SPECULATION WITH MEMORY

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Applied Mathematical Sciences is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Abstract

This thesis considers asset price evolution in financial markets, deriving the price dynamics from micro-economic considerations. In contrast to most models of speculative price evolution in which the Efficient Market Hypothesis is assumed valid, and thus the price process is Markovian, we introduce dependence on past prices via the action of speculators endeavouring to profit by extracting information from the price history.

Such speculators, called chartists, form a large portion of the agents in financial markets. However, hitherto no satisfactory continuous time model of their impact on market behaviour has been developed. This thesis attempts to close that gap.

By modelling price evolution by a linear stochastic integro-differential equation and making exogenous allowances for the fluctuation of agents' participation levels and the periodicity present in such markets, we show that several properties of financial markets can be qualitatively mimicked: the relationship between heterogeneity and fat-tailed returns distributions, the autocorrelation term structure of the returns, the relationship between volatility and volume and the relative success of chart speculators. Our model outlines mechanisms by which speculative bubbles or crashes arise, and demonstrates that certain types of derivative pricing are robust to the violation of the Efficient Market Hypothesis.

These applied results are based on new findings in the theory of stochastic integro-differential equations which are developed in this thesis. We establish that such equations have unique, continuous solutions whose paths have the same local topology as those of Brownian Motion and which can be expressed in terms of the resolvent of a related deterministic integro-differential equation. By using the techniques of stochastic analysis (in particular the Itô Calculus) and the theory of deterministic integro-differential equations, we determine the pathwise asymptotics, a.s. growth of the extrema and asymptotic distributional character of the evolution.

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Chapter 1

Introduction

If the reader interjects that there must surely be large profits to be gained ... in the long run by a skilled individual who ... purchase[s] investments on the best genuine long-term expectation he can frame, he must be answered ... that there are such serious-minded individuals and that it makes a vast difference to an investment market whether or not they predominate. ... But we must also add that there are several factors which jeopardise the predominance of such individuals in modern investment markets. Investment based on genuine long-term expectation is so difficult ... as to be scarcely predictable. He who attempts it must surely ... run greater risk than he who tries to guess better than the crowd how the crowd will behave.
Keynes, 1936 [42], in DeLong et al. [18].

The central purpose of this thesis is to examine the implications of chartist trading in a financial market; this chapter seeks to outline and summarise that examination.

The justification for our study detailed in Section 1 of this chapter comes from the growing literature in behavioural financial economics. More specifically, we reprise several aspects of traders activity, together with some observed features of financial market behaviour, summarising the connections financial economists have drawn between the two.

In Section 2, I offer an outline of my mathematical synthesis of this review. The substance of this synthesis is that financial asset prices evolve due to the interaction of feedback traders and well–

informed investors; that the feedback introduces present dependence on past prices; and that this dependence can be modelled dynamically by hypothesising that prices are governed by a stochastic integro–differential equation. The inclusion of a secondary hypothesis, which attempts to model the fluctuations in agents' participation and the seasonality in financial markets, is also sketched.

Finally, the third section of this introduction comprises a brief summary of the dissertation's main findings and contents.

1.1 Heterogeneous Agents in Financial Markets: a Review

In recent times a growing body of financial economists have begun to re-examine the behaviour of financial markets in the light of bubbles, crashes, and other anomalies of asset prices. Events such as the sharp decline in stock prices in October 1987 have called into question theories postulating that market participants form rational expectations based on complete information—such theories have difficulty explaining what fresh information relating to economic fundamentals could have precipitated such large revisions in expectations. A further anomaly which attracted great attention was the appreciation of the dollar between 1981 and 1985, (and its subsequent sharp decline) during which forecasters issued recommendations to buy the dollar, while simultaneously maintaining that it was over–priced relative to its fundamental value (see Frankel and Froot [28]). This fact and its implications are at the heart of the forthcoming discussion: firstly, are expectations rational? Secondly, how prevalent is the type of feedback trading advocated and practised by the above forecasters? Thirdly, what other forms might such feedback take, and why might traders seek to practise them? Fourthly, might the presence of diversity amongst agents, together with feedback trading, be responsible for many aspects of financial market behaviour?

1.1.1 The Case Against Rational Expectations and in Favour of Heterogeneity

The indirect evidence against rational expectations has recently been supported by more direct survey evidence reported in a recent series of papers: Frankel and Froot [27], [28], [29], Froot and Ito [32], and Ito [40]. In these papers, not only is the hypothesis of rationality rejected, but agents in the foreign exchange markets are found to have extrapolating but inelastic expectations over the short run and mean-reverting expectations over the longer term. Following Frankel and Froot [27], we say expectations are inelastic when the change in the exchange rate is expected to cause smaller future change. By extrapolative expectations, we mean that an increase (resp. decrease) in the exchange rate is expected to induce a further increase (resp. decrease).

The case against rationality is made in Froot and Ito [32] and Ito [40]. Froot and Ito ask whether short term expectations over-react. They ask whether agents expectations at different forecast horizons lead to equivalent predictions of the level of the exchange rate far into the future. If agents have such expectations, the expectations are said to be consistent. Short term expectations are said to be inconsistent relative to long term expectations, if a positive shock to the exchange rate leads agents to expect a higher long run future spot rate when iterating forward their short term expectations, than when thinking directly about the long run. Expectations will be said to be rational if the expectation over the market's subjective conditional density function at each time t is equal to the objective density function conditional on all information available at time t. Therefore, we see that rationality implies consistency. Consistency is a weaker restriction than rationality, since it does not require that the expectations process match the stochastic process generating the exchange rate. Having made these definitions and observations, Froot and Ito test statistically for consistency, and they find that agents' expectations do exhibit inconsistencies: relative to longer term expectations, shorter term expectations invariably over-react to an exchange rate shock.

Ito [40] analyses the twice monthly surveys of the yen–dollar exchange rate expectations of Japanese banks, securities companies, trading companies and export and import oriented companies. He finds that the participants' expectation formation displays significant individual effects which exhibit to some degree wishful thinking, with exporters expecting yen depreciation and importers yen appreciation (relative to others). This heterogeneity damages the hypothesis of rational expectations: that hypothesis would require that market participants be homogeneous in their formation of expectations, since the true stochastic process is unique. Ito rejects the hypothesis that forecast errors are random; moreover, he shows that forecast errors are correlated to information available at the time the forecast is made. This is incompatible with the hypothesis of rational expectations, since the information correlated with the ex post error could have been exploited to make a better forecast. As in [32], the forecast term structure is shown to be inconsistent—forecasts with long horizons showed less yen appreciation than those with short horizons.

Having rejected rationality in price formation expectations, Frankel and Froot [28] first analyse the Money Market Services' (M.M.S.) survey which, either weekly or bi–weekly, collects the exchange rate expectations of market participants.

Their testing shows bandwagon hypotheses at horizons of three months or less; for example, a 10% increase in prices over the past week by itself leads to an expectation that prices will rise by 1.35% over the next week—a current appreciation generates a self-sustaining expectation of future appreciation. Long run expectations seem to be mean-regressing; for example, a 10% appreciation over the last 12 months by itself generates an expectation of a 2.02% depreciation over the coming 12 months, so longer run expectations are in this sense stabilising. The same findings are reported for regressive and adaptive expectations.

Frankel and Froot then present a strong case for the prevalence of, and rationale behind, short term positive feedback and chart trading, particularly during the early 1980s. The 1987 Euromoney survey notes that most forecasting services were using technical analysis:

"[T]he surveys appeared to have convinced many ... [traders] that forecasts could be used profitably and that most profitable forex forecasters were technical rather than those who focused on economic fundamentals."

This is perhaps not surprising in the light of Schulmeister's 1987 paper [66]: he looks at various rules of technical analysis in widespread use and calculates that all the rules would have made money over the period since the 1973 float, and moreover, that they were profitable in each of the 18-month sub-periods up to 1986. In fact, he cites a 1985 statistic that 97% of banks and 87% of securities houses report the belief that "the use of technical models has had an increasingly significant impact on the market". This is in agreement with the quote of Goodhart [36] also included in Frankel and Froot's paper:

"Traders, so it is claimed, consistently make profits from their position taking (and those who do not get fired), over and above their return from straight dealing, owing to the bid/ask spread."

The banks report that their speculation doesn't take place in the forward market (only 4–5% of their large customers were prepared to take open positions in the forward market). In [29], Frankel and Froot note that this may be because bankers recall the Franklin National crisis and other bank failures caused by open foreign positions that were held too long; this is not a unique situation, as in the late 1970s banks were also unwilling to hold large net positions in foreign currency.

Instead, banks take very short term open positions in the spot market. Taking long term positions, based on fundamentals or positions in the forward market, is viewed as being "too speculative". They are, however, prepared to trust their spot traders to take large open positions, provided they close most of them out by the end of the day, because these operations are profitable on aggregate.

This leads Frankel and Froot to comment:

"There seems to be some sort of a breakdown of the economists' rule of rationality that the long run is the sum of a series of expected short runs. Even though the market is not taking adequate account of the fact that the exchange rate must return to equilibrium eventually, there is no easy way for an investor to make expected profits from this mistake, unless he has sufficient patience, and sufficiently low risk aversion to wait through the short term volatility."

Thus, because traders put too much weight on the current rate in forming their expectations, and insufficient weight on the fundamentals,

"[t]he result is that economic fundamentals do not enter into most traders' behaviour, even if fundamentals must win out in the long run. Indeed, most traders are so young and have been at their jobs so short a time, that they may not remember the preceeding major upswing or downswing four years earlier."

Similar points relating to the memory and/or youth of feedback traders are made in DeLong et al. [19]. There are, moreover, two good reasons why it may not be irrational for individual banks to adopt this short term perspective. First, in Froot and Frankel [28], the authors note that allowing its traders to take a sequence of many short term open positions in the spot market may be the only way a bank has of learning which traders can make money doing so and which cannot.

Secondly, in [29] it is remarked that a year may be a statistically significant period of time to determine whether a particular spot trader is good at her job, can be rewarded and given greater discretion, or let go. In the case of portfolio investment on the other hand, a year may be insufficient to judge whether a given analyst is good or bad at picking currencies or securities that are incorrectly valued. Given high short-term volatility, many years of data may be necessary to discern a slowly disappearing mis-valuation. It may therefore be rational for a bank executive to restrict the size of the investment portfolio on the grounds of risk aversion, while simultaneously allowing spot traders to take a sequence of large open positions.

Having provided this evidence, Frankel and Froot, in their 1988 paper, discount what they call "the one reasonable economic reason" for the dollar appreciation in the 1981–85 period—overshooting. Having done so, they propose an irrational bubble mechanism as the cause of the dollar appreciation. In [29], they claim it better explains the dollar's path than a rational bubble, since for a rational bubble, all agents must know the correct model, and the cause of such a bubble is not explained. Their evidence seems to support this contention since in reality participants neither know about nor agree upon the model.

In this model, there are three classes of actors—fundamentalists, chartists and portfolio managers. None of these agents act in an utterly irrational manner, but rather perform their tasks in a reasonable and realistic fashion. The portfolio managers start with complete confidence in the fundamentalists' expectations of price changes. However, if chartists provide better forecasts, the portfolio managers are prepared to give their views greater weight. As the currency appreciates (which it does in their model), the portfolio managers give progressively less weight to the fundamentalists' forecasts of the currency's depreciation towards fundamental values. The model predicts that the weight given to the fundamentalist opinion declines to zero, at which time a new "equilibrium" is reached. They propose a mechanism for bursting this bubble by including the effects of persistent current account defecits. The decline in the exchange rate is driven by a revival in the fundamentalists' forecasts by the portfolio managers, arising from a greater sensitivity of the exchange rate to the current account when the fundamentalists' weight is low. Comparing this with the U.S. current account defecits over the early 1980s, the model predicts a decline towards fundamental values similar to that which occured after the 1985 Plaza Accord.

There is also some evidence that heterogeneity has been increased by recent innovations in financial markets. For example, in Guillaume et al. [38], some evidence is provided for the FX market. They state that daily turnover of global foreign exchange markets stood at $832 billion in April 1992, representing a tripling in turnover from 1986. The rapid growth in transaction volume has increasingly been made up of short term, intra–daily transactions (now more than 75% of volume) and results from the interaction of traders with different time horizons, risk profiles, and regulatory constraints. The movement of FX activity of institutional investors (such as pension funds) from long term investment strategies has been enhanced by the development of real–time information systems (such as that operated by Reuters) and the reduction of transaction costs arising from the liberalisation of global capital. These developments have also increased the trading of financial institutions in the wholesale market, whose reasons for short term trading we have already outlined. Heterogeneity is also increased in this market by the presence of central banks, who, in contrast to other institutions, can take relatively large open positions.

1.1.2 Feedback Trading: Evidence and Consequences

As mentioned in DeLong et al [19], a wide variety of trading strategies call for buying stocks when their prices rise and selling them when their prices fall. Such strategies include the use of stop-loss orders, which prompt selling in response to price declines, and the liquidation of the positions of investors unable to meet margin calls. It is also exhibited by buyers of portfolio insurance, whose willingness to bear risk increases rapidly with wealth. A common form of negative feedback trading is so called "profit–taking". The results of DeLong et al [18] indicate that in the presence of noise traders, the contrarian investment strategy of buying after prices fall or selling after they have risen is recommended. Negative feedback trading is also seen in the actions of central banks when they "lean against the wind" in defense of their exchange rate objectives.

Very strong experimental evidence for both trend chasing and negative feedback trading arising from charting is supplied by the papers of Smith, Suchanek and Williams [71] (in which, in an experimental asset market, a price bubble endogeneously inflates and then bursts), those of Andreassen [3] and Andreassen and Kraus [4], [5]. We summarise the results of the latter group of papers here.

In [3], subjects with some training in economics were divided into two groups. Half the subjects received news stories about a real stock, together with its price, while the other group received only the price signal. He shows that when causal attributions are supplied to explain recent changes, the group given the news tends to make less regressive predictions. Andreassen remarks that by explaining prices changes the media should cause prices to stay high after they have risen, and low after they fall—"it is difficult to imagine that investors would long be satisfied with a news service that ascribed down changes to good news or up changes to bad news". The author draws some other interesting conclusions—first, it transpires that large changes in prices require greater numbers or stronger explanations than small changes do. Thus with large price changes, more potent explanations are offered by the media and more vigorously sought out; alternatively, there may be a limit as to what news can explain. If this is the case, we should expect to find that the autocorrelations of price changes will be more negative during volatile periods than quiescent ones. Two empirical findings seem to support these views. First, the autocorrelation of changes in the Dow was more negative in the 1920s than in the 1950s when prices were less volatile, even though prices rose in both periods. The negative autocorrelation at the trade to trade level observed in Niederhoffer and Osborne [58] and Guillaume et al. [38] happen at a level at which transactions occur too quickly for the media (or any financial agency) to explain, and at which the effects of tracking should predominate.

In [4] and [5] experimental evidence is furthered for trend chasing. The subjects in the experiment, who have some training in economics, are shown real stock price patterns and asked to trade as price takers. If, over some period, the price level does not change much relative to the period-to-period variability, the subjects track the average price level, selling when prices rise and buying when they fall. If prices exhibit a trend relative to the variability, subjects begin to chart the trend, buying when prices rise and selling when they fall. Instead of extrapolating price levels to arrive at forecasts, they extrapolate price changes. This switch occurred in practically all subjects in the experiment. The switch seems to happen in response to significant changes in the price level over many observations, and not merely the most recent price change.

A further reason to view positive feedback strategies as being rational, but also leading to price bubbles, is offered by George Soros [72]. His success over the past decades has apparently been based on betting not on the fundamentals, but on future crowd behaviour.

In his view, the 1960s saw a number of poorly informed investors become excited about rises in conglomerate earnings. Soros believes that the truly informed investment strategy in this case was not to sell short, in anticipation of the eventual collapse of prices, but to buy in anticipation of future price rises caused by uninformed investors. Initial rises in conglomerate stocks, driven in part by buying by speculators such as Soros, created a trend of increasing prices. This signal was observed by uninformed investors causing them to buy, thereby amplifying the price increase. In 1970, the price increases stopped, uninformed investors' expectations of conglomerate earnings went unrealised, and prices plummeted. Although shorting by fundamentalists helped in bringing prices down to fundamentals, the initial buying by smart money, by raising the expectations of uninformed investors about future returns, may have exacerbated the deviation from fundamentals. Similar trends can be seen in recent times, for instance, in the rise and fall of emerging market stocks, or those of some biotechnology companies.

This view of self–feeding bubbles is not new: for example see the Bagehot quote dating from 1872 in DeLong et al. [19], and also the essays of Mackay [56] (1852) in which is given a detailed account of the South Sea Bubble of 1720 and the Tulipomania of 1636. A more modern account of financial panics may be found in Kindleberger [43].

