6} a^{-1/12} w^{-1/4} \exp\left[ -\frac{2}{3} \left( \delta^{-2/3} a^{1/3} w \right)^{3/2} \right] \qquad [3.2.16]$$

In order to obtain the outgoing wave type solution in [3.2.16] , we must make the choice $-n = n\, e^{-\pi i}$ in calculating $a^{1/3}$.

[3.2.16] is valid if two criteria are satisfied. First we require $|w| \ll 1$ as $\delta \to 0^+$ so that the Airy Equation is a good approximation to the differential equation.

Second, the use of asymptotic approximation to the Airy Equation requires that $t = \delta^{-2/3}a^{1/3}w$ be large, i.e $|w| \gg \delta^{2/3}$, so that the region of validity of [3.2.16] is

$$\delta^{2/3} \ll |w| \ll 1, \qquad \delta \to 0^+ .$$

Thus we have

$$y_{II}(w) \sim \frac{1}{2}A_1\pi^{-1/2}\delta^{1/6}a^{-1/12}w^{-1/4}\exp\left[-\frac{2}{3}\left(\delta^{-2/3}a^{1/3}w\right)^{3/2}\right]$$

$$\delta^{2/3} \ll |w| \ll 1, \qquad \delta \to 0^+ \qquad [3.2.17]$$

Notice that in the common region of validity $\delta^{2/3} \ll w \ll \delta^{2/5}$, $\delta \to 0^+$, $y_I(w)$ in [3.2.13] and $y_{II}(w)$ in [3.2.17] have same functional form and can therefore be matched asymptotically. The requirement that [3.2.13] and [3.2.17] match asymptotically determines the constant $A_1$:

$$A_1\pi^{-1/2}\delta^{1/6}a^{-1/12}/2 = Ca^{-1/4}$$

Thus $\qquad A_1 = 2\pi^{1/2}\delta^{-1/6}a^{-1/6}C$ .

Hence we have

$$y_{II}(w) \sim 2\pi^{1/2}\delta^{-1/6}a^{-1/6}C\ \mathrm{Ai}(\delta^{-2/3}a^{1/3}w),$$

$$\delta^{2/3} \ll |w| \ll 1, \qquad \delta \to 0^+ \qquad [3.2.18]$$

Finally in Region III:

The WKB solution of the problem is

$$y_{III}(w) = F\left[(w+\delta x_0)^n + \lambda\right]^{-1/4} \exp\left[ i/\delta \int_w^0 [(t+\delta x_0)^n + \lambda]^{1/2} dt \right]$$

$$+ G\left[(w+\delta x_0)^n + \lambda\right]^{-1/4} \exp\left[ - i/\delta \int_w^0 [(t+\delta x_0)^n + \lambda]^{1/2} dt . \right]$$

$$\mathrm{Re}\, w < 0, \quad -w \gg \delta^{2/3}, \quad \delta \to 0^+ .$$

On the overlap region $\delta^{2/3} \ll -w \ll \delta^{2/5}, \quad \delta \to 0^+$

w is so small that $(w+\delta x_0)^n + \lambda \sim -aw$ .

Hence on the overlap region

$$y_{III}(w) \sim F(-aw)^{-1/4} \exp\left[ i/\delta \int_w^0 (-aw)^{1/2} dw \right]$$

$$+ G(-aw)^{-1/4} \exp\left[ - i/\delta \int_w^0 (-aw)^{1/2} dw \right]$$

$$\sim Fe^{-i\pi/4} a^{-1/4} w^{-1/4} \exp\left[ \frac{2}{3} (\delta^{-2/3} a^{1/3} w)^{3/2} \right]$$

$$+ \quad Ge^{-i\pi/4} a^{-1/4} w^{-1/4} \exp\left[ -\frac{2}{3} (\delta^{-2/3} a^{1/3} w)^{3/2} \right] \quad .$$

[3.2.20]

Consider $y_{II}(w)$ on the overlap region $\delta^{2/3} << -w << \delta^{2/5}$ $\delta \to 0^+$.