A further reason for extrapolative expectations and charting may be given by its profitability: as is shown in DeLong et al. [18], if traders' mistakes cause them to take riskier positions than rational investors, they can earn higher returns, which may be a deterrent to learning the full structure of the model needed to make a rational forecast. A similar point is made in Day and Huang [16]: in their model, prices increase as trend chasing speculators buy into a rising market and decrease as they sell into a falling market. In bull markets, price increases generally exceed price decreases in number, while the opposite is true in a bear market. In this sense, the trend chasing investors are right more often than they are wrong: when they buy in, the market usually goes up; when they sell, the market usually goes down. Buying near the peak and selling near the trough can convince an investor that the only mistake made was one of timing. Thus the market tends to re-inforce the behaviour of the trend chasing investors most of the time.

Grounds for considering crowd– or trend–following imitative behaviour has been proffered by Shiller and Pound [69], who asked investors for their motivation for buying specific assets. In the same vein, Lux [55] references the book of Shiller [67] in which Shiller quotes a survey taken after the 1987 crash where most of the investors reported experiencing a "contagion of fear" and reacted on the price drop itself rather than on new information. Shiller's is led to claim that herd behaviour need not be foolish, but may be perfectly sensible in the absence of private information.

Lastly, DeLong et al. [19], make a good case that positive feedback can occur at horizons of differing length:

"Investment pools whose organizers buy stock, spread rumors, and then sell the stock slowly as positive feedback demand picks up rely on extrapolative expectations over a few days. Frankel and

Froot's forcasters have a horizon of several months, which also appears relevant for bubbles like those that may have occured in 1929 and 1987. The conglomerate boom by contrast lasted several years... As long as people expect a price rise over the particular horizon on which they focus to continue, they form expectations that may support positive feedback trading patterns."

The most important type of feedback trading, namely charting, or technical analysis, is discussed separately in the next subsection.

1.1.3 Technical Analysis

As already mentioned in section 1.1.1, Schulmeister [66] claims that, as mentioned in Isard [39] that "various types of trading strategies based on technical analysis generate statistically significant profits". For details of this evidence see Dooley and Shafer [20], Sweeney [75], Cumby and Modest [13], Dunis and Feeney [21], Neftci [57], Surajaras and Sweeney [74] and Levich and Thomas [52]. We therefore enquire more closely into the nature of charting, the spread of its use and its possible effects on price. Some answers to these questions are furnished by the survey conducted by Allen and Taylor [1], which we sumarise below.

The essential difference between chartists and fundamentalists according to Allen and Taylor, is that "at least in principle chartists study only the price action of a market whereas fundamentalists attempt to look at the reasons behind the action". Basic chart analysis involves identifying recurring patterns in time series price data: for example, chartists will identify levels within which prices are supposed to trade, the upper and lower limits called "resistance" and "support" levels respectively. Chartists will also generally employ one or more 'mechanical indicators' when forming a view of prices. These might be trend following (e.g.,'buy when a shorter moving average cuts a longer moving average from below') or of negative feedback type (e.g., 'oscillators' which calculate the rate of change of prices, assuming that the market's tendency to over-react leaves assets "overbought" or "oversold").

The survey attempts to determine the influence of chart trading in the London FX market by sending a questionnaire to chief foreign exchange dealers in that market. They found that a majority of those dealers use some chart input into their trading decisions, particularly at short time horizons. Moreover, at the intra-day to one week level, 90% used some chart input, and fully 60% viewed charts to be at least as important as fundamentals, with importance dropping off at longer horizons: at the one year horizon, for instance, 85% thought fundamentals more important than charts, while 30% used fundamentals alone at this horizon.

The respondents were also given an opportunity to add their own testimonies in this survey; Allen and Taylor report that the remark that charts might obscure the fundamentals over the short run was often made. For instance, one dealer commented:

"As a trading tool, they are useful because they are widely used and therefore can be self–fulfilling",

a view to which 40% of respondents concurred with explicitly. Such a view was held even by dealers believing fundamentals to be more important, and amounts to saying that chartists generate temporary excursions from fundamentals—fads. This remark supports the model of DeLong et al. [19], in that rational speculators may adapt their strategies in the presence of feedback traders.

A further indirect manner in which technical analysis can be self–fulfilling is remarked upon in DeLong et al. [18]. If the noise traders in their model take the pseudo-signals of volume, price pattern, indices of market sentiment and pronouncements of investment gurus seriously, the price pattern charted may become self–fulfilling not singly through the actions of chartists, but at one remove through the actions of uninformed investors.

Allen and Taylor also discovered that chart and fundamental analysis were viewed as complementary and not conflicting methods of forecasting. We observe that this fact will be less compatible with the analysis presented in this thesis than with, for example, the models of Fankel and Froot [28], Day and Huang [16] and De Grauwe et al. [17]. However, our model of chart speculators, using moving averages for price decisions, is supported by the fact that 65% of respondents use trend following systems such as moving averages. Allen and Taylor also find evidence for the inelastic extrapolative expectations reported in the various papers of Frankel, Froot and Ito [27], [28], [32], [29] and [40]. However as Allen and Taylor note:

"Logically separate from this issue, however, is the question of whether chartist advice may be destabilising in the sense of leading the market away from the underlying fundamentals. The most that can be said, given the present evidence, is that chart advice may at the most cause mean-reverting, or stationary deviations from the fundamentals ... i.e., fads," upon which they refer the reader to a preprint of the paper of Poterba and Summers [62], in which some further motivation for the effectiveness of trading using past market behaviour as a guide, is presented. We will presently review this paper focussing on the autocorrelation of stock price returns over horizons of varying length.

In this paper, some interesting comments are made about the implications for financial practice, relating to the authors' findings of significant transitory price components. In particular, it may be desirable to use investment strategies involving the purchase of securities which have recently declined in value, which is clearly a chart strategy. It may also justify the practice of some institutions of spending on the basis of a weighted average of past endowment values rather than current market value.

In Cutler, Poterba and Summers [15], it is mentioned that if asset returns are positively correlated at short horizons and negatively correlated at longer horizons (a pattern they find in a very wide variety of speculative assets), then procedures which involve the crossing of two moving averages as a trading signal are optimal.

1.1.4 Implications of Heterogeneous Agents for Speculative Market Behaviour

In section 1.1.2 we explored the manner in which feedback trading and trend chasing can provide a plausible mechanism for irrational speculative bubbles, and from the analysis of Andreassen and Kraus [3], [4], [5], saw that regressive and extrapolative expectations might be responsible for particular types of autocorrelation in the returns of financial asset prices. In this subsection, we will, as promised, expand upon the claim of Cutler, Poterba and Summers in [62], [15], [14] that the reversal of sign in the autocorrelation of asset returns can be generated by a market comprising of diverse agents. Furthermore, we will expand upon the relationships between trade volume, volatility of asset prices and a proxy for agent heterogeneity, degree of dispersion of forecasts. For the time being, we merely note the comments of Frankel and Froot [28] that "[w]hen a new piece of information becomes available, if all investors process the information in the same way and are otherwise identical no trading need take place... To explain the volume of trading some heterogeneity is required". A similar opinion is expressed in [15].

In the course of this dissertation, we will establish links between agent heterogeneity and each of the phenomena listed above, in conjunction with a qualitative relationship between heavy tails in asset price returns and heterogeneous agents.

In Poterba and Summers [62] and in Cutler, Poterba and Summers [15], [14], evidence relating to the auto-correlation of asset returns and their possible connection with agent diversity is presented. They claim that if market and fundamental values diverge, but beyond some range the differences are eliminated by speculative forces, then stock prices will revert to their mean. Returns must be negatively serially correlated at some frequency if erroneous market moves are eventually corrected. The authors confirm the findings of Fama and French [24]— real and excess returns at long horizons are negatively serially correlated, while the null hypothesis of serial independence of returns is rejected. It is found for stock price returns for a wide variety of countries that there exists positive return autocorrelation at horizons of up to one year (see also Lo and MacKinlay [53]) and negative serial correlation at horizons of 13 to 24 months. This data is robust to the inclusion or exclusion of the Depression era. This study concludes that variation in ex ante returns are best explained as by-products of price deviations caused by noise traders, rather than by changes in interest rates or volatility.

In [14], the alternating autocorrelation structure noted in [62] is also observed for bonds, metals and exchange rates. This fact is consistent with the findings of Frankel and Froot [27]. Thus it is rational for investors to have extrapolative expectations over the short run and mean-regressing expectations over the long run if short run autocorrelation is positive and long run autocorrelation is negative; this pattern is also the premise of technical strategies which seek to catch trends in short-term investing.

In [15] a model of asset market equilibrium is sketched, in which there is interaction between rational investors (fundamentalists) who base demand on expected future returns and feedback traders who base demand on past returns. In this framework, positive short run serial correlation results if the fundamentalists learn news with a lag, or if feedback traders "lean into the wind". Feedback traders who respond to such positive autocorrelation and who base their demand on past returns can generate the observed autocorrelation pattern. Furthermore, by prolonging the impact of fundamental news, positive feedback traders can lengthen the horizon over which returns are positively serially correlated, causing prices to overshoot, and thereby inducing a negative correlation at some horizon.

The heavy level of trading volume and its relationship with agents' diversity and volatility has been noted by many authors. We will concentrate on the papers of Frankel and Froot [29] and Tauchen and Pitts [76]. Frankel and Froot pose the following alternatives: does higher volume of trading increase the efficiency with which news regarding economic fundamentals is processed and reduce the unnecessary volatility in price, or might it be that trading volume is irrelevant to price movements, or even that much trading is based on "noise" rather than "news", and leads to excessive volatility?

Frankel and Froot forward evidence that trading volume, exchange rate volatility, and the dispersion of expectations among forecasters are all positively related. They define trading volume by the weekly number of futures contracts (nearest to maturity) traded on the I.M.M. of the Chicago Mercantile Exchange; volatility is measured by the squared percentage change each 15 minutes in the futures price averaged over the week; dispersion is defined by the percentage standard deviation of the forecasts across respondents in the weekly survey of market participants conducted by M.M.S..

Given these definitions, they find that dispersion Granger–causes volume at the 90% significance level in three out of four currencies they examine: it also Granger–causes volatility in all currencies at the one week horizon and three of four at the one month horizon. Moreover, the contemporaneous correlation between volume and volatility is high: 0.417 for the dollar–yen is representative. Frankel and Froot then say the following:

"The existence of conflicting forecasts leads to noise trading— the causation runs from dispersion to the volume of trading, and then from trading to volatility."

It may also be remarked that higher volatility should cause higher dispersion of expectations because forecasters use different models of price information. It is interesting and instructive to keep this information in mind in the light of our results in Chapter 2.

In the Tauchen–Pitts paper a positive relationship between price variability (as measured by the squared price change) and the trading volume is empirically observed and some theoretical explanations proposed.

In the paper of Clark [12], the author assumes that the number of intra-day transactions is random, so the daily price change is the sum of a random number of within-day price changes. The variance of the daily price change is thus a random variable with a mean proportional to the number of transactions. Clark argues that the trading volume is related positively to the number of within-day transactions, and hence to the variability of the price change.

Another explanation Tauchen and Pitts supply is due to Epps and Epps [23], who hypothesise that change in the market price on each within-day transaction or market clearing is the average of the changes in all the traders' reservation prices. They then assume there is a positive relationship between the extent to which traders disagree when they revise their reservation prices, and the absolute value of the change in market price. The price variability–volume relationship arises because the volume of trading is positively related to the extent to which traders disagree about their reservation prices.

Tauchen and Pitts develop a model having characteristics of both the above models but closer to Clark in spirit. They construct an intra–day equilibrium model where the number of equilibria is random. By assuming first that the number of traders is constant and that the intra–equilibria changes in price are identically and normally distributed, they prove that the daily changes in price and volume are positively correlated. They find the data from the T-Bills futures market to be consistent with their theoretical prediction. Their theory also indicates that the extent to which traders disagree, the number of active traders and the flow of information, all increase the volume of trade.

1.2 Synthesis: Memory, Integro-differential Equations and Periodicity

We see from the above review that there is ample evidence from studies and surveys of the prevalence of feedback trading, and in particular technical analysis, in financial markets. However, very little work has been devoted to studying the effects of feedback trading using the techniques of the Itô Calculus (see for example Schweizer and Föllmer [26], Platen and Schweizer [61] and Frey and Stremme [31]). The work that does exist tends to concentrate on the effects that such heterogeneity has on the pricing of options. This topic, though of central importance in Mathematical Finance, will not dominate the subject matter in this thesis, although we will demonstrate that the pricing, replication and hedging of options is possible in the model economy we propose to study. Instead, we will develop from microeconomic foundations, a model of price evolution which will allow us to study the feedback effects of charting. As in the literature mentioned in Section 1, we will include a second class of investors with mean-reverting price expectations, which we will call fundamentalists. These speculators base their estimates of fair value for a financial asset on factors they believe to underly its long term value. Whenever prices rise above that value, they sell the asset; whenever they fall below, the fundamentalists buy. In our formulation, the chartists will study the deviation of the price from a moving average of past prices as a signal to trade. Most usually, in this thesis, we will assume that the chartists are positive feedback traders, in the sense outlined in Section 1 above. That is to say, prices above the moving average are signals to buy, whereas prices below the moving average provide the chartists with sell signals. However, taking note of the experimental evidence of Andreassen [3] and Andreassen and Kraus [4], [5], we will sometimes allow these signals to be reversed, so that chartists can change from being positive to negative feedback traders, and vice versa. For reasons of tractability, we make this switch occur exogenously.

1.2.1 Stochastic Integro–Differential Equations

The plan sketched above affords us an opportunity to study the strategy of chartists and their effect on price behaviour, but carries with it some added technical problems. The assumption that by virtue of their trading, chartists effect the price, signifies that the price process cannot be a Markov process, since some portion of the price history before the present is necessary to determine the evolution thereafter, not merely the current value. Consequently, the price process cannot be the solution of a stochastic differential equation. In Chapter 4, we distill the economic hypotheses into the following rule which governs log-price (X_t) evolution:

$$X_t = X_0 + \int_0^t \lambda(s) \left(X_s - \int_0^s w(u,s) X_u \, du \right) ds - \int_0^t \beta(s) \left(X_s - k(s) \right) ds + \int_0^t \sigma(s) \, dB_s,$$

where $(B_t)_{t \geq 0}$ is a standard Brownian motion. Adopting the usual notional formalism for stochastic differential equations, we may write:

$$dX_t = \lambda(t) \left(X_t - \int_0^t w(s,t) X_s \, ds \right) dt - \beta(t)(X_t - k(t)) \, dt + \sigma(t) \, dB_t, \tag{1.1}$$

where X_0 is known. Stochastic processes evolving according to equations such as (1.1) are called stochastic integro-differential equations by Berger and Mizel [7], which we will frequently abbreviate to S.I.D.E. The existence and uniqueness of a.s. continuous solutions to S.I.D.Es was proved by Berger and Mizel [7] using similar ideas to those used to prove existence and uniqueness of strong solutions of stochastic differential equations. Identical results for linear S.I.D.Es were independently established by Vespri [77]. For example, equation (1.1) above has a unique a.s. continuous solution on the interval $[0, T]$ for any $T > 0$ if $w \in \mathbf{L}^1(\mathbb{R}^+ \times \mathbb{R}^+)$ and $\lambda(\cdot)$, $\beta(\cdot)$, $\sigma(\cdot)$, $k(\cdot)$ and $w(\cdot, \cdot)$ are continuous functions. We will study the behaviour of solutions of (1.1) above under quite similar circumstances, giving suitable economic interpretations of each term in the S.I.D.E. Berger and Mizel also established rules for manipulating iterated stochastic integrals with non-adapted integrands [6]. Results relating to the existence and uniqueness of stochastic integral equations with non-adapted integrands have been proved by Protter [63] and Kleptsyna and Veretennikov [46]. More recent existence, uniqueness and regularity results of anticipating stochastic Volterra equations may be found in Pardoux and Protter [60] and Alòs and Nualart [2].

Despite these theoretical foundations and some results giving conditions for mean-square stability of zero solutions of S.I.D.Es (see Pachpatte [59] and Zan Kan and Zhang [80]), the theory of S.I.D.Es is substantially underdeveloped. This thesis attempts to rectify this defiency somewhat, at least for the linear theory. In order to be able to consider important applied problems in the context of our model, I have established some new results relating to the representation, local regularity, pathwise asymptotics and the pathwise growth rate of the extrema of this system. We will also consider the asymptotic distributional character of this equation. To prove these results, we exploit the fact that the solution of such equations can be shown to be normally distributed, arising from their "stochastic variation of parameters" representation, and use a combination of the techniques of stochastic analysis and the theory of deterministic integro-differential equations, and in particular, the theory of Liapunov functionals for deterministic integro-differential equations.

1.2.2 Periodicity

Our concern with investigating the effects of memory on asset prices will be matched with a parallel concern with the effects of seasonality or periodicity.

The motivation for introducing periodicity into our analysis is twofold: most simply, it permits us to develop a tractable theory of price evolution when the agents' confidence or liquidity is varying. This question will be considered thoroughly in Chapter 2, in the absence of memory effects. We will show that the greater this variation becomes, the more unstable the behaviour of prices, and the heavier the tails of the returns distribution will be. Moreover, the tails are always guaranteed to be heavier than those of a normal distribution with the same variance. In Chapter 7, we prove weaker results of a similar character when the feedback effects of chartists are included. This mechanism corresponds closely to that mooted in Guillaume et al. [38]. Periodicity with higher amplitude can also be interpreted as arising from increased feedback trading, or greater heterogeneity generally. In this context, it is interesting to be able to report that volume of trade increases with increasing amplitude: this result is consistent with the findings of Froot and Frankel [29], Tauchen and Pitts [76] etc. It can also be consistent with the alternating sign of the autocorrelation of the returns discussed in Cutler, Poterba and Summers [14], [15] and Poterba and Summers [62].

A second reason for studying the effects of periodicity is given by Guillaume et al. [38], who detected significant seasonality in the global foreign exchange market. They find a strong seasonality corresponding to the hour of the day or day of the week in the volatility and volume of trade (as measured by the tick frequency). For example, trade picks up after midnight G.M.T. as the Tokyo and Sydney exchanges open, and subsequently with the opening of the Hong Kong and Singapore markets, with a sharp fall in business coinciding with lunch in these markets at 4.00 G.M.T. The intensity of trading continues at a high level throughout the afternoon in the Far East and continues as Hong Kong and Singapore close and London and Frankfurt open. Trade in Europe falls back prior to the opening of the New York market, but increases again as both European and North American markets are simultaneously open. Trade then declines steadily after New York closes until the Far East markets re-open. Furthermore, different currencies are traded at differing intensities according to geographical location; in general, currencies are traded more heavily when their home exchange is open. Periodic structure is also imposed, for example, by traders having to close their positions each day (recall the remarks above of Froot and Frankel [28]). Over longer horizons, the appraisal of investment analysts' performance at fixed time increments should also introduce periodicity, as should the regular announcement of currency-, industry- or stock-specific news. For example, economic data which might effect a currency – for instance inflation, money supply growth, levels of public borrowing – are published in most countries on the same day each month.

1.3 Outline of the Thesis

In this section we will briefly summarise the contents of each chapter of the thesis.

In Chapter 2, as noted above, the effects of fluctuating agents confidence and periodicity are able to mimic qualitatively several properties of real asset prices. This periodicity has recently been remarked upon by financial economists, but has not been treated theoretically. The analysis we consider here is a tentative step towards such a treatment. In Chapter 2, we model log–prices as evolving according to a stochastic differential equation, which we arrive at by applying an invariance principle to a discrete time price process which results from a market equilibrium. This equation is given by

$$dX_t = -a(t)X_t\,dt + \sigma\,dB_t, \tag{1.2}$$

where $a(\cdot)$ is a T-periodic function and

$$A := \frac{1}{T}\int_0^T a(s)\,ds > 0.$$

We prove that (1.2) has more unstable dynamics when $a(\cdot) \not\equiv A$ than when $a(\cdot) \equiv A$, and that while the diffusion with $a(\cdot)$ non-constant does not have an asymptotic invariant density, the limiting empirical distribution converges almost surely to a distribution function whose density function is a continuous superposition of Gaussian densities, a result which appears absent from the literature. This result has implications for the returns distribution, autocorrelation of returns, volume of trade, and the agents' behaviour and confidence.

In Chapter 3, we prove that there is a unique a.s. continuous solution to (1.1), even in the presence of an integrable singularity in $w(\cdot,\cdot)$ and that the solution possesses a variation of parameters representation given in terms of the resolvent of a related deterministic integro-differential equation. Finally, we notice that the solution has the same regularity as a Brownian motion—it is locally Hölder continuous of all orders less than 1/2, but not of any order 1/2 or greater.

In Chapter 4, we interpret the function $w(\cdot,\cdot)$ in (1.1) as the weight chartists give to previously observed asset prices. This interpretation leads to choosing some very natural and general properties for $w(\cdot,\cdot)$, which yields a very rich form of chartist behaviour: for example, they either chase the trend or take profits at the top of the market, do not trade if the price is flat, and accurately track periodically oscillating and growing prices. We give an incomplete construction of the program which, in Chapter 2, builds a continuous price process in continuous time from a discrete time market equilibrium. However, it is demonstrated informally how the log-price evolution (1.1) results. Following this, we prove that European options can be priced, replicated and hedged under the same conditions as imposed on the standard Black-Scholes case with deterministic time dependent volatility. In these circumstances, the hedging strategy of a small investor is the same as for Black-

Scholes dynamics with the same volatility and time dependent interest rates. This result shows that the Efficient Market Hypothesis is inessential for the correct pricing of certain classes of derivative securities. Results are presented relating volatility positively with the volume of trade and indicating the stabilising effect of fundamentalist speculators.

In Chapter 5, we prove under some particular assumptions on the structure of the chartists' memory, that prices follow the consensus reservation price of the fundamentalists asymptotically, if the fundamentalists are the dominant class of investor. If the chartists are dominant and they are positive feedback speculators, the prices can either form a bubble, or crash, and both events have a positive probability of occuring. The possibility of a bubble is greater if the price starts at a higher level or if the fundamentalists are more optimistic about future fundamentals. This result will be seen to tally very well both with the picture of bubble formation as outlined in De Long et al. [18] and real speculative bubbles (and crashes). However, in our case, true feedback is being introduced for the first time via the explicit dependence of past prices.

In Chapter 6, we show that the results of the previous chapter are subject to considerable generalisation. If we suppose once more that price evolves according to (1.1) with $w(\cdot, \cdot)$ satisfying only the properties endowed upon it in Chapter 4, the log-price is then asymptotic to the fundamentals almost surely whenever the fundamentalists are dominant. This pathwise convergence is the strongest convergence result yet proved for stochastic integro-differential equations, albeit the equation is linear. Moreover, pathwise convergence is assured if the chartists are negative feedback traders, or if the fundamentals are determined by a stochastic process independent of the standard Brownian motion $(B_t)_{t\geq 0}$ which drives (1.1), and are subject to the same asymptotic growth condition as before. For example, if fundamentals grow linearly with time, the price has a well defined growth rate, just as in the Black-Scholes case. We remark that these asymptotic convergence results imply that both chartists and fundamentalists are relatively successful at estimating the price level, so that the price dynamics do not tend to undermine the continued existence of both groups of speculators.

In the penultimate chapter, we prove analogous results for the price and returns distributions as were established in Chapter 2: that is to say, if one assumes that the fundamentalists are dominant, that some of the functions $\lambda(\cdot)$, $\beta(\cdot)$ and $\sigma(\cdot)$ are periodic and $k(\cdot) = K^*$, a constant, then the price and returns distributions do not converge, but that their empirical distribution functions converge in some manner to continuous superpositions of Gaussian densities, causing the tails of the returns distribution to be heavier than a normal density with the same variance.

The concluding chapter of the thesis considers the almost sure asymptotics of the extremes of $(X_t)_{t \geq 0}$ when the same conditions as the previous chapter apply. In the process, we supply an extension to the theory of Klass and Barndorff-Nielsen in establishing the almost sure asymptotic growth rate of the extrema of a dependent sequence of random variables. We can use this result to observe that faster asymptotic growth of the extrema of X can be associated with a market in which the fundamentalists are weaker or less confident in the equilibrium price level, but that the asymptotic growth rate of the extreme deviations from the fundamentals is smaller than is experienced for Black-Scholes price dynamics, even though the returns distribution may have heavier tails.

Chapter 2

Heterogeneous Markets, Time Horizons and Heavy Tails

2.1 Introduction

A recent trend in models of financial markets has been the relaxation of the assumption of homogeneous agents. Surveys have pointed to a diversity of agents with differing price expectations hypotheses and differing investment time horizons, with particular reference to the foreign exchange market e.g., Allen and Taylor [1], Itô [40], and Froot and Frankel [29].

Specifically, classes of traders with extrapolative expectations or positive feedback strategies have been identified—trend chasing, the use of stop-loss orders, certain aspects of charting or technical analysis and dynamic trading strategies such as portfolio insurance. These aspects of speculative behaviour are discussed at length in De Long et al. [18], Day and Huang [16] and De Grauwe et al. [17].

The work of Guillaume et al. [38] indicates that seasonal and short term deterministic periodic strucures exist in foreign exchange markets. This chapter attempts to show that heterogeneity amongst the agents, together with a speculative strategy which is periodic in character, leads to return and log-price processes which have heavier tails than normal distributions, even when the log-price dynamics obey an Ornstein-Uhlenbeck equation with time varying coefficients. This also has consequences for the autocorrelation of the return process which agrees with the empirical findings of [38] and Porteba and Summers [62]. We can think of the periodic behaviour as arising from a sequence of investments each of which has finite time horizon T, positing that the investors face similar situations at similar stages during the lifetime of the investment.

This chapter consists of several sections. Section 2 develops a microeconomic model of prices in discrete time and, using the notion of weak convergence outlined in Schweizer and Föllmer [26], produces a diffusion model in continuous time. This leads to a stochastic differential equation of the form

$$dX_t = -a(t)X_t\, dt + \sigma\, dB_t \tag{2.1}$$

where B is a standard Brownian motion, $X_t = \log S_t$ and S_t is the price of the asset at time t.

In Section 3, we see that the diffusion does not have a stationary invariant distribution, although there is stationarity in a periodic sense. The large deviations of this process are considered, and shown to be greater than those experienced by the diffusion resulting when the classes of agents considered have time-independent strategies. In Section 4, we show that both greater heterogeneity among the agents and greater prevalence of agents with extrapolative expectations increase the sample moments of the log-price. This is shown to lead to heavier tails by the results of Sections 5 and 6. In Section 5, we prove that the sample moments of X_t converge almost surely, and then in Section 6 use this result to show that the empirical distribution of X_t converges almost surely. Section 7 addresses properties of the density of the limit distribution and of the returns process. Section 8 produces a rough heuristic argument relating agent heterogeneity, the volume of trade and level of price variability. An Appendix, containing the proof of the key supporting lemma on which the a.s. convergence of the empirical distribution function relies, concludes the chapter.

This chapter aims to demonstrate that heterogeneity among agents' perceptions (even under recurrent Ornstein-Uhlenbeck log-price dynamics) leads to heavier tails in the returns and the limiting empirical distribution function. This indicates that heavy tails should be a consequence of heterogeneity.

Moreover, we show that by endowing a recurrent time scale on investors' actions, it is possible to replicate the negative autocorrelation in returns reported by several authors, as well as to mimic the seasonal/intra-day fluctuations experienced in the global foreign exchange market. However, the tails of the returns process limiting empirical distribution function have infinite tail index, whereas the tail index is finite and positive for observed asset returns.

2.2 Building the Diffusion

To derive the form of the price dynamics and returns processes, we employ the methodology introduced by Schweizer and Föllmer [26]. We follow their notation closely here in the derivation of the equilibrium price and our analysis differs little. For convenience we paraphrase their argument. Let A be a finite set of agents active in the market. Each agent $a \in A$ forms an excess demand $e_a(p)$ given an asset price p. The equilibrium asset price S_k is determined at a sequence of times $t_k, k = 0, 1, \ldots$. Other factors influencing the agents decisions are summarised by the variable ω, which may be viewed as an outcome in the underlying probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. The equilibrium process thus becomes a discrete time stochastic process defined implicitly by zero total excess demand. Assume that the excess demand functions are given by

$$e_{a,k}(p, \omega) = \alpha_{a,k} \log \frac{\hat{S}_{a,k}(\omega)}{p} + \delta_{a,k}(\omega). \tag{2.2}$$

where $\alpha_{a,k} \geq 0$, $\delta_{a,k}$ is liquidity demand and $\hat{S}_{a,k}$ denotes an individual reference level of agent a for period k. As in Schweizer and Föllmer, we note that log–linear excess demand functions arise quite frequently in monetary models, see for example Cagan [11], Gourieroux, Laffont and Monfort [37] and Laidler [48]. The implicit zero total demand condition now reads

$$\log S_k(\omega) = \sum_{a \in A_k} \bar{\alpha}_{a,k} \log \hat{S}_{a,k}(\omega) + \delta_k(\omega) \tag{2.3}$$

with

$$\bar{\alpha}_{a,k} = \left(\sum_{a \in A_k} \alpha_{a,k} \right)^{-1} \alpha_{a,k}, \qquad \delta_k = \left(\sum_{a \in A_k} \alpha_{a,k} \right)^{-1} \sum_{a \in A_k} \delta_{a,k}. \tag{2.4}$$

Schweizer and Föllmer remark that static expectations lead to the price process behaving as a geometric Brownian motion, under suitable rescaling and passage from discrete to continuous time.

For a *fundamentalist* or *information trader*, the individual reference level is determined according to his perception of the fundamental value of the asset and by his belief that the price should be attracted to that value. Take the fundamental value to be 1 and specify

$$\log \hat{S}_{a,k} = \log S_{k-1} + \beta_{a,k} \log S_{k-1} \tag{2.5}$$

where $\beta_{a,k} \leq 0$.

The *technical analyst* or *chartist* will have reference level given by

$$\log \hat{S}_{a,k} = \log S_{k-1} + \gamma_{a,k}(\log S_{k-1} - \log S_k) \tag{2.6}$$

where $\gamma_{a,k} \leq 0$. Let $X_k = \log S_k$ and set

$$\gamma_k = \sum_{a \in A_k} \bar{\alpha}_{a,k}\gamma_{a,k} \tag{2.7}$$

and

$$\beta_k = (1+\gamma_k)^{-1} \sum_{a \in A_k} \bar{\alpha}_{a,k}\beta_{a,k}, \qquad \varepsilon_k = (1+\gamma_k)^{-1}\delta_k. \tag{2.8}$$

Then with m_1 chartists and m_2 fundamentalists, the process X_k satisfies

$$X_k - X_{k-1} = \beta_k X_{k-1} + \varepsilon_k. \tag{2.9}$$

As is stated in [26], it is now possible to obtain a continuous asset price process S by a passage from the discrete-time equilibrium price process X. We use weak convergence on the Skorohod space $D_{\mathbb{R}}$ of all $\mathbb{R}$-valued RCLL functions on $[0, \infty)$ endowed with the Skorohod topology.

For each n consider the process $\{X_k^n\}_{k=0,1,\ldots}$ given by

$$X_k^n - X_{k-1}^n = \beta_k^n X_{k-1}^n + \varepsilon_k^n$$

with $X_0^n = x_0$ fixed. Notice the RCLL version of $\{Z_k^n\}_{k=0,1,\ldots}$ is given by $Z_t^n = Z_{[nt]}^n$. Define

$$\tilde{Z}_t^n = \sum_{k=1}^{[nt]} \beta_k^n, \qquad Z_t^n = \sum_{k=1}^{[nt]} \varepsilon_k^n. \tag{2.10}$$

Then notice that (2.10) gives us

$$dX_t^n = X_{t-}^n \, d\tilde{Z}_t^n + dZ_t^n. \tag{2.11}$$

Now let us describe the additional properties that we require of the process δ, and the deterministic sequences α, β, γ. If every $1/n$ time units, speculators update their reference levels and the market clears, it is necessary to determine reasonable orders of magnitude for the sequences $\alpha^n, \beta^n, \gamma^n$ and process δ^n where, as with X above, processes and sequences superscripted with n satisfy the corresponding equations.

It is reasonable to assume that in finite time intervals that reference level changes will be $O(1)$. From (2.5) we thus have

$$\beta^n_{a,k} = O(\frac{1}{n}),$$

and from (2.6) we have

$$\gamma^n_{a,k} = O(1).$$

In finite time intervals, we assume that each agent a has $O(1)$ excess demand. From (2.2), we have

$$e^n_{a,k} = O(\frac{1}{n}).$$

By considering (2.5), (2.6), we see that

$$\alpha^n_{a,k} = O(1), \qquad \mathbb{E}\left[{\delta^n_{a,k}}^2\right] = O(\frac{1}{n}).$$

Additionally, suppose that for each a, $\delta_{a,k}$ are iid with mean 0. To make these assumptions concrete, we assume for $a = 1, 2, \ldots, m_1 + m_2$ that there exist $\alpha_a, \delta_a \in \mathbf{C}([0, \infty))$; and in addition for $a = 1, 2, \ldots, m_2$, there exist $\beta_a \in \mathbf{C}([0, \infty))$ and for $a = m_2 + 1, \ldots, m_1 + m_2$ there exist $\gamma_a \in \mathbf{C}([0, \infty))$ such that

$$\begin{aligned}
\alpha^n_{a,k} &= \alpha_a(k/n), \\
\beta^n_{a,k} &= \frac{1}{n}\beta_a(k/n), \\
\gamma^n_{a,k} &= \gamma_a(k/n), \\
\delta^n_{a,k} &= \frac{1}{\sqrt{n}}\delta_a(k/n)\zeta^k_n.
\end{aligned}$$

where ζ^k_n are a sequence of iid random variables with 0 mean, unit variance.