On the overlap region $|\delta^{-2/3}a^{1/3}w| \to +\infty$ as $\delta \to 0^+$

The leading terms of the full asymptotic expansion of Ai are

$$Ai(z) \sim C_1 z^{-1/4}\exp(\tfrac{2}{3}z^{3/2}) + C_2 z^{-1/4}\exp(-\tfrac{2}{3}z^{3/2})$$
$$|z| \to +\infty . \qquad [3.2.21]$$

The Stokes' multiplier $C_1$ and $C_2$ really depends on the region of validity arg(z) is in. We now carry out a careful analysis of the argument of w in the Airy function in [3.2.15].

$$\arg(\delta^{-2/3}a^{1/3}w) = \arg\left[(-n)^{1/3}(-\lambda)^{(n-1)/3n}w\right]$$

$$= \arg(-n)/3 + \frac{n-1}{3n}\arg(-\lambda) + \arg(x-x_0) .$$

We notice that $\arg(-\lambda)=\Theta_1(\delta) \to 0^+$ as $\delta \to 0^+$ .

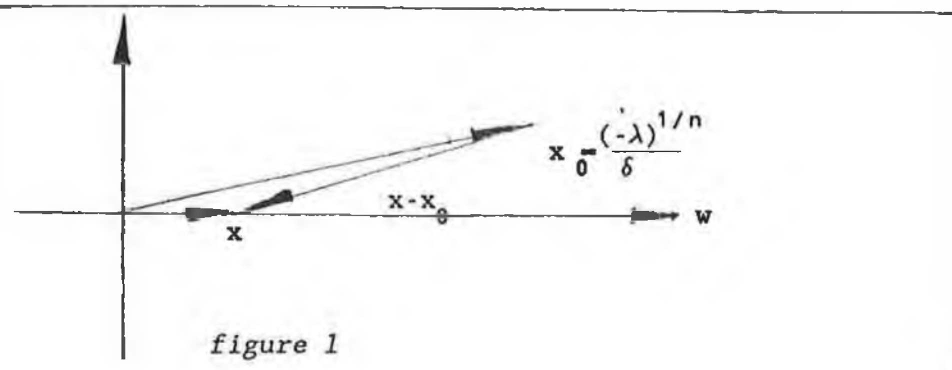


*figure 1*

Since $\qquad \delta^{2/3} << -w << \delta^{2/5}, \quad \delta \to 0^+$

we have $\qquad \delta^{-1/3} << -(x - \frac{(-\lambda)^{1/n}}{\delta}) << \delta^{-3/5} \quad , \quad \delta \to 0^+$

Thus $\arg(x - x_0) = -\pi + \Theta_2(\delta), \; \Theta_2(\delta) \to 0^+ \quad , \quad \delta \to 0^+$

To be consistent with [3 2.16] we then write $-n = ne^{-\pi i}$,

so that

$$\arg(\delta^{-2/3} a^{1/3} w) \to -\frac{4\pi^+}{3} \quad , \; \delta \to 0^+ \quad . \qquad [3.2.22]$$

Thus in the limit, by our result [2.8.4] using Berry's theory, we have Stokes' multipliers,

$$C_1 = -\frac{1}{2} i \pi^{-1/2} \quad \text{and} \quad C_2 = \frac{1}{4} \pi^{-1/2}$$

$$Ai(z) \sim -\frac{1}{2} i \pi^{-1/2} z^{-1/4} \exp(\frac{2}{3} z^{3/2})$$

$$+ \frac{1}{4} \pi^{-1/2} z^{-1/4} \exp(-\frac{2}{3} z^{3/2})$$

$$|z| \to +\infty \; , \quad \arg(z) \to -\frac{4\pi^+}{3} \quad .$$

Hence on the common region of validity,

$$y_{II}(w) \sim -i \, a^{-1/4} C \, w^{-1/4} \left\{ \exp\left[\frac{2}{3}(\delta^{-2/3} a^{1/3} w)^{3/2}\right] \right.$$

$$\left. + \frac{i}{2} \exp\left[-\frac{2}{3}(\delta^{-2/3} a^{1/3} w)^{3/2}\right] \right\} \quad .$$