Further assume for each $a = 1, 2, \ldots, m_1 + m_2$, there exists $T_a > 0$ such that $\alpha_a(\cdot)$ is T_a-periodic, while for $a = 1, 2, \ldots, m_2$, $\beta_a(\cdot)$ is T_a-periodic, and for $a = m_2 + 1, \ldots, m_1 + m_2$, $\gamma_a(\cdot)$ is T_a-periodic. This endows a periodic structure on the demand of the agents. Suppose, moreover, that there exists a minimal $T > 0$ such that for all $a = 1, 2, \ldots, m_1 + m_2$,

$$\frac{T}{T_a} \in \mathbb{N}.$$

This assumption is satisfied if e.g., the agents' investment schedules are denominated in a given time unit. Since these horizons are all likely to be denominated in hours or days for all but the most myopic speculators, this assumption, although stylised, is not completely unreasonable.

We will require the following variation of the Donsker Invariance Principle.

Lemma 2.2.1 *Let $\varepsilon : [0,\infty) \to \mathbb{R}$ and define*

$$Z_t^n = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \varepsilon(k/n)\zeta_n^k$$

where ζ_n^k are iid with zero mean, unit variance. Then

$$Z_t^n \xrightarrow{\mathcal{D}} \int_0^t \varepsilon(s)\, dB_s \text{ as } n \to \infty$$

where $\{B_t : t \geq 0\}$ is a standard one-dimensional Brownian motion.

Proof: Rework the argument in Chapter 2.4.D of [41]. $\diamond$

From (2.10) and Lemma 2.2.1 we notice that there exist continuous $T-$periodic functions $a(\cdot), \sigma(\cdot)$ such that

$$\tilde{Z}_t^n \to -\int_0^t a(s)\, ds \qquad \text{as } n \to \infty \tag{2.12}$$

$$Z_t^n \xrightarrow{\mathcal{D}} \int_0^t \sigma(s)\, dB_s \qquad \text{as } n \to \infty. \tag{2.13}$$

Then from (2.11), (2.12), (2.13) we can use the analysis of Kurtz and Protter referenced in [26] to prove

Theorem 2.2.1 *Under the above assumptions the process X^n converges in distribution to the strong solution X of the stochastic differential equation*

$$dX_t = -a(t)X_t\, dt + \sigma(t)\, dB_t \tag{2.14}$$

where $a(\cdot), \sigma(\cdot) > 0$ are T-periodic functions, and B is a standard Brownian motion. Moreover, the price processes $S^n (n = 1, 2, \ldots)$ converge in distribution to the process

$$S_t = \exp X_t.$$

Proof of Theorem 2.2.1: See Theorem 3.1 in [26]. $\diamond$

Remark 2.2.1

As observed in Schweizer and Föllmer [26], price dynamics of Ornstein–Uhlenbeck type are common in mathematical finance; see for example Froot and Obstfeld [33], Shiller [68], Summers [73] and West [78]. In this case, however, the equation is specified on the basis of microeconomic considerations.

Remark 2.2.2

We observe that $a(\cdot)$ is on average positive (resp. negative) when the fundamentalists (resp. chartists) are dominant (in the sense of large contributions $\beta_{a,k}$ (resp. $\gamma_{a,k}$)). Furthermore, $a(\cdot)$ is a constant function when the agents have time independent excess demand functions, and time independent reference levels.

Remark 2.2.3

This model, taken on a suitable time scale satisfies Fact 8 in [38] viz., seasonality in intra day statistics, which is found in, for example, the volatility and the tick frequency. We may think of the fluctuations in the reference levels and excess demands as coming from different speculators in the different geographical markets participating in the market according as to whether their local exchange is open or closed.

Remark 2.2.4

In the analysis that follows we take $\sigma(\cdot) = \sigma$, a constant, with little loss of generality.

Since our interest is centered on price processes which exhibit stationary or "stationary-on-average" properties we need only study the case where X is recurrent. Towards this end define

$$A = \frac{1}{T}\int_0^T a(t)dt. \tag{2.15}$$

Then one immediately has

Proposition 2.2.1 *X is recurrent or transient according as $A > 0$ or $A < 0$ in (2.15). Indeed, if $A < 0$ one has*

$$\lim_{t\to\infty} \frac{1}{t} \log |X_t| = -A, \qquad a.s.. \tag{2.16}$$

Proof: Proposition 2.3.2 shows that X is recurrent when $A > 0$. Transience for $A < 0$ obviously follows from (2.16), which follows from a direct calculation of the solution to (2.14). $\diamond$

2.3 Distribution and Maximal Deviations

We fix our model thus. Suppose $a(\cdot)$ is a T-periodic function, continuous on $[0,T]$, and satisfying (2.15) with $A > 0$ and let X evolve according to

$$dX_t = -a(t)X_t\, dt + \sigma\, dB_t \tag{2.17}$$

where B is a standard Brownian motion, and $X_0 = x_0$. Let $\tilde{v}(0) = 0$ and define

$$\tilde{v}'(t) = -2a(t)\tilde{v}(t) + 1 \tag{2.18}$$

also let $m(0) = x_0$, and define $m'(t) = -a(t)m(t)$, so that

$$X_t \sim \mathcal{N}(m(t), \sigma^2\tilde{v}(t)). \tag{2.19}$$

Let $v(\cdot)$ satisfy (2.18) but with initial condition

$$v(0) = \frac{\int_0^T e^{2\int_0^s a(u)\,du} ds}{e^{2AT} - 1}$$

so that $v(\cdot)$ is the periodic solution to (2.18).

Proposition 2.3.1 *Let $(X_t)_{t\geq 0}$ be given by (2.17). Then:*

(i) X_t does not converge in distribution as $t \to \infty$.

(ii) For all $t \in [0,T]$ there exists $\bar{X}_t \sim \mathcal{N}(0, \sigma^2 v(t))$ such that X_{t+nT} converges in distribution to $\bar{X}_t$ as $n \to \infty$.

Proof: Note that (i) follows directly from (ii). We prove (ii). Writing $w = v - \tilde{v}$, one uses Floquet theory to show $\lim_{t\to\infty} w(t) = 0$. Similarly, $\lim_{t\to\infty} m(t) = 0$. Thus for all $t \in [0,T]$, we have

$$\begin{aligned} \lim_{n\to\infty} \tilde{v}(t+nT) &= \lim_{n\to\infty} (\tilde{v}(t+nT) - v(t+nT)) + \lim_{n\to\infty} v(t+nT) \\ &= v(t). \end{aligned}$$

Part (ii) of the proposition follows directly from (2.19). $\diamond$

The non-existence of an invariant distribution indicates that a more delicate analysis is required to determine the asymptotic character of the diffusion. To motivate the investigation of heavy-tailed properties, we compare the extreme deviations and moments of $(X_t)_{t\geq 0}$ and the diffusion $(\tilde{X}_t)_{t\geq 0}$ defined by

$$d\tilde{X}_t = -A\tilde{X}_t\,dt + \sigma\,dB_t \tag{2.20}$$

and $\tilde{X}_0 = x_0$. This recurrent Ornstein-Uhlenbeck process has the same average attraction towards 0 as X, but with a normal invariant distribution with mean 0, variance $\frac{\sigma^2}{2A}$. It follows from the martingale time change theorem (see Karatzas and Shreve [41], p.174) and the law of the iterated logarithm that the asymptotic maximal deviations of $\tilde{X}$ are given by

$$\begin{aligned}\limsup_{t\to\infty}\frac{\tilde{X}_t}{\sqrt{2\log t}} &= \sigma\sqrt{\frac{1}{2A}}\ \text{a.s.},\\ \liminf_{t\to\infty}\frac{\tilde{X}_t}{\sqrt{2\log t}} &= -\sigma\sqrt{\frac{1}{2A}}\ \text{a.s..}\end{aligned}$$

To prove an analogous result for X we introduce the function $u(\cdot)$

$$u(t) = \int_0^t e^{2\int_0^s a(u)\,du}ds \tag{2.21}$$

and

$$t_0(t) = t - T\left\lfloor\frac{t}{T}\right\rfloor \tag{2.22}$$

where $\lfloor x\rfloor$ is the floor of the number x, namely the largest integer less than or equal to x. One then easily obtains

$$\lim_{t\to\infty}\frac{v(t_0(t))u'(t)}{u(t)} = 1. \tag{2.23}$$

Proposition 2.3.2 *(i) X is recurrent.*

(ii) The following hold almost surely:

$$\limsup_{t\to\infty}\frac{X_t}{\sqrt{2\log t}} = \sigma\sqrt{\max_{0\leq s\leq T} v(s)}, \tag{2.24}$$

$$\liminf_{t\to\infty}\frac{X_t}{\sqrt{2\log t}} = -\sigma\sqrt{\max_{0\leq s\leq T} v(s)}. \tag{2.25}$$

Proof: Note that (i) is a direct consequence of (ii). We prove part (ii) in the limsup case from which the liminf case follows by symmetry. Using (2.17), (2.21) and the martingale time change theorem (Karatzas and Shreve [41], p.174) one has

$$\begin{aligned}\frac{X_t}{\sqrt{2v(t_0(t))\log\log u(t)}} &= \frac{1}{\sqrt{\frac{v(t_0(t))u'(t)}{u(t)}}}\frac{x_0}{\sqrt{2u(t)\log\log u(t)}} +\\ &\quad \frac{1}{\sqrt{\frac{v(t_0(t))u'(t)}{u(t)}}}\frac{\sigma\tilde{B}_{u(t)}}{\sqrt{2u(t)\log\log u(t)}}.\end{aligned} \tag{2.26}$$

Applying (2.23), the law of the iterated logarithm and $\log\log u(t) \sim \log t$ we obtain

$$\limsup_{t\to\infty} \frac{X_t}{\sqrt{2v(t_0(t))\log t}} = \sigma. \tag{2.27}$$

Define $\hat{t} \in [0,T]$ so that $v(\hat{t}) = \max_{0\leq s\leq T} v(s)$. The proposition is now proven provided we can show

$$\limsup_{n\to\infty} \frac{X_{\hat{t}+nT}}{\sqrt{2\log(\hat{t}+nT)}} = \sigma\sqrt{v(\hat{t})}.$$

Expanding this expression in a similar way to that used in (2.26), one requires that

$$\limsup_{n\to\infty} \frac{\tilde{B}_{u(\hat{t}+nT)}}{\sqrt{2u(\hat{t}+nT)\log\log u(\hat{t}+nT)}} = 1.$$

This can be established by letting $a_n = u(\hat{t}+nT)$ and $b_n = \lfloor a_n \rfloor$. Therefore, by the law of the iterated logarithm

$$\limsup_{n\to\infty} \frac{\tilde{B}_{b_n}}{\sqrt{2b_n \log\log b_n}} = 1.$$

Writing $\tilde{B}_{a_n} = \left(\tilde{B}_{a_n} - \tilde{B}_{b_n}\right) + \tilde{B}_{b_n}$, and using the relationship between a_n and b_n proves (2.3). $\diamond$

Remark 2.3.1

$$\begin{aligned} \max_{0\leq s\leq T} v(s) &> \frac{1}{2A} && (2.28) \\ \min_{0\leq s\leq T} v(s) &< \frac{1}{2A} && (2.29) \\ \frac{1}{T}\int_0^T v(t)\,dt &> \frac{1}{2A} && (2.30) \end{aligned}$$

unless $a(t) \equiv A$.

Proof of Remark: Let $f(x) = \log x$. Integrating by parts gives

$$0 = f(v(T)) - f(v(0)) = -2AT + \int_0^T \frac{1}{v(t)}\,dt$$

demonstrating (2.29), (2.28). Using the Cauchy-Schwarz inequality and the above gives (2.30). $\diamond$

Remark 2.3.1, and Propositions 2.3.1, 2.3.2 show that the process X will undergo larger deviations than $\tilde{X}$, but will be less variable on some time intervals (see Figures 2.1 and 2.2 over). This makes us suspect that X will manifest some type of heavy tailed behaviour, and that the greater variability of X than $\tilde{X}$ has its source in increased variation in $a(\cdot)$. We address the latter suspicion in the next section.

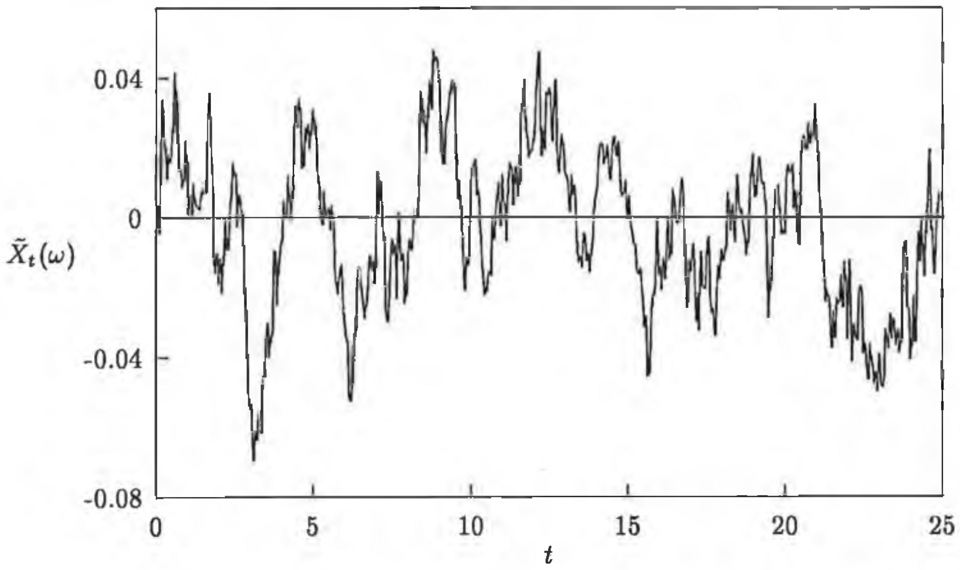

Figure 2.1: Sample path of $d\tilde{X}_t = -A\tilde{X}_t\,dt + \sigma\,dB_t$.

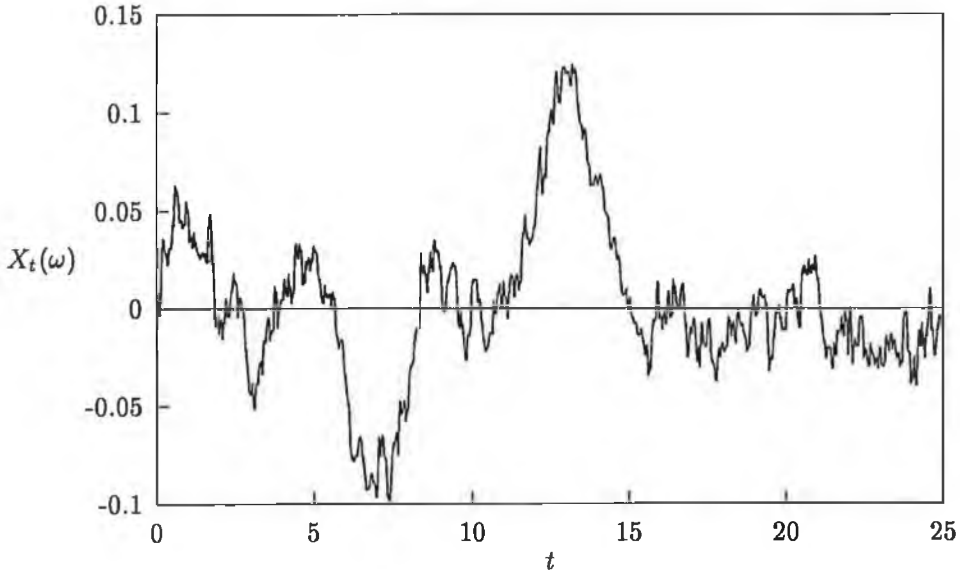

Figure 2.2: Sample path of $dX_t = -a(t)X_t\,dt + \sigma\,dB_t$.

Remark 2.3.2

We can also classify the behaviour of X when $A = 0$. Using the technique of Proposition 2.3.2 we have

$$\begin{aligned}
\limsup_{t\to\infty} \frac{X_t}{\sqrt{2t\log\log t}} &= \sigma\sqrt{\frac{1}{T}\int_0^T e^{2\int_0^s a(u)\,du}\,ds \cdot \max_{0\le s\le T} e^{-2\int_0^s a(u)\,du}} \\
\liminf_{t\to\infty} \frac{X_t}{\sqrt{2t\log\log t}} &= -\sigma\sqrt{\frac{1}{T}\int_0^T e^{2\int_0^s a(u)\,du}\,ds \cdot \max_{0\le s\le T} e^{-2\int_0^s a(u)\,du}}
\end{aligned}$$

while

$$\begin{aligned}
\limsup_{t\to\infty} \frac{\tilde{X}_t}{\sqrt{2t\log\log t}} &= \sigma \\
\liminf_{t\to\infty} \frac{\tilde{X}_t}{\sqrt{2t\log\log t}} &= -\sigma
\end{aligned}$$

so since

$$\frac{1}{T}\int_0^T e^{2\int_0^s a(u)\,du}\,ds \cdot \max_{0\le s\le T} e^{-2\int_0^s a(u)\,du} \ge 1$$

(with equality if and only if $a(\cdot) \equiv A = 0$) it follows that the asymptotic maximal deviations of X are greater than those of $\tilde{X}$, and that X is recurrent but has no stationarity, even on average.

2.4 Variation of $a(\cdot)$ and Variance

We now seek to elucidate the connection between variation in $a(\cdot)$ and higher variance, and explain the behaviour of the function $v(\cdot)$ in terms of the mixture of agents with mean-reverting and extrapolative price expectations present in the market at any instant.

It is possible to interpret greater variation in $a(\cdot)$ as representing the agents' uncertainty as to the permanence of the equilibrium level 0. At some time t_1, they may be confident that 0 represents a correct equilibrium: we interpret this mathematically as $a(t_1) > A$. At another time t_2 they are less sure that 0 is a good equilibrium value, so $a(t_2) < A$. This condition is not arbitrary: rather, from (2.15) it reflects that their confidence is either above or below its average level, or indeed the constant level of confidence in 0 if prices evolved according to (2.20). Moreover, observe from (2.18), (2.29), (2.28) that minimum variance occurs for some time satisfying $a(\cdot) > A$ and maximum variance for some time satisfying $0 < a(\cdot) < A$. Thus, lowest variability occurs when confidence in the equilibrium is above average; greatest variability when confidence is below average.

The effect on price dynamics of agents using extrapolative expectations has been studied by several authors. In the case of this model, such agents would contribute to negative values of $a(\cdot)$, with greater negative values of $a(\cdot)$ corresponding to greater certainty amongst such agents that deviations above 0 are a signal of higher future prices, deviations below 0 a signal of lower future prices. If $a(t) > A$, then these agents have below average influence at time t: if $a(t) < A$, they have above average influence. Note from the above paragraph that minimum variance occurs at a time of below average influence, where $a(\cdot)$ is decreasing, while maximum variance occurs at a time of above average influence where $a(\cdot)$ is increasing. Returning to the issue of variability, since $a(\cdot)$ must be positive on average, the larger these negative contributions, the greater will be the variability in $a(\cdot)$.

We will now formalise the proposed positive link between variation of $a(\cdot)$ and variance. To do this, we construct parameterised families of maps. Let $f(\cdot)$ be a continuous T-periodic function, which is not identically 0, and satisfies $\int_0^T f(t)\,dt = 0$. Now let

$$a(f;\alpha,t) = A\,(1-\alpha f(t)) \tag{2.31}$$

so that for any f that satisfies the above properties $a(f;\alpha,t)$ plays the role of $a(\cdot)$ in the foregoing analysis. Now define the sets of functions $\mathcal{A}_f^+$, $\mathcal{A}_f^-$:

$$\mathcal{A}_f^+ \;=\; \{g : \exists\alpha \geq 0 \text{ s.t. } g(t) = a(f;\alpha,t)\forall t \geq 0\}\,, \tag{2.32}$$

$$\mathcal{A}_f^- \;=\; \{g : \exists\alpha \leq 0 \text{ s.t. } g(t) = a(f;\alpha,t)\forall t \geq 0\}\,. \tag{2.33}$$

Let $\mathcal{A}_f = \mathcal{A}_f^+ \cup \mathcal{A}_f^-$. We call $v(a(\cdot);t)$ the periodic solution to (2.18), and define for $p \in \mathbb{N}$ the functionals

$$\mathcal{M}_p : C[0,T] \to \mathbb{R}^+ : a \mapsto \mathcal{M}_p(a) \stackrel{\text{def}}{=} \frac{1}{T}\int_0^T v(a(\cdot);t)^p\,dt \tag{2.34}$$

and

$$\mathcal{M}_\infty : C[0,T] \to \mathbb{R}^+ : a \mapsto \mathcal{M}_\infty(a) \stackrel{\text{def}}{=} \max_{0\leq t\leq T} v(a(\cdot);t). \tag{2.35}$$

We introduce the following as our chacterisation of variation

$$\mathcal{V} : C[0,T] \to \mathbb{R}^+ : a \mapsto \mathcal{V}(a) \stackrel{\text{def}}{=} \frac{1}{T}\int_0^T \left| a(t) - \frac{1}{T}\int_0^T a(s)\,ds \right|^r dt$$

where $r \geq 1$.

Proposition 2.4.1 *(a) Let $a_1,\ a_2 \in \mathcal{A}_f^+$ (or $a_1,\ a_2 \in \mathcal{A}_f^-$).*

(i) For $p \in \mathbb{N}$, $\mathcal{V}(a_1) > \mathcal{V}(a_2)$ if and only if $\mathcal{M}_p(a_1) > \mathcal{M}_p(a_2)$.

(ii) $\mathcal{V}(a_1) > \mathcal{V}(a_2)$ implies $\mathcal{M}_\infty(a_1) \geq \mathcal{M}_\infty(a_2)$.

(b) For $p = 1, 2, \ldots$

$$\lim_{\mathcal{V}(a)\to\infty} \mathcal{M}_p(a) = \infty.$$

Proof: (a) (i) We can consider $a_1,\ a_2 \in \mathcal{A}_f^+$. The proof for $\mathcal{A}_f^-$ is identical. For $a \in \mathcal{A}_f^+$ such that $a(t) = a(f;\alpha,t)$ define

$$\begin{aligned}
\mathcal{V}(\alpha) &= \mathcal{V}(a(f;\alpha,\cdot)),\\
v(\alpha;t) &= v(a(f;\alpha;t)),\\
\mathcal{M}_p(\alpha) &= \mathcal{M}_p(a(f;\alpha;\cdot)),\\
\mathcal{M}_\infty(\alpha) &= \mathcal{M}_\infty(a(f;\alpha;\cdot)).
\end{aligned}$$

The hypothesis that $a \in \mathcal{A}_f^+$ means that the function $v(\cdot;t) \in \mathbf{C}^2[0,\infty)$ for each $t \in [0,T]$. More particularly let

$$\begin{aligned}
g_1(\alpha;t) &= e^{-2\int_0^t a(f;\alpha,s)\,ds} \int_0^t e^{2\int_0^s a(f;\alpha,u)\,du}\,ds,\\
g_2(\alpha;t) &= e^{-2\int_0^t a(f;\alpha,s)\,ds} v(\alpha;0)
\end{aligned}$$

so that $v(\alpha;t) = g_1(\alpha;t) + g_2(\alpha;t)$. Differentiating under the integral sign twice and using the fact that $f \not\equiv 0$, leads to $\frac{\partial^2 g_1(\alpha;t)}{\partial\alpha^2} > 0$, $\frac{\partial^2 g_2(\alpha;t)}{\partial\alpha^2} > 0$. Thus $\frac{\partial^2 v(\alpha;t)}{\partial\alpha^2} > 0$. Now recall (2.30) and use Jensen's inequality to show

$$\left(\frac{1}{2A}\right)^p \leq \frac{1}{T}\int_0^T v(s)^p\,ds \tag{2.36}$$

where equality is achieved if and only if $v(\cdot)$ is constant, so that necessarily one has $a(t) \equiv A$. Naturally, this argument does not depend on $a \in \mathcal{A}_f$. The critical observation is this: $\mathcal{M}_p(\alpha)$ has a global minimum at $\alpha = 0$. The same conclusion can be drawn for $\mathcal{M}_\infty$. Furthermore, since $v(\cdot;t) \in \mathbf{C}^2(-\infty,\infty)$, we can differentiate under the integral sign successively to derive the following expressions for $p = 1$ and $p \geq 2$

$$\begin{aligned}
\mathcal{M}_1''(\alpha) &= \frac{1}{T}\int_0^T \frac{\partial^2 v(\alpha;t)}{\partial\alpha^2}\,dt,\\
\mathcal{M}_p''(\alpha) &= \frac{1}{T}\int_0^T p(p-1)v(\alpha;t)^{p-2}\left(\frac{\partial v(\alpha;t)}{\partial a}\right)^2 + pv(\alpha;t)^{p-1}\frac{\partial^2 v(\alpha;t)}{\partial\alpha^2}\,dt.
\end{aligned}$$

Plainly, this means that $\mathcal{M}_p''(\alpha) > 0$ for all p. Since $\mathcal{M}_p'(0) = 0$, it follows that $\mathcal{M}_p'(\alpha) > 0$ for all $\alpha > 0$. A straightforward calculation shows that $\mathcal{V}'(\alpha) > 0$ for all $\alpha > 0$. This establishes part (a) (i) of the proposition.

(a) (ii) To prove part (ii), we note the following: If $f_p(\alpha)$ converges pointwise to $f(\alpha)$ as $p \to \infty$ for every α, and each of the $f_p(\cdot)$ are increasing, then $f(\cdot)$ is non-decreasing. Now, since $\mathcal{M}_p(\cdot)$ is increasing on $\mathbb{R}^+$, so is $\mathcal{M}_p(\cdot)^{\frac{1}{p}}$. Therefore, one can put $f_p(\alpha) = \mathcal{M}_p(\alpha)^{\frac{1}{p}}$, and $f(\alpha) = \mathcal{M}_\infty(\alpha)$ in the above to conclude that $\mathcal{M}_\infty(\cdot)$ is non-decreasing. This proves (ii).

(b) For $p < \infty$, this comes from the fact that $\mathcal{M}_p(\cdot)$ is convex. For $p = \infty$, it follows from $\mathcal{M}_q(\alpha) \leq \mathcal{M}_\infty(\alpha)$ for all $q < \infty$. $\diamond$

Remark 2.4.1

If f is $\mathbf{C}^1[0, T]$, we may take $\mathcal{V}$ to be the total variation, with no change in conclusion.

Remark 2.4.2

The convexity of $\mathcal{M}_p(\cdot)$ shows that the average and maximum variance are quite sensitive to changes in α. As a consequence, small changes in the heterogeneity or psychology of the market could reasonably lead to greater instability than might be expected. We will shortly see that (a) (i) shows that all moment time averages of X are increased by increases in the variation of $a(\cdot)$. This will allow us to conclude that a heavier tailed asymptotic structure is generated by more uncertain market, or one in which, at least temporarily, a majority of agents take prices above a certain level as a signal of future rises.

We have shown that increases in variation of $a(\cdot)$ increases variance and that the maximum and minimum variance occur at times of low and high reversion to 0 respectively. Specialising the choice of $a(\cdot)$ and relaxing the continuity assumptions on $a(\cdot)$ somewhat allow a more precise result in this direction. To this end, suppose $a(\cdot)$ admits a jump discontinuity at T. We now may consider non-increasing functions on $[0, T]$ as candidates for $a(\cdot)$, interpreting this as an increase in the influence of chart traders as time advances across the period. It can then be shown that the maximum variance occurs at the times at which the chartists are most influential (viz. times nT, $n \in \mathbb{N}$). To show this define

$$h(t) = \frac{\int_0^t e^{2\int_0^s a(u)\,du}\,ds}{e^{2\int_0^t a(s)\,ds} - 1}.$$

Then $1 - 2a(t)h(t) \geq 0$ so $h'(t) \geq 0$. Thus $h(t) \leq h(T) = v(0)$. Rearranging this inequality yields $v(t) \leq v(0)$. Thus

$$\max_{0 \leq t \leq T} v(t) = v(T).$$

Similarly, if $a(\cdot)$ is increasing, the minimum variance occurs when the chartists are least influential.

To conclude this section, we make the following observations which exemplify the close inverse dependence of $v(\cdot)$ on $a(\cdot)$.

Remark 2.4.3

(a) If $a(\cdot)$ is non-increasing, one notices that the first derivative conditions on the internal extrema of $v(\cdot)$ implies that there exists $t_m \in [0,T]$ such that $v(\cdot)$ is decreasing on $[0, t_m]$ and non-decreasing on $[t_m, T]$. Analogous statements are true when $a(\cdot)$ is non-decreasing, increasing or decreasing.

(b) One can use basic calculus techniques to show that if $a(\cdot)$ is increasing on $(0, T_1)$ and decreasing on (T_1, T) then there exist $t_M < T_1$, $t_m > T_1$ so that $v(\cdot)$ is non-decreasing on $[0, t_M)$, non-increasing on (t_M, t_m) and non-decreasing on $(t_m, T]$.

2.5 An Ergodic Theorem

To establish an almost sure description of the asymptotic distributional character of the logarithm of the asset price we show that its p^{th}-moment time averages converge almost surely as time increases to infinity.

Theorem 2.5.1 *If X is the solution to (2.17), and $v(\cdot)$ is the periodic solution to (2.18), then the following holds for $p \in \mathbb{N}$ almost surely:*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X_s^{2p}\,ds = \sigma^{2p}\frac{(2p)!}{2^p p!}\frac{1}{T}\int_0^T v(s)^p\,ds, \tag{2.37}$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X_s^{2p+1}\,ds = 0. \tag{2.38}$$

The proof of Theorem 2.5.1 follows later in this section.

Remark 2.5.1

We remark that departure from strict periodicity is possible with no loss of generality. Let dynamics of X_t^b be governed by

$$dX_t^b = -b(t)X_t^b\,dt + \sigma dB_t, \tag{2.39}$$

with X_0 deterministic, where $b(\cdot)$ satisfies

$$\lim_{t\to\infty} (b(t) - a(t)) = 0, \tag{2.40}$$

and $a(\cdot)$ satisfies all the usual properties. If $v(\cdot)$ is the periodic solution to (2.18), and $v^b(\cdot)$ satisfies $v^b(0) = 0$, and

$$v^{b'}(t) = -2b(t)v^b(t) + 1, \tag{2.41}$$

(so that then $X_t^b \sim \mathcal{N}\left(X_0 e^{-\int_0^t b(s)\,ds}, \sigma^2 v^b(t)\right)$) then one can prove

$$\lim_{t\to\infty} v^b(t) - v(t) = 0.$$

Then we have

Theorem 2.5.2 *If X^b is the solution to (2.39), and $v(\cdot)$ is the periodic solution to (2.18), then the following holds for $p \in \mathbb{N}$ almost surely:*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t {X_s^b}^{2p}\,ds = \sigma^{2p}\frac{(2p)!}{2^p p!}\frac{1}{T}\int_0^T v(s)^p\,ds, \tag{2.42}$$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t {X_s^b}^{2p+1}\,ds = 0. \tag{2.43}$$

Proof: Analogous to Theorem 2.5.1. $\diamond$

Remark 2.5.2

The even power time averages increase as the disturbance increases, from Proposition 2.4.1, confirming the claim in Remark 2.4.2. Moreover, for each $p \in \mathbb{N}$ the asymptotic $2p^{th}$ order sample moments of X are greater than the corresponding asymptotic sample moments of $\tilde{X}$. This shows that the pathwise behaviour of the price will be increasingly unstable under conditions of increasing market heterogeneity and increasing uncertainty of agents. It is not implausible that this tendency will reinforce the agents' behaviour.

We notice that this result shows that more unstable price dynamics (and as we will presently show, heavier tails in the returns distribution), has its roots in increased agent heterogeneity. It therefore adds some theoretical substance to the mechanism that Guillaume et al. [38] propose to be responsible for heavy tails in exchange rate returns.

They state that extreme events rely on the presence or absence of certain market participants such as medium term investors or pure speculators due to changing market conditions. They claim that fat tails result from the activity of relatively long term speculators reacting after some news, (corresponding to $a(\cdot)$ positive) upon which short term traders become even more active (corresponding roughly to $a(\cdot)$ changing from positive to negative), thereby reinforcing the initial fluctuations.

The price distribution always has kurtosis greater than 3. More generally, if

$$\kappa_p(\omega;t) = \frac{\frac{1}{t}\int_0^t \left(X_s(\omega) - \frac{1}{t}\int_0^t X_u(\omega)\,du\right)^{2p} ds}{\left(\frac{1}{t}\int_0^t \left(X_s(\omega) - \frac{1}{t}\int_0^t X_u(\omega)\,du\right)^{2} ds\right)^{p}}. \tag{2.44}$$

and

$$\kappa_p = \frac{\mathbb{E}[Y^{2p}]}{\mathbb{E}[Y^2]^p},$$

where $Y \sim \mathcal{N}(0,1)$, then one uses Jensen's inequality to prove

$$\lim_{t\to\infty} \kappa_p(\omega;t) \geq \kappa_p \qquad \text{a.s.}$$

with equality iff $a(t) \equiv A$.

To prove Theorem 2.5.1, we first introduce some notation and results that will be used frequently in the proof. For brevity, we write $v_{max} = \max_{0\leq s\leq T} v(s)$.