Mathching asymptotically with $y_{III}(w)$ in [3.2.20],

we have

$$F = -ie^{i\pi/4}C = e^{-i\pi/4}C$$

$$G = -ie^{i\pi/4}(\frac{i}{2})C = e^{-i\pi/4}(\frac{i}{2})C$$

Thus

$$y_{III}(w) \sim e^{-i\pi/4}C$$

$$\left\{ \left[(w+\delta x_0)^n + \lambda\right]^{-1/4} \exp\left[ i/\delta \int_w^0 [(t+\delta x_0)^n + \lambda]^{1/2} dt \right] \right.$$

$$\left. + \frac{i}{2}\left[(w+\delta x_0)^n + \lambda\right]^{-1/4} \exp\left[ - i/\delta \int_w^0 [(t+\delta x_0)^n + \lambda]^{1/2} dt \right] \right\}$$

$$w < 0, \quad -w \gg \delta^{2/3}, \quad \delta \to 0^+. \qquad [3.2.23]$$

In summary, we have found the asymptotic solution to the transformed model equation in these overlapping regions as follows:

$$y_I(w) \sim C\left[-(w+\delta x_0)^n-\lambda\right]^{-1/4} \exp\left[1/\delta \int_0^w [-(t+\delta x_0)^n-\lambda]^{1/2}\,dt\right]$$

$$w \gg \delta^{2/3}, \quad \delta \to 0^+. \qquad [3.2.24]$$

$$y_{II}(w) \sim 2\pi^{1/2}\delta^{-1/6}a^{-1/6}C\,\mathrm{Ai}(\delta^{-2/3}a^{1/3}w)$$

$$\delta^{2/3} \ll |w| \ll 1, \qquad \delta \to 0^+. \qquad [3.2.25]$$

$$y_{III}(w) \sim e^{-i\pi/4}C$$

$$\left\{\left[(w+\delta x_0)^n+\lambda\right]^{-1/4} \exp\left[i/\delta \int_w^0 [(t+\delta x_0)^n+\lambda]^{1/2}dt\right]\right.$$

$$\left.+\frac{i}{2}\left[(w+\delta x_0)^n+\lambda\right]^{-1/4} \exp\left[-\,i/\delta \int_w^0 [(t+\delta x_0)^n+\lambda]^{1/2}dt\right]\right\}$$

$$\mathrm{Re}\, w < 0, \quad -w \gg \delta^{2/3}, \quad \delta \to 0^+.$$

[3.2.26]

# CHAPTER 4

# Using Matched Asymptotics Solution To Obtain Eigenvalues

By using the boundary condition at the origin in the w-variable, we can obtain an eigenvalue relation from which we can solve to obtain the imaginary part of $\lambda$.

## 4.1 Obtaining the eigenvalue relation

Since the derivative of $y_{III}(w)$ at the boundary point with n=1 differs from that of case $n \geq 2$, we discuss them separately.

Case n=1:

$$y_{III}(w) \sim e^{-i\pi/4} C\, w^{-1/4} \left\{ \exp\left[ i/\delta \int_w^0 t^{1/2} dt \right] + i/2 \exp\left[ -i/\delta \int_w^0 t^{1/2} dt \right] \right\}$$

$$= e^{-i\pi/4} C\, w^{-1/4} \left\{ \exp\left[ -2iw^{3/2}/3\epsilon \right] + i/2 \exp\left[ 2iw^{3/2}/3\epsilon \right] \right\}. \qquad [4.1.1]$$

$$y'_{III}(w) \sim e^{-i\pi/4} C \Big\{ -w^{-5/4}/4 \exp\left[-2iw^{3/2}/3\epsilon\right]$$

$$- iw^{1/4}/\epsilon \exp\left[-2iw^{3/2}/3\epsilon\right] - iw^{-5/4}/8 \exp\left[2iw^{3/2}/3\epsilon\right]$$

$$-w^{1/4}/2\epsilon \exp\left[2iw^{3/2}/3\epsilon\right]\Big\} \qquad [4.1.2]$$

Substitute [4.1.1] and [4.1.2] into boundary condition [3.1.6]:

$$\delta\, y'_{III}(-\delta x_0) + h\, y_{III}(-\delta x_0) = 0$$

By using dominant balance we have the eigenvalue relation:

$$i\lambda^{1/2} \sim h\frac{(1+\frac{1}{2}ie^{-2\tau})}{(1-\frac{1}{2}ie^{-2\tau})} \qquad [4.1.3]$$

where

$$\tau = i/\delta \int_{-\delta x_0}^{0} \left[(t+\delta x_0)^n + \lambda\right]^{1/2} dt = i/\epsilon \int_{-\delta x_0}^{0} t^{1/2} dt = -2i\,\lambda^{3/2}/3\epsilon$$

Case n $\geq$ 2:

$$y'_{III}(w) \sim e^{-i\pi/4} C$$

$$\Bigg\{ -1/4 \left[ (w+\delta x_0^n) + \lambda \right]^{-5/4} n(w+\delta x_0)^{n-1} \exp \left[ i/\delta \int_{-\delta x_0}^{0} \left[ (t+\delta x_0)^n + \lambda \right]^{1/2} dt \right]$$

$$- \; i/\delta \left[ (w+\delta x_0^n)+\lambda \right]^{1/4} \exp \left[ i/\delta \int_{-\delta x_0}^{0} [ (t+\delta x_0)^n + \lambda]^{1/2} dt \right]$$

$$- \; i/8 \left[ (w+\delta x_0^n) + \lambda \right]^{-5/4} n(w+\delta x_0)^{n-1} \exp \left[ -i/\delta \int_{-\delta x_0}^{0} [ (t+\delta x_0)^n + \lambda]^{1/2} dt \right]$$

$$-1/2\delta \left[ (w+\delta x_0)^n + \lambda \right]^{1/4} \exp \left[ -i/\delta \int_{-\delta x_0}^{0} [ (t+\delta x_0)^n + \lambda]^{1/2} dt \right] \Bigg\}$$

From the boundary condition [3.1.6] we have

$$\left[ -i\lambda^{1/2} + h \right] \exp \left[ i/\delta \int_{-\delta x_0}^{0} [ (t+\delta x_0)^n + \lambda]^{1/2} dt \right]$$

$$+ \left[ -\frac{\lambda^{1/2}}{2} + ih/2 \right] \exp \left[ -i/\delta \int_{-\delta x_0}^{0} [ (t+\delta x_0)^n + \lambda]^{1/2} dt \right] = 0 \quad .$$

By setting $\quad \tau = i/\delta \displaystyle\int_{-\delta x_0}^{0} \Big[ (t+\delta x_0)^n + \lambda \Big]^{1/2} dt$

we have eigenvalue relation:

$$i\lambda^{1/2} \sim h \frac{(1+ \frac{1}{2} ie^{-2\tau})}{(1- \frac{1}{2} ie^{-2\tau})} \quad . \qquad [4.1.4]$$

We notice here that [4.1.4] is the same as the eigenvalue relation

for the case n = 1.

With the assumption that $e^{-2\tau}$ is exponentially small as $\delta \to 0^+$,

from [4.1.2] and [4.1.3] we have

$$i\lambda^{1/2} = h(1+1/2ie^{-2\tau})[1+1/2ie^{-2\tau} + O(e^{-4\tau})]$$

$$i\lambda^{1/2} \sim h(1 + ie^{-2\tau})$$

$$\lambda \sim - h^2(1 + 2ie^{-2\tau}) \quad , \qquad \delta \to 0^+ \qquad [4.1.5]$$

[4.1.5] is the eigenvalue relation we use to find the imaginary part of eigenvalue $\lambda$ for any n = 1,2,3,...