(1) Recall the definition of $u(\cdot)$ in (2.21), and that $u(\cdot)$ is invertible. If $u(t) = T^*$, then one can show that there exists $\infty > M_1 \geq 0$ such that

$$-M_1 + \frac{1}{2A}\log T^* \leq u^{-1}(T^*) \leq M_1 + \frac{1}{2A}\log T^*. \tag{2.45}$$

To prove (2.45), write down an $\epsilon - \delta$ version of (2.23) and use the fact that

$$\log u'(t) \sim 2AT \left\lfloor \frac{t}{T} \right\rfloor .$$

(2) Recall (2.22) and define

$$G_0(w) = \left(\frac{v(t_0(u^{-1}(w)))}{w} \right)^{1+\frac{p}{2}} \tag{2.46}$$

and recursively for $k = 0, 1, \ldots, \frac{p}{2} - 1$ ($\frac{p-1}{2}$ for p odd)

$$G_{k+1}(w) = -\int_w^\infty G_k(r)\, dr. \tag{2.47}$$

Notice for future use that

$$G'_{k+1}(w) = G_k(w). \tag{2.48}$$

One also has the bound

$$|G_0(w)| \leq v_{max}^{1+\frac{p}{2}} \frac{1}{w^{1+\frac{p}{2}}}.$$

Induction on (2.47) yields

$$|G_k(w)| \leq v_{max}^{1+\frac{p}{2}} \frac{1}{\prod_{j=0}^{k-1} \left(\frac{p}{2} - j\right)} \frac{1}{w^{1+\frac{p}{2}-k}}. \tag{2.49}$$

(3) Using (2.45), (2.49) and the law of the iterated logarithm, we have for any standard Brownian motion $\tilde{B}$

$$\lim_{T^* \to \infty} \frac{\tilde{B}_{T^*}^{p-2k} G_{k+1}(T^*)}{u^{-1}(T^*)} = 0. \tag{2.50}$$

(4) Define for some as yet to be specified $w_0 > 0$ the integrals

$$I_k(p, T^*) = \frac{1}{u^{-1}(T^*)} \int_{w_0}^{T^*} \tilde{B}_w^{p-2k} G_k(w)\, dw. \tag{2.51}$$

For p even we have

$$\lim_{T^* \to \infty} I_{\frac{p}{2}}(p, T^*) = (-1)^{\frac{p}{2}} \frac{1}{(\frac{p}{2})!} \frac{1}{T} \int_0^T v(s)^{\frac{p}{2}}\, ds. \tag{2.52}$$

To show (2.52), define for $j = 0, 1, \ldots, \frac{p}{2}$

$$J^j(T^*) = \frac{1}{j!} \int_{w_0}^{T^*} w^j G_{\frac{p}{2}-j}(w)\, dw. \tag{2.53}$$

Thus $I^0(\cdot) = I_{\frac{p}{2}}(p, \cdot)$. Using the periodicity of $v(\cdot)$, (2.23) and the change of variable $w = u(s)$, for sufficiently large $s_0 > 0$, one can calculate

$$
\begin{aligned}
& \frac{1}{(\frac{p}{2})!}\frac{1}{T}\int_0^T v(s)^{\frac{p}{2}}\,ds \\
= & \lim_{t\to\infty}\frac{1}{(\frac{p}{2})!}\frac{1}{t}\int_0^t v(s)^{\frac{p}{2}}\,ds \\
= & \lim_{T^*\to\infty}\frac{1}{(\frac{p}{2})!}\frac{1}{u^{-1}(T^*)}\int_{s_0}^t v(t_0(s))^{\frac{p}{2}}\,ds \\
= & \lim_{T^*\to\infty}\frac{1}{(\frac{p}{2})!}\frac{1}{u^{-1}(T^*)}\int_{w_0}^{T^*} v(t_0(u^{-1}(w)))^{\frac{p}{2}}\frac{v(t_0(u^{-1}(w)))}{w}\,dw \\
= & \lim_{T^*\to\infty} I^{\frac{p}{2}}(T^*).
\end{aligned}
$$

Integrating (2.53) by parts and observing from (2.45), (2.49) that

$$\lim_{T^*\to\infty}\frac{T^{*j+1}G_{\frac{p}{2}-j}(T^*)}{u^{-1}(T^*)} = 0$$

gives on taking limits and using a reverse induction argument that

$$\lim_{T^*\to\infty} I^{\frac{p}{2}}(T^*) = (-1)^{\frac{p}{2}}\lim_{T^*\to\infty} I^0(T^*).$$

Equation (2.52) is now immediate.

(5) Define for $k = 0, 1, \ldots, \frac{p}{2} - 1$.

$$J_k(T^*) = \frac{1}{u^{-1}(T^*)}\int_{w_0}^{T^*} \tilde{B}^{p-(2k+1)}G_{k+1}(w)\,d\tilde{B}_w. \tag{2.54}$$

If $Y_k(t) = J_k(u(t))$, then almost surely, for $t \in \mathbb{N}$

$$\lim_{t\to\infty} Y_k(t) = 0. \tag{2.55}$$

To show (2.55), let $q = p - 2k$ and observe by the Burkholder-Davis-Gundy inequalities that there exists a constant $0 < K_4 < \infty$ such that

$$
\begin{aligned}
\mathbb{E}\left[Y_k(t)^4\right] &\le \frac{1}{u^{-1}(T^*)^4}\mathbb{E}\left[\sup_{w_0\le r\le T^*}\left(\int_{w_0}^{r}\tilde{B}_w^{p-(2k+1)}G_{k+1}(w)\,d\tilde{B}_w\right)^4\right] \\
&\le \frac{K_4}{u^{-1}(T^*)^4}\mathbb{E}\left[\left(\int_{w_0}^{T^*}\tilde{B}_w^{2(p-(2k+1))}G_{k+1}(w)^2\,dw\right)^2\right] \\
&\le \frac{K_4C_q}{t^2}\mathbb{E}\left[\left(\frac{1}{u^{-1}(T^*)}\int_{w_0}^{T^*}\frac{\tilde{B}_w^{2(q-1)}}{w^q}\,dw\right)^2\right]
\end{aligned} \tag{2.56}
$$

where we used (2.49) in (2.56) and $0 < C_q < \infty$. It is possible to show, using an induction argument similar to that used in the proof of the theorem, that the expectation term in (2.56) is bounded by a constant $0 < C_q^1 < \infty$: thus

$$\mathbb{E}\left[Y_k(t)^4\right] \leq \frac{K_4 C_q C_q^1}{t^2}.$$

A standard Borel–Cantelli argument now establishes (2.55).

Proof of Theorem 2.5.1: Without loss of generality, we may assume $X_0 = 0$, and $\sigma = 1$. Using the Martingale Time Change Theorem, the same type of argument as used in part (4) of the preamble gives us

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X_s^p\,ds = \lim_{T^*\to\infty} I_0(p, T^*) \tag{2.57}$$

provided the limit on the right-hand side of (2.57) exists. To calculate it we have using (2.48), (2.51), (2.54) the following by stochastic integration by parts for $k = 0, 1, \ldots, \frac{p}{2} - 1$ ($\frac{p-1}{2}$ for p odd)

$$\begin{aligned}\frac{\tilde{B}_{T^*}^{p-2k} G_{k+1}(T^*)}{u^{-1}(T^*)} - \frac{\tilde{B}_{w_0}^{p-2k} G_{k+1}(w_0)}{u^{-1}(T^*)} &= \\ I_k(p, T^*) + (p-2k)J_k(T^*) + \frac{1}{2}(p-2k)(p-(2k+1))I_{k+1}(p, T^*).\end{aligned} \tag{2.58}$$

Write $T^* = u(t)$, let $t \in \mathbb{N}$ and take the limit as t goes to infinity both sides of (2.58). By (2.45), (2.50), (2.55) one has

$$I_k(p, T^*) + \frac{1}{2}(p-2k)(p-(2k+1))I_{k+1}(p, T^*) \to 0. \tag{2.59}$$

When p is even, an induction argument on (2.59) together with (2.52) gives, when substituted into (2.57)

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X_s^p\,ds = \frac{p!}{2^{\frac{p}{2}}(\frac{p}{2})!}\frac{1}{T}\int_0^T v(s)^{\frac{p}{2}}\,ds. \tag{2.60}$$

When p is odd, put $k = \frac{p-1}{2}$ into (2.58), remembering that (2.55) suffices to show $I_{\frac{p-1}{2}}(p, T^*) \to 0$. Induction on (2.59) proves $I_0(p, T^*) \to 0$, and hence

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X_s^p\,ds = 0. \tag{2.61}$$

The proof is not quite complete, since (2.60), (2.61) only give a.s. convergence for $t \in \mathbb{N}$. For $t \notin \mathbb{N}$, let $n = \lfloor t \rfloor$, and write (for any $p \in \mathbb{N}$)

$$\begin{aligned}\frac{1}{t}\int_0^t X_s{}^p\,ds - \mathbb{E}\left[\frac{1}{t}\int_0^t X_s{}^p\,ds\right] &= \\ \frac{n}{t}\left(\frac{1}{n}\int_0^n X_s{}^p\,ds - \mathbb{E}\left[\frac{1}{n}\int_0^n X_s{}^p\,ds\right]\right) &+ \frac{1}{t}\int_n^t X_s{}^p - \mathbb{E}[X_s{}^p]\,ds.\end{aligned} \tag{2.62}$$

The first term on the right-hand side of (2.62) goes to 0 by (2.60), (2.61), the second by Proposition 2.3.2, part (ii). $\diamond$

2.6 Convergence of the Empirical Distribution Function

We construct a family of empirical distribution functions $\{F_t(x)\}_{t\geq 0}$ as follows:

$$F : (\mathbb{R}^+, \mathbb{R}, \Omega) \rightarrow [0,1] : (t, x, \omega) \mapsto F_t(x)(\omega) = \frac{1}{t}\int_{s=0}^{t} I_{\{X_s(\omega)\leq x\}}\, ds.$$

We now prove the following

Theorem 2.6.1

$$\lim_{t\rightarrow\infty} \frac{1}{t}\int_{s=0}^{t} I_{\{X_s(\omega)\leq x\}}\, ds = F(x), \tag{2.63}$$

almost surely, where $F(\cdot)$ is a distribution function with continuous density

$$f(x) = \frac{1}{T}\int_0^T \frac{1}{\sigma\sqrt{v(s)}\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2 v(s)}}\, ds. \tag{2.64}$$

Remark 2.6.1

We remark that a result of a similar spirit is presented in Cabrales and Hoshi [10]. In their paper, asset price dynamics arise from an equilibrium model in which there are two types of investors: optimists and pessimists, who share different opinions as to the stochastic evolution of prices. By solving an optimal consumption problem for each class of investors, the authors show that the wealth of the investors fluctuates according to the success of their strategies, with each class gaining temporary dominance over the other. This dominance is to an extent self–feeding, as the dominant speculators' beliefs are translated to prices more consistent with their beliefs than with those of the weaker group. However, random shocks can disturb this process and the weaker group of speculators in turn become dominant. Therefore, there is an endogeneous mechanism for price dynamics shifting between one regime and the other: the proportion of wealth fluctuates according to the relative success of each investor. This in turn influences the asset price dynamics, which influences the success of each strategy and therefore the distribution of wealth, and so on. The authors thus show that for certain parameter values, the proportion of wealth held by each class of investor converges to an ergodic distribution. This model is also responsible for stochastic volatility in the asset price dynamics.

The similarity with our model, and the ones we later describe in which memory has a role, can be seen as follows. Cabrales and Hoshi see their asset price process as belonging to a Smooth Transition

Autoregressive (STAR) class of models, which has apparently been applied to aggregate variables in macroeconomics. A STAR process is a mixture of two autoregressive processes where the relative weight of the processes changes over time. The log-price evolution we have considered is clearly closely related to such processes, since it is in a sense a mixture of infinitely many A.R. processes. It also has the character of a STAR process in that it smoothly changes between the distinct regimes where $a(\cdot) > 0$, when the price process is mean–reverting, and where $a(\cdot) < 0$, in which case the log–price can temporarily deviate from the fundamental value 0. The model of Cabrales and Hoshi is clearly preferrable in terms of realism by virtue of the endogeneous mechanism by which the switches in dynamics occur, in contrast to the exogeneous mechanism we use here. We can exploit the tractability our model possesses, however, to prove more precise qualitative results than in [10].

Remark 2.6.2

If μ_t is the ordinary Lebesgue measure on $[0, t]$, then

$$\lim_{t\to\infty} \frac{\mu_t(\{s : a \leq X_s \leq b\})}{\mu_t([0,t])} = \int_a^b f(x)\, dx$$

almost surely. Thus, although X does not have an invariant distribution, it has an invariant occupation distribution. We remark that in surveys empirical distribution functions are constructed, not invariant distributions.

In the sequel we denote

$$\begin{aligned} f_n(p)(\omega) &= \frac{1}{n}\int_0^n X_s^p(\omega)\, ds \\ f(p) &= \begin{cases} \sigma^p \frac{(p)!}{2^{\frac{p}{2}}(\frac{p}{2})!} \frac{1}{T} \int_0^T v(s)^{\frac{p}{2}}\, ds & \text{if } p \text{ is even} \\ 0 & \text{if } p \text{ is odd} \end{cases} \end{aligned}$$

To prove Theorem 2.6.1 we require the following Lemma

Lemma 2.6.1 *There exists a non-negative summable sequence $C_p(\omega)$ such that for all $|\alpha| < \infty$ and $\omega \in \Omega^*$*

$$\limsup_{n\to\infty} \frac{1}{p!} |\frac{\sqrt{n}}{\sqrt{\log\log n}} (f_n(p)(\omega) - f(p))| |\alpha|^p \leq C_p(\omega)$$

where $\mathbb{P}[\Omega^*] = 1$.

We relegate the proof of this result to the Appendix of this chapter. The following continuous modification of a well known result is also used.

Lemma 2.6.2 *For each $s \in [0, n]$ let $F(\cdot; s)$ be a distribution function with characteristic function $\phi(\alpha; s)$. Then the distribution function F*

$$F(x) = \frac{1}{n} \int_0^n F(x; s)\, ds$$

has characteristic function ϕ given by

$$\phi(\alpha) = \frac{1}{n} \int_0^n \phi(\alpha; s)\, ds$$

Sketch proof of Lemma 2.6.2: Develop a Stieltjes sum for $\phi(\cdot)$ over $[-N, N]$. Using Fubini's theorem this can be written as an integral over s of Stieltjes sums for $\phi(\cdot; s)$ over $[-N, N]$. Letting the mesh of the partition decrease to 0 one can use a dominated convergence argument and let N go to infinity to establish the result. $\diamond$

Proof of Theorem 2.6.1: For each $n \in \mathbb{N}$ consider $F_n(\cdot)(\omega)$. Using Lemma 2.6.2 we conclude that $F_n(\cdot)(\omega)$ has characteristic function

$$\phi_n(\alpha)(\omega) = \frac{1}{n} \int_0^n e^{i\alpha X_s(\omega)}\, ds = \sum_{p=0}^{\infty} \frac{1}{p!} f_n(p)(\omega)(i\alpha)^p.$$

Further define

$$\phi(\alpha) = \frac{1}{T} \int_0^T e^{\frac{-\sigma^2 v(s)}{2}\alpha^2}\, ds = \sum_{p=0}^{\infty} \frac{1}{p!} f(p)(i\alpha)^p.$$

We make the following observation: if $b_n \to \infty$, and

$$\limsup_{n\to\infty} b_n |g_n(p)(\omega) - g(p)(\omega)| \leq C_p(\omega),$$

where $C_p(\omega)$ is non–negative and summable and $\omega \in \Omega^*$ where $\mathbb{P}[\Omega^*] = 1$, then

$$\lim_{n\to\infty} \sum_{p=0}^{\infty} g_n(p) = \sum_{p=0}^{\infty} g(p) \quad \text{a.s.}$$

Using this observation and Lemma 2.6.1 we conclude that for each $|\alpha| < \infty$

$$\lim_{n\to\infty} \phi_n(\alpha)(\omega) = \phi(\alpha)$$

for all $\omega \in \Omega^*$, where $\mathbf{P}[\Omega^*] = 1$. Notice that $\phi(\cdot)$ is continuous at 0 so the Levy continuity theorem (see e.g., Feller p.508 [25]), allows us to deduce for each $\omega \in \Omega^*$ that

$$\lim_{n\to\infty} F_n(x)(\omega) = F(x)$$

where $\phi(\cdot)$ is the characteristic function of the distribution function $F(\cdot)$. Since Ω^* has full measure the convergence is almost sure. Moreover, one can verify (2.63) by using an interpolation argument as in (2.62). It is easy to check that $\phi \in L^1$, so one can immediately use the Levy inversion theorem (see e.g., Feller p. 509 [25]) to conclude that $F(\cdot)$ has bounded density $f(\cdot)$ given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} \phi(\alpha)\, d\alpha.$$

Standard contour integration and Fubini arguments show that $f(\cdot)$ is given by (2.64). $\diamond$

Corollary 2.6.1 *Suppose $\int_{-\infty}^{\infty} g(x) f(x)\, dx$ exists. Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t g(X_s)\, ds = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

almost surely.