Now we verify our assumption by calculating $\tau$ :

$$\tau = i/\delta \int_{-\delta x_0}^{0} [(t+\delta x_0)^n+\lambda]^{1/2} dt$$

$$= i/\delta \int_{0}^{\delta x_0} [u^n+\lambda]^{1/2} du$$

$$= i\lambda^{1/2}/\delta \int_{0}^{\delta x_0} [u^n/\lambda+1]^{1/2} du$$

$$= \frac{(-\lambda)^{\frac{(n+2)}{2n}}}{\epsilon^{1/n}} \quad S(n)$$

where $\quad S(n) = \int_0^1 (1-v^n)^{1/2} dv$

$$= 1/n \int_0^1 w^{1/n-1} (1-w)^{3/2-1} dw \quad .$$

We recognise this as a betafunction ingegral (see Bender & Orszag p 575) and write

$$S(n) = 1/n \ B(1/n,3/2)$$

$$= \frac{\Gamma(1/n + 1)\Gamma(3/2)}{\Gamma(3/2 + 1/n)}$$

Thus in the case $n = 1$, $S(n) = 2/3$

and in the case $n = 2$ $S(n) = \pi/4$ on using the formula for the gamma function.

As $n \to +\infty$, we see that $S(n) \to 1$ on using the result $\Gamma(1) = 1$.

Now we offer an alternative method to calculate the $\tau$ which gives the same result as above without using the beta function integral.

When $n = 1$ and $n = 2$, $\tau$ can be calculated directly

Case $n = 1$:

$$\tau = i/\delta \int_{-\delta x_0}^{0} [(t+\delta x_0)^{?} + \lambda]^{1/2} dt$$

$$= i/\epsilon \int_{-\delta x_0}^{0} t^{1/2} dt$$

$$= -\frac{2i}{3\epsilon} \lambda^{3/2}$$

$$= -\frac{2i}{3\epsilon} \lambda^{3/2}$$

$$= \frac{2}{3\epsilon} (-\lambda)^{3/2} \qquad [4.1.6]$$

Case n = 2:

$$\tau = i/\delta \int_{-\delta x_0}^{0} \left[(t+\delta x_0)^n+\lambda\right]^{1/2} dt$$

$$= i/\delta \int_{0}^{\delta x_0} [u^n+\lambda]^{1/2} du$$

$$= i/\epsilon^{1/2} \left[u[u^n+\lambda]^{1/2}/2 + \lambda/2 \ln[u+(u^n+\lambda)]^{1/2}\right]_0^{(-\lambda)^{1/2}}$$

$$= i\lambda/2\epsilon^{1/2} \left[\ln(-\lambda)^{1/2} - \ln(-\lambda)^{1/2}\right]$$

$$= i\lambda/2\epsilon^{1/2} \left[\ln(i)\right]$$

$$= i\lambda/2\epsilon^{1/2} \left[\ln|i| + i\pi/2\right]$$

$$\frac{(-\lambda)\pi}{4\epsilon^{1/2}} \quad .$$

Case $n \geq 1$: we can calculate $\tau$ by binomial expansion

$$\tau = i/\delta \int_{0}^{\delta x_0} [u^n+\lambda]^{1/2} du \quad .$$

Since $0 \leq u \leq \delta x_0 = (-\lambda)^{1/n}$, we have

$$|u^n/\lambda| \leq |-\lambda/\lambda| = 1 \quad .$$

According to binomial expansion

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \ldots \qquad |x| \leq 1$$

$$\tau = \frac{(-\lambda)^{1/2}}{\epsilon^{1/n}} \int_0^{(-\lambda)^{1/n}} \left(1 + \frac{1}{2}\left(\frac{u^n}{\lambda}\right) \quad \frac{1\cdot 1}{2\cdot 4}\left(\frac{u^n}{\lambda}\right)^2 \right.$$

$$\left. + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\left(\frac{u^n}{\lambda}\right)^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\left(\frac{u^n}{\lambda}\right)^4 + \ldots\right) du.$$