Proof: This is an immediate consequence of Theorem 2.6.1. $\diamond$

2.7 Properties of the Density Function and Returns Process

2.7.1 Properties of the Density Function of the Asymptotic EDF of X

We seek to emphasise several salient properties of the density function $f(\cdot)$: first, its symmetry and unimodality about 0; second, that the probability of the process being close to 0 is higher than for a normal density with the same variance; third, that the tail of the distribution is heavier than a normal distribution with the same variance. From this we notice that the process has the desirable property of being relatively quiescent when close to the mean but capable of experiencing larger extreme deviations.

Proposition 2.7.1 *Suppose $F(\cdot), f(\cdot)$ are as defined in Theorem 2.6.1. Then*

(a) $F(\cdot)$ is a symmetric, unimodal distribution with mode at 0.

(b) If

$$Y \sim \mathcal{N}\left(0, \sigma^2 \frac{1}{T} \int_0^T v(s)\, ds\right),$$

with distribution function $\bar{F}(\cdot)$ then

(i) there exists $\underline{a} > 0$ such that for all $|x| \leq \underline{a}$

$$\lim_{t\to\infty} \frac{\mu_t(\{0 \leq s \leq t : |X_s| \leq x\})}{\mu_t([0,t])} > \bar{F}(x) - \bar{F}(-x). \tag{2.65}$$

(ii) there exists $\bar{a} > \underline{a}$ such that for all $x \geq \bar{a}$

$$\lim_{t\to\infty} \frac{\mu_t(\{0 \leq s \leq t : X_s \geq x\})}{\mu_t([0,t])} > 1 - \bar{F}(x). \tag{2.66}$$

(c)

$$\lim_{x\to\infty} \frac{1}{x^2} \log f(x) = -\frac{1}{2\sigma^2 \max_{0\leq s\leq T} v(s)}. \tag{2.67}$$

Proof: Part (a) is obvious. For (b)(i), we show $\underline{a}^2 > (3-\sqrt{6})\sigma^2 \min_{0\leq s\leq T} v(s)$. Let $\bar{f}(\cdot) = \bar{F}'(\cdot)$. Fix x so that $|x|^2 < (3-\sqrt{6})\sigma^2 \min_{0\leq s\leq T} v(s)$, and define $h(y) = e^{-\frac{1}{2y}x^2}/\sqrt{y}$. Then h is convex on the interval $[\sigma^2 \min_{0\leq s\leq T} v(s), \sigma^2 \max_{0\leq s\leq T} v(s)]$ so Jensen's inequality gives

$$h\left(\int_0^1 \sigma^2 v(sT)\, ds\right) \leq \int_0^1 h(\sigma^2 v(sT))\, ds$$

so $\bar{f}(x) \leq f(x)$. Theorem 2.6.1 concludes the proof of (2.65). Part (b)(ii) follows identically with $\bar{a}^2 < (3+\sqrt{6})\sigma^2 \max_{0\leq s\leq T} v(s)$. To show part (c) recall the spectral property of the sup norm

$$\lim_{y\to\infty}\left(\int_0^T \left(e^{-\frac{1}{2\sigma^2 v(s)}}\right)^y ds\right)^{\frac{1}{y}} = \sup_{0\leq s\leq T} e^{-\frac{1}{2\sigma^2 v(s)}}. \diamond$$

Remark 2.7.1

Other characterisations of heavier tails than a normal with the same variance can be developed:

(i)

$$\lim_{x\to\infty} \frac{1-F(x)}{1-\bar{F}(x)} = \infty.$$

(ii) For all $v < \sigma^2 \max_{0\leq s\leq T} v(s)$, let $f_v(\cdot)$ be the density of a normal with variance v. Then

$$\lim_{x\to\infty} \frac{f(x)}{f_v(x)} = \infty.$$

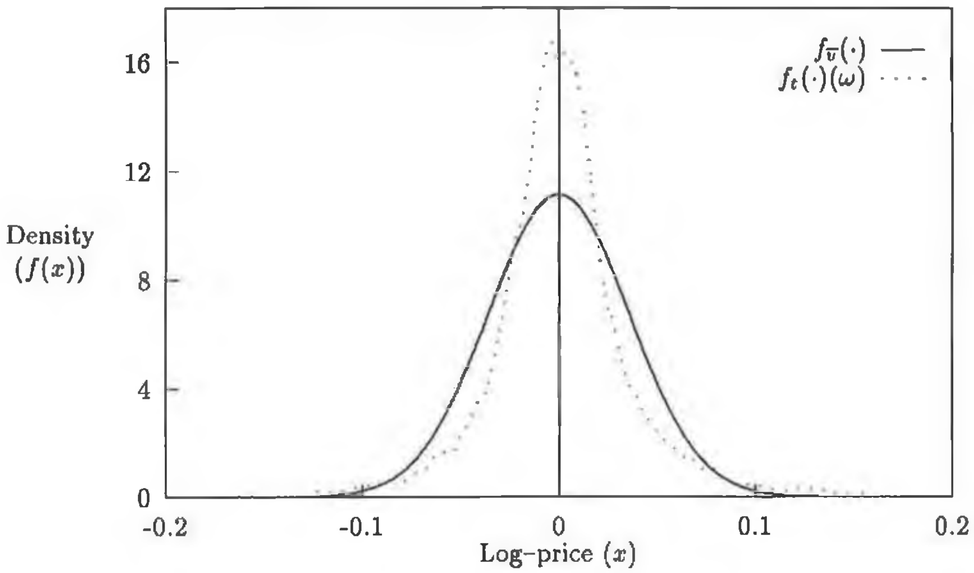

Figure 2.3: Empirical Density function $(f_t(\cdot)(\omega))$ in periodic case and normal density with same variance $(f_{\overline{v}}(\cdot))$.

The result of Proposition 2.7.1 can be observed from the Figure above, where we define

$$\overline{v} = \sigma^2 \frac{1}{T} \int_0^T v(s)\, ds,$$

and f_v to be the density of a normal with zero mean and variance v, and $f_t(\cdot)(\omega)$ to be the "empirical density function" whose distribution function is $F_t(\cdot)(\omega)$.

Remark 2.7.2

Notice from (2.67) that the tail of the distribution behaves in a similar manner to a normal distribution with variance $\sigma^2 \max_{0 \le s \le T} v(s)$. This result is not too surprising in the light of Proposition 2.3.2, part (ii). From our previous analysis, larger maximum variance results from increasing uncertainty of or changes of opinion among the agents. Numerical evidence suggests that the maximum variance increases faster than the average variance when the variation of $a(\cdot)$ increases. Thus the tails can be very much heavier than a normal with the same average variance. Similar numerical evidence is available for the kurtosis.

Remark 2.7.3

Using the method of proof of part (b) of Proposition 2.7.1, it can be shown that if

$$\max_{0\le s\le T} v(s) < \frac{3+\sqrt{6}}{3-\sqrt{6}} \min_{0\le s\le T} v(s)$$

then for

$$(3-\sqrt{6})\sigma^2 \max_{0\le s\le T} v(s) \le x^2 \le (3+\sqrt{6})\sigma^2 \min_{0\le s\le T} v(s)$$

we have $\bar{f}(x) \ge f(x)$.

2.7.2 Returns Process

We now turn our attention to properties of the return process. In particular, we wish to study the convergence of the empirical distribution function of the Δ-return, and also the autocorrelation of the T-returns viz., the autocorrelation of the returns across an investment period. We clarify our terminology with the following definition.

Definition 2.7.1 *The Δ-return of the price process S at time $t+\Delta$, $r(\Delta;t)$ is given by*

$$r(\Delta;t) = X_{t+\Delta} - X_t, \tag{2.68}$$

where $X_t = \log S_t$.

Proposition 2.7.2

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{t}\int_{s=0}^{t} I_{\{r(\Delta;s)(\omega)\le x\}}\, ds\right] = F_\Delta(x) \tag{2.69}$$

where $F_\Delta(\cdot)$ is a distribution function with density $f_\Delta(\cdot)$ given by

$$f_\Delta(x) = \frac{1}{T}\int_0^T \frac{1}{\sigma\sqrt{q_\Delta(s)}\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2 q_\Delta(s)}}\, ds \tag{2.70}$$

and $q_\Delta(\cdot)$ satisfies

$$q_\Delta(t) = v(t+\Delta) + v(t) - 2v(t)e^{-\int_t^{t+\Delta} a(s)\, ds}. \tag{2.71}$$

Proof of Proposition 2.7.2: Let $n \in \mathbb{N}$ and set

$$F_{n,\Delta}(x) = \mathbb{E}\left[\frac{1}{n}\int_{s=0}^{n} I_{\{r(\Delta;s)(\omega)\le x\}}\, ds\right]$$

and since by Definition 2.7.1 and (2.17), $r(\Delta;t) \sim \mathcal{N}(\mu_\Delta(t), g_\Delta(t))$, $F_{n,\Delta}(\cdot)$ has characteristic function given by

$$\phi_{n,\Delta}(\alpha) = \frac{1}{n}\int_{s=0}^{n} e^{i\mu_\Delta(s)\alpha - \frac{1}{2}\sigma^2 g_\Delta(s)\alpha^2}\, ds$$

where we can use the properties of the diffusion X to calculate

$$\begin{aligned}
\mu_\Delta(t) &= x_0 e^{-\int_0^t a(s)\,ds}\left(e^{-\int_t^{t+\Delta} a(s)\,ds} - 1\right), \\
g_\Delta(t) &= \tilde{v}(t+\Delta) + \tilde{v}(t) - 2\tilde{v}(t)e^{\int_t^{t+\Delta} a(s)\,ds}.
\end{aligned}$$

We see that $\mu_\Delta(t) \to 0$ as $t \to \infty$, so that by defining

$$\tilde{\phi}_{n,\Delta}(\alpha) = \frac{1}{n}\int_{s=0}^{n} e^{-\frac{1}{2}\sigma^2 g_\Delta(s)\alpha^2}\, ds,$$

we notice that

$$\lim_{n\to\infty} |\phi_{n,\Delta}(\alpha) - \tilde{\phi}_{n,\Delta}(\alpha)| = 0. \tag{2.72}$$

Now let

$$\phi_\Delta(\alpha) = \frac{1}{T}\int_0^T e^{-\frac{1}{2}\sigma^2 q_\Delta(s)\alpha^2}\, ds$$

It is easy to use the fact that $q_\Delta(t) - g_\Delta(t) \to 0$ as $t \to \infty$ to show

$$\lim_{n\to\infty} \frac{1}{n}\int_{s=0}^{n} e^{-\frac{1}{2}\sigma^2 g_\Delta(s)}\, ds = \frac{1}{T}\int_0^T e^{-\frac{1}{2}\sigma^2 q_\Delta(s)}\, ds,$$

from which it follows that

$$\lim_{n\to\infty} |\phi_\Delta(\alpha) - \tilde{\phi}_{n,\Delta}(\alpha)| = 0. \tag{2.73}$$

The result is now immediate from (2.72), (2.73) and the line of analysis in Theorem 2.6.1. ⋄

We have been unable thus far to apply the method of proof of Theorem 2.6.1 to prove the analagous result for the Δ-return process. Motivated by Proposition 2.7.2, we claim

Conjecture 2.7.1

$$\lim_{t\to\infty} \frac{1}{t}\int_{s=0}^{t} I_{\{r(\Delta;s)(\omega)\le x\}}\, ds = F_\Delta(x) \qquad a.s. \tag{2.74}$$

where $F_\Delta(\cdot)$ is a distribution function with continuous density $f_\Delta(\cdot)$ given by (2.70).

Simulations indicate that Conjecture 2.7.1 is correct.

Remark 2.7.4

Some empirical studies have tested for asset returns being distributed as a mixture of normal distributions, and have not rejected this hypothesis. It is possible to view Proposition 2.7.2 and Conjecture 2.7.1 in this light, since the limiting empirical distribution function of returns is a mixture of infinitely many normal distributions. In the paper of Boothe and Glassman [9], for example, Student distributions, Stable Paretian distributions, and mixtures of two normal distributions were fitted to exchange rate returns. The authors found that the mixture of normals and Student distributions gave the best fit, with evidence the distribution parameters were time varying.

Remark 2.7.5

We remark that the Δ-returns process will have heavier tails than a normal distribution unless $q(\cdot)$ is constant, which is equivalent to

$$\frac{d}{dt}\left(a(t+\Delta)+a(t)\right)=a(t+\Delta)^2-a(t)^2. \tag{2.75}$$

Notice that $a(t)\equiv A$ is a solution of (2.75) and that if $a(\cdot)$ is determined on $[0,\Delta]$, the solution of (2.75) is uniquely determined on $[0,\infty)$. Particularly note that the T-returns process is normal iff $a(t)\equiv A$; otherwise the T-returns process has heavier tails than a normal distribution in the way specified in Proposition 2.7.1. Thus a diversity of opinion among the agents leads to heavier-than-normal tails in the returns process. Crucially, we note that it is unnecessary for chartists to be present for this effect to be manifest, merely heterogeneity.

The negative autocorrelation of asset returns has been noted in several empirical surveys e.g., Fama and French [24], Porteba and Summers [62] and Guillaume et al. [38]. Using Conjecture 2.7.1, we show that the T-returns are negatively autocorrelated. Given that this happens in the presence of heterogeneous agents and on a time-scale (T) in which speculators have as a whole mean-reverting price expectations, it may be seen as a theoretical reinforcement of the mechanism proposed as Fact 5 in [38], and also in [62]. Let us fix a continuous time definition

Definition 2.7.2 *For all $t\geq 0$, $\omega\in\Omega$, the autocorrelation of the Δ-return process $(r(\Delta;t))_{t\geq 0}$ is* *given by (suppressing ω dependence):*

$$\rho(\Delta;t) = \tag{2.76}$$
$$\frac{\frac{1}{t}\int_0^t r(\Delta;s)r(\Delta;s+\Delta)\,ds - \left(\frac{1}{t}\int_0^t r(\Delta;u)\,du\right)\left(\frac{1}{t}\int_0^t r(\Delta;u+\Delta)\,du\right)}{\sqrt{\left(\frac{1}{t}\int_0^t\left(r(\Delta;s)-\frac{1}{t}\int_0^t r(\Delta;u)\,du\right)^2 ds\right)\left(\frac{1}{t}\int_0^t\left(r(\Delta;s+\Delta)-\frac{1}{t}\int_0^t r(\Delta;u+\Delta)\,du\right)^2 ds\right)}}.$$

We first show that the sample correlation of the Δ-returns converges almost surely.

Proposition 2.7.3 *Let*

$$\varrho(\Delta;t) = \left(v(t+\Delta) - v(t)e^{-\int_t^{t+\Delta} a(s)\,ds}\right)\left(e^{-\int_{t+\Delta}^{t+2\Delta} a(s)\,ds} - 1\right). \tag{2.77}$$

Then

$$\lim_{t\to\infty}\rho(\Delta;t) = \frac{\int_0^T \varrho(\Delta;s)\,ds}{\int_0^T q_\Delta(s)\,ds} \quad a.s. \tag{2.78}$$

where $q_\Delta(\cdot)$ is given by (2.71).

Proof: Conjecture 2.7.1 allows us to calculate limits of the form

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t X_{s+\Delta_1}X_{s+\Delta_2}\,ds.$$

Thus

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\int_0^t r(\Delta;s)r(\Delta;s+\Delta)\,ds &= \sigma^2\frac{1}{T}\int_0^T \varrho(\Delta;s)\,ds,\\
\lim_{t\to\infty}\frac{1}{t}\int_0^t r(\Delta_1;s)\,ds &= 0,\\
\lim_{t\to\infty}\frac{1}{t}\int_0^t r(\Delta;s)^2\,ds &= \sigma^2\frac{1}{T}\int_0^T q_\Delta(s)\,ds
\end{aligned}$$

almost surely. These results together with (2.76) give (2.78). ⋄

Corollary 2.7.1 *If $\rho(\cdot,\cdot)$ is defined by (2.76), then*

$$\lim_{t\to\infty}\rho(T;t) < 0,$$

while if

$$\int_0^T a(t)^2 v(t)\,dt > AT \tag{2.79}$$

there exists $0 < \Delta < T$ such that

$$\lim_{t\to\infty}\rho(\Delta;t) > 0. \tag{2.80}$$

Proof: From Proposition 2.7.3, we see that the sign of the correlation is the same as the sign of the numerator on the r.h.s. of (2.78). However,

$$\begin{aligned}
\varrho(T;t) &= \left(v(t+T) - v(t)e^{-\int_t^{t+T} a(s)\,ds}\right)\left(e^{-\int_{t+T}^{t+2T} a(s)\,ds} - 1\right) \\
&= -v(t)(1-e^{-AT})^2 < 0,
\end{aligned}$$

so the numerator is negative. To show the autocorrelation of the Δ-returns is positive for some $\Delta \in (0,T)$, we notice that the left hand side of (2.78) has the following expansion for small $\Delta > 0$:

$$\Delta \frac{1}{2T}\int_0^T -2a(t)(1-a(t)v(t))\,dt + O(\Delta^2),$$

so if (2.79) is true, then so is (2.80). $\diamond$

Remark 2.7.6

We see that (2.79) will be true whenever the disturbance in $a(\cdot)$ is sufficiently great, as this will increase the level of $v(\cdot)$. Thus whenever the agents are sufficiently heterogeneous, the returns will have positive autocorrelation at short horizons, when fads may occur, and have negative autocorrelation at horizons over which fundamentals dominate.

Remark 2.7.7

The sample autocorrelation of the squared Δ-returns converge a.s. since

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t}\int_0^t r(\Delta;s)^2\,ds &= \frac{1}{T}\int_0^T q_\Delta(s)\,ds \quad a.s. \\
\lim_{t\to\infty} \frac{1}{t}\int_0^t r(\Delta;s)^4\,ds &= 3\frac{1}{T}\int_0^T q_\Delta(s)^2\,ds \quad a.s. \\
\lim_{t\to\infty} \frac{1}{t}\int_0^t r(\Delta;s)^2 r(\Delta;s+\Delta)^2\,ds &= \frac{1}{T}\int_0^T q_\Delta(s)q_\Delta(s+\Delta) + 2\varrho(\Delta;s)^2\,ds \quad a.s.
\end{aligned}$$

Taking $\Delta = T$ in the above, the numerator of the autocorrelation of the squared T-returns converges almost surely to

$$4(1-e^{-AT})^2\left(\frac{1}{T}\int_0^T v(s)^2\,ds - \left(\frac{1}{T}\int_0^T v(s)\,ds\right)^2\right) + 2(1-e^{-AT})^4\frac{1}{T}\int_0^T v(s)^2\,ds.$$

Therefore the autocorrelation of the squared T-returns is strictly positive, almost surely. The positive autocorrelation of empirical squared asset returns has been remarked upon by Frey [30], for example.

Remark 2.7.8

If X^b is the solution to (2.39),

$$r^b(\Delta;t) = X^b_{t+\Delta} - X^b_t,$$

is the Δ-return of the price process $S^b = exp(X^b)$, and $\rho^b(\Delta;\cdot)$ is the autocorrelation of the Δ-return process $(r^b(\Delta;t))_{t\geq 0}$ we then have

Proposition 2.7.4

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{t}\int_{s=0}^{t} I_{\{r^b(\Delta;s)(\omega)\leq x\}}\, ds\right] = F_\Delta(x)$$

where $F_\Delta(\cdot)$ is a distribution function with density $f_\Delta(\cdot)$ given by (2.70).

Furthermore, the corresponding analogues to Conjecture 2.7.1, Proposition 2.7.3 and Corollary 2.7.1 are true. The statements of these results differ only in that $r^b(\Delta;\cdot)$ replaces $r(\Delta;\cdot)$, and $\rho^b(\Delta;\cdot)$ replaces $\rho(\Delta;\cdot)$.

2.8 Volume, Volatility and Heterogeneity

In this short section, we will give a very rough heuristic result relating the level of trade to the volatility of the price and degree of heterogeneity in the market.

Let us discard the limiting argument we used to derive the continuous price process. Suppose that the log–price responds to excess demand in a Walrasian linear fashion so that

$$X_{t+\Delta t} - X_t = \alpha D(t, t+\Delta t)$$

where $D(t,t+\Delta t)$ is excess demand over $(t, t+\Delta t)$ and $\alpha > 0$. Comparing this to (2.17), we see that

$$D(t,t+\Delta t) = \frac{1}{\alpha}\int_t^{t+\Delta t} -a(s)X_s\, ds + \frac{1}{\alpha}\int_t^{t+\Delta t} \sigma\, dB_s.$$

Thus the instantaneous planned excess demand at time t, V_t is given by

$$V_t(\omega) = \frac{1}{\alpha}|a(t)X_t(\omega)|.$$

Cabrales and Hoshi [10] take instantaneous excess demand as representing volume, from which they prove a volume-volatility relationship, and we proceed similarly. Our proxy for volume will be the asymptotic average of squared instantaneous planned excess demand. Let this be $\overline{V}(a(\cdot))$, so

$$\overline{V}(a(\cdot))(\omega) = \lim_{t\to\infty} \frac{1}{t}\int_0^t V_s^2(\omega)\,ds.$$

Let our proxy for heterogeneity be the variation in $a(\cdot)$, $\mathcal{V}(a(\cdot))$.

We let the proxy for volatility be the asymptotic variance for log-price, $V(a(\cdot))$;

$$V(a(\cdot))(\omega) = \lim_{t\to\infty} \frac{1}{t}\int_0^t \left(X_s(\omega) - \frac{1}{t}\int_0^t X_u(\omega)\,du\right)^2 ds.$$

Then we have

Proposition 2.8.1 *Let $\mathcal{V}(a(\cdot))$, $V(a(\cdot))$ and $\overline{V}(a(\cdot))$ be defined as above.*

(i) $V(a(\cdot))$ and $\overline{V}(a(\cdot))$ are almost surely constant.

(ii) $\mathcal{V}(a(\cdot))$, $V(a(\cdot))$ and $\overline{V}(a(\cdot))$ are all minimised for $a(\cdot) \equiv A$.

Proof: We know from (2.5.1) that $V(a(\cdot))$ is almost surely constant and is indeed minimised for $a(\cdot) \equiv A$, as is $\mathcal{V}(a(\cdot))$. Next observe that

$$\int_0^T a(s)^2 v(s)\,ds$$

is minimised for $a(\cdot) \equiv A$. To do this notice that by the Cauchy-Schwarz inequality we have

$$(AT)^2 = \left|\int_0^T a(s)\,ds\right|^2 \leq \left(\int_0^T |a(s)|\,ds\right)^2 \leq \int_0^T \frac{1}{v(s)}\,ds \int_0^T a(s)^2 v(s)\,ds,$$

so

$$\int_0^T a(s)^2 v(s)\,ds \geq \frac{1}{2}AT,$$

with equality if and only if $|a(t)|v(t)$ is constant, and $a(t) \geq 0$. These conditions lead to $v(\cdot)$ being constant, and so $a(\cdot)$ must be constant. However by Theorem 2.6.1, $\overline{V}(a(\cdot))$ converges almost surely to its asymptotic expected value:

$$\overline{V}(a(\cdot)) = \frac{1}{\alpha^2}\cdot\sigma^2\frac{1}{T}\int_0^T a(s)^2 v(s)\,ds \quad \text{a.s.}$$

This ensures that the Proposition is true. $\diamond$

Remark 2.8.1

The proposition hints in a very crude fashion that there is a positive relationship market heterogeneity, activity and volatility, in the sense that all three are simultaneously minimised. Moreover, the chain of causation seems to run from heterogeneity to volatility and volume, consistent with the observation of Froot and Frankel [29] cited in the Chapter 1. We will prove later in the thesis that increases in the volatility coefficient in (1.1) increases the volume of trade.

2.9 Appendix

Proof of Lemma 2.6.1 : Writing $M_{T^*} = u^{-1}(T^*)J_k(T^*)$ we note that M_{T^*} is a martingale with $\lim_{T^*\to\infty}\langle M\rangle_{T^*} = \infty$ so that if we write

$$R_k(T^*) = \frac{1}{u^{-1}(T^*)}\int_{w_0}^{T^*} B_w^{2(p-(2k+1))} G_{k+1}(w)^2\, dw$$

then

$$\limsup_{T^*\to\infty}\left|\frac{u^{-1}(T^*)J_k(T^*)}{\sqrt{2u^{-1}(T^*)}\sqrt{R_k(T^*)}\sqrt{\log\log(u^{-1}(T^*)R_k(T^*))}}\right| = 1.$$

Using (2.49) we show that

$$\limsup_{T^*\to\infty} R_k(T^*) \leq 2Av_{max}^{p+2}C_{p,k}$$

where

$$C_{p,k} = \frac{1}{\left(\prod_{j=0}^{k}\frac{p}{2}-j\right)^2}\frac{(2(p-(2k+1)))!}{(p-(2k+1))!2^{p-(2k+1)}}.$$

Thus

$$\limsup_{T^*\to\infty}\frac{\sqrt{\log T^*}J_k(T^*)}{\sqrt{\log\log\log T^*}} \leq 2\sqrt{2}Av_{max}^{1+\frac{p}{2}}\sqrt{C_{p,k}}. \tag{2.81}$$

As similar arguments are used below, we omit the proof that there exists a non–negative summable sequence $C_p^2(\omega) < \infty$ such that for all $|\alpha| < \infty$.

$$\limsup_{n\to\infty}\frac{1}{p!}|n(f_n(p) - I_0(p,u(n)))| \leq C_p^2(\omega), \tag{2.82}$$

where $\omega \in \Omega_1$ with $\mathbb{P}[\Omega_1] = 1$. Define for p odd $\tilde{I}_{\frac{p-1}{2}}(p) = 0$, for p even

$$\tilde{I}_{\frac{p}{2}}(p) = (-1)^{\frac{p}{2}}\frac{1}{(\frac{p}{2})!}\frac{1}{T}\int_0^T v(s)^{\frac{p}{2}}\, ds$$

and $\tilde{I}_k(p) + \frac{1}{2}(p-2k)(p-(2k+1))\tilde{I}_{k+1}(p) = 0$. Notice $f(p) = \tilde{I}_0(p)$. Let

$$D_k(p) = \limsup_{T^*\to\infty} \frac{1}{2\sqrt{2}Av_{max}^{1+\frac{p}{2}}} \frac{\sqrt{\log T^*}}{\sqrt{\log\log\log T^*}} |I_k(p,T^*) - \tilde{I}_k(p)|.$$

Then for all $|\alpha| < \infty$, we can use (2.82) to show

$$\limsup_{n\to\infty} \frac{1}{p!} \frac{\sqrt{n}}{\sqrt{\log\log n}} |f_n(p) - f(p)||\alpha|^p \leq \frac{1}{p!} D_0(p)|\alpha|^p 2\sqrt{A} v_{max}^{1+\frac{p}{2}}. \tag{2.83}$$

To establish the lemma, we need only show the r.h.s. of (2.83) is summable over p. Using (2.58) we can achieve the following iterative inequality:

$$D_k(p) \leq (p-2k)\sqrt{C_{p,k}} + \frac{1}{2}(p-2k)(p-(2k+1))D_{k+1}(p)$$

with boundary condition for p odd

$$D_{\frac{p-1}{2}}(p) \leq \frac{1}{\prod_{j=0}^{\frac{p-1}{2}} \frac{p}{2} - j}.$$

and as we prove below for p even, $D_{\frac{p}{2}} = 0$. An induction argument yields for p odd

$$\begin{aligned}\frac{1}{p!}D_0(p) \quad &\leq \quad \frac{1}{2^{\frac{p-1}{2}}} D_{\frac{p-1}{2}}(p) + \sum_{l=1}^{\frac{p-3}{2}} \frac{1}{2^{\frac{p-3}{2}+l-1}} \frac{1}{(2l)!} \frac{1}{\prod_{r=l+\frac{1}{2}}^{\frac{p}{2}} r} \sqrt{\frac{(4l)!}{(2l)!2^{2l}}} \\ &+ \quad \frac{2}{p!}\sqrt{\frac{(2(p-1))!}{(p-1)!2^{p-1}}} \end{aligned} \tag{2.84}$$

with a similar inequality for p even. Let $a_p = 2\sqrt{A}v_{max}^{1+\frac{p}{2}}|\alpha|^p$ times the summation on the r.h.s of (2.84). One can then use Stirling's formula together with the ratio test to show that

$$\sum_{p\in odd} a_p < \infty.$$

The same argument works when p is even. Let $b_p = 2\sqrt{A}v_{max}^{1+\frac{p}{2}}|\alpha|^p$ times the last term on the r.h.s of (2.84). Again,

$$\sum_{p\in odd} b_p < \infty.$$

We proceed analogously for p even. This proves the summability of the right hand side of (2.83). ◇

To prove that $D_{\frac{p}{2}}(p) = 0$ for p even, we partition integrals to obtain the following bound

$$\left| I^{\frac{p}{2}}(T^*) - \frac{1}{(\frac{p}{2})!}\frac{1}{T}\int_0^T v(s)^{\frac{p}{2}}\,ds \right| \leq \frac{1}{(\frac{p}{2})!} v(0) v_{max}^{\frac{p}{2}} \frac{1}{u^{-1}(T^*)} \int_{u^{-1}(w_0)}^{u^{-1}(T^*)} \frac{1}{u(s)}\,ds + \\ \frac{1}{(\frac{p}{2})!} \left| \frac{1}{u^{-1}(T^*)} \int_{u^{-1}(w_0)}^{u^{-1}(T^*)} v(t_0(s))^{\frac{p}{2}}\,ds - \frac{1}{T}\int_0^T v(t_0(s))^{\frac{p}{2}}\,ds \right|$$

and in a similar vein we have

$$\left| \frac{1}{u^{-1}(T^*)} \int_{u^{-1}(w_0)}^{u^{-1}(T^*)} v(t_0(s))^{\frac{p}{2}}\, ds - \frac{1}{T} \int_0^T v(t_0(s))^{\frac{p}{2}}\, ds \right|$$

$$\leq \frac{1}{u^{-1}(T^*)} v_{max}^{\frac{p}{2}} \left(2T + u^{-1}(w_0)\right).$$

Integrating (2.53) by parts successively we obtain the bound

$$|I_{\frac{p}{2}}(p, T^*) - \tilde{I}_{\frac{p}{2}}(p)| \leq \left| I^{\frac{p}{2}}(T^*) - \frac{1}{(\frac{p}{2})!} \frac{1}{T} \int_0^T v(s)^{\frac{p}{2}}\, ds \right| +$$

$$\frac{1}{u^{-1}(T^*)} \sum_{j=0}^{\frac{p}{2}-1} \left(\frac{T^{*j+1}}{(j+1)!} |G_{\frac{p}{2}-j}(T^*)| + \frac{w_0{}^{j+1}}{(j+1)!} |G_{\frac{p}{2}-j}(w_0)| \right).$$

Applying (2.49) and recalling the definition of D_k, we see from the above that $D_{\frac{p}{2}}(p) = 0$.

Chapter 3

The Existence, Uniqueness, Regularity and Representation of Solutions of Linear Stochastic Integro-differential Equations

3.1 Introduction

In this chapter we determine some of the fundamental properties of a linear stochastic integro-differential equation of the form

$$dX_t = \left(\lambda(t)\left(X_t - \int_0^t w(s,t)X_s\,ds\right) - \beta(t)\,(X_t - k(t))\right)dt + \sigma(t)\,dB_t, \tag{3.1}$$

where we assume $\lambda(\cdot), \beta(\cdot), \sigma(\cdot), k(\cdot)$ are continuous. Additionally, we impose some continuity and integrability conditions on $w(\cdot,\cdot)$; most importantly, we require that w has an integrable singularity at $(0,0)$, in the sense that for $t > 0$,

$$\int_0^t w(s,t)\,ds = 1.$$

By using Picard iteration, and mimicking the proof used to show uniqueness and existence for stochastic differential equations, we show that there exists a unique, continuous solution to (3.1). Some additional analysis is merited to handle the complications arising from the necessity of adjoining memory, and also dispatching the difficulties arising from the singular behaviour of the kernel $w(\cdot,\cdot)$.

Furthermore, the continuous solution has a stochastic variation of parameters representation, which can be expressed in terms of the resolvent of the deterministic integro-differential equation given by

$$x'(t) = \lambda(t)\left(x(t) - \int_0^t w(s,t)x(s)\,ds\right) - \beta(t)x(t).$$

Using the Kolmogorov-Čentsov theorem, one can show that the continuous solution has a modification which is locally Hölder continuous of order $\gamma \in (0, \frac{1}{2})$, but is not locally Hölder continuous of any order $\frac{1}{2}$ or greater.

We will later use this equation to model the evolution of log–prices of an asset in which some agents have memory and others believe prices revert to levels determined by economic fundamentals.

3.2 Preliminaries

For the present, we assume

$$k \in \mathrm{C}[0,\infty), \tag{3.2}$$

and impose the following properties on $w(\cdot,\cdot)$: let

$$D := \{(s,t) \,:\, 0 < s \le t\}$$

Then

$$w \in \mathrm{C}(D,(0,\infty)). \tag{3.3}$$

For all $t > 0$

$$\int_0^t w(s,t)\,ds = 1. \tag{3.4}$$

Let $\lambda, \beta, \sigma \in \mathbf{C}(\mathbb{R}^+ \cup \{0\})$. Define

$$\begin{aligned} M_1 &= \max_{t\geq 0} |\lambda(t)|, && (3.5)\\ M_2 &= \max_{t\geq 0} |\beta(t)|, && (3.6)\\ M_3 &= \max_{t\geq 0} |\sigma(t)|. && (3.7) \end{aligned}$$

We further assume there exists $\underline{\sigma} > 0$ such that

$$\inf_{t\geq 0} |\sigma(t)| = \underline{\sigma}. \tag{3.8}$$

3.3 Representation