Since binomial expansion is uniformly convergent, we can calculate $\tau$ by addition of integration of each term.

$$\tau = \frac{(-\lambda)^{1/2}}{\epsilon^{1/n}} \left( u + \frac{1}{2(n+1)}\left(\frac{u^n}{\lambda}\right)u - \frac{1\cdot 1}{2\cdot 4(2n+1)}\left(\frac{u^n}{\lambda}\right)^2 u \right.$$

$$\left. + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6(3n+1)}\left(\frac{u^n}{\lambda}\right)^3 u - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8(4n+1)}\left(\frac{u^n}{\lambda}\right)^4 u + \ldots \right) \Bigg|_0^{(-\lambda)^{1/n}}$$

$$= \frac{(-\lambda)^{\frac{(n+2)}{2n}}}{\epsilon^{1/n}} \left\{ 1 - \frac{1}{2(n+1)} - \frac{1\cdot 1}{2\cdot 4(2n+1)} \right.$$

$$\left. - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6(3n+1)} - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8(4n+1)} - \ldots \right]$$

By setting

$$S(n) = \left[ 1 - \frac{1}{2(n+1)} - \frac{1\cdot 1}{2\cdot 4(2n+1)} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6(3n+1)} - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8(4n+1)} - \cdots \right]$$

we have

$$\tau = \frac{(-\lambda)^{\frac{(n+2)}{2n}}}{\epsilon^{1/n}} S(n) \qquad [4.1.7]$$

By comparing the coefficients with those of thebinomial expansion, we know that the infinite series S(n) is a convergent series.

Case $n = 1$, S(n) is convergent to 2/3 .

Case $n = 2$, S(n)is convergent to $\pi/4$ .

Case $n \rightarrow +\infty$

$$S(n) = \left[ 1 - \frac{1}{2(n+1)} - \sum_{m=2}^{\infty} \frac{1\cdot 3\cdot 5\cdots (2m-3)}{2\cdot 4\cdot 6\cdots (2m-2)2m(mn+1)} \right]$$

$$\sum_{m=2}^{\infty} \frac{1\cdot 3\cdot 5\cdots (2m-3)}{2\cdot 4\cdot 6\cdots (2m-2)2m(m*n+1)} \leq \sum_{m=2}^{\infty} \frac{2\cdot 4\cdot 6\cdots (2m-2)}{2\cdot 4\cdot 6\cdots (2m-2)2m(m*n+1)}$$

$$\le \sum_{m=2}^{\infty} \frac{1}{2nm^2} = \frac{1}{2n} \sum_{m=2}^{\infty} \frac{1}{m^2} < \pi^2/12n$$

$$\sum_{m=2}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2m-3)}{2\cdot 4\cdot 6\cdots(2m-2)2m(m+n+1)} \to 0\ , \qquad \text{as } n \to +\infty \qquad [4.1.8]$$

Hence $\quad S(n) \to 1, \quad$ as $n \to +\infty$ . $\qquad$ [4.1.9]

According to regular perturbation in [1.2]

n=1: $\quad \lambda = -h^2 - \epsilon/(2h) - \epsilon^2/(8h^4) - 5\epsilon^3/(32h^7) + O(\epsilon^4)$

[4.1.10]

n=2: $\quad \lambda = -h^2 - \epsilon/(2h^2) - 7\epsilon^2/(8h^6) - 121\epsilon^3/(16h^{10}) + O(\epsilon^4)$

[4.1.11]

n≥2: $\quad \lambda = -h^2 - n!\epsilon/(2h)^n + O(\epsilon^2)$ . $\qquad$ [4.1.12]

Therefore our assumption that $\exp(-2\tau)$ is exponentially small as $\delta \to 0^+$ is correct, hence eigenvalue relation [4.1.5] is correct.

$$\lambda \sim -h^2[1 + (\text{real power series in ascending powers of } \epsilon) + 2ie^{-2\tau}] \quad \delta \to 0^+$$

[4.1.13]

Combining [4.1.7] and [4.1.12] and [4.1.13] we have the general formula for the imaginary part of eigenvalue which is as expected exponentially small as $\epsilon \to 0^+$,

$$\mathrm{Im}(\lambda) \sim -2h^2 \exp\left[-2(-\lambda)^{(n+2)/2n} S(n)/\epsilon^{1/n}\right]$$

$$\sim -2h^2 \exp\left[-2[h^2 + n!\epsilon/(2h)^n + O(\epsilon^2)]^{(n+2)/2n} S(n)\right]$$

$$\epsilon \to 0^+, \quad n = 1,2,3,\ldots \quad . \qquad [4.1.14]$$

When $n = 1$,

$$\mathrm{Im}(\lambda) \sim -\frac{2h^2}{e} \exp\left(-\frac{4h^3}{3\epsilon}\right), \quad \epsilon \to 0^+ \qquad [4.1.15]$$

When $n = 2$,

$$\mathrm{Im}(\lambda) \sim -2h^2 \exp\left(-\frac{\pi h^2}{2\epsilon^{1/2}}\right), \quad \epsilon \to 0^+ \qquad [4.1.16]$$

When n > 2,

$$\mathrm{Im}(\lambda) \sim -2h^2 \exp\left\{-\frac{2h^{(n+2)/n}}{\epsilon^{1/n}} S(n)\right\}, \quad \epsilon \to 0^+ \qquad [4.1.17]$$

Thus by using asymptotic matching techniques, we finally find the general formula for the imaginary part of eigenvalue $\lambda$ which includes the earlier result of Paris and Wood and Brazel, thus concluding our thesis.

## REFERENCES


[1] M. Abramowitz and I. Stegun (ed.) (1970) *Handbook of mathematical Functions*, Dover, New York.

[2] C. M Bender and S. A. Orszag (1978) *Advanced Mathematical Methods for Scientist and Engineers*, McGraw-Hill, London.

[3] F. W. J. Olver (1990) "On Stokes phenomenon and converging factors" in Asymptotic and Computational Analysis(ed. Wong). Marcel Dekker, Ner York.

[4] R. B. Paris and A. D. Wood (1989) "A Model Equation for Optical Tunnelling",IMA *Journal of Applied Math*. 273-284.

[5] A. W. Snyder and J.D. Love (1983) *Optical Waveguide Theory* Chapman and Hall, London.

[6] A. D. Wood and R. Paris (1990) "*On eigenvalue with exponentially small imaginary part*" in Asymptotic and Computational Analysis (ed.R.Wong).Marcel Dekker, New York.

[7] R. E. Meyer (1980) "Exponential Asymptotics", SIAM *Review* 22,2, pp. 213-224.

[8] W. L. Kath and G. A. Kriegsmann (1988) "Optical Tunnelling: Radiation losses in bent fibre optic waveguides" IMA *Journal of Applied Math*. 41, pp.85-103.

[9] M. V. Berry (1989) "Uniform asymptotic smoothing of Stokes discontinuities". *Proc. Roy. Soc. London* A422, pp.7-21.

[10] N. Brazel, F. R. Lawless and A. D. Wood (1990) "Exponential asymptotics for the eigenvalues of a problem involving parabolic cylinder functions" Submitted for publication.

[11] C. M. Bender and T. T. Wu (1973) "Anharmonic Oscillator II" Physical Review D, 7,6 pp 1620-1636.

[12] F. W. J. Olver (1974) *Asymptoticsand Special Functions*, Academic Press, New York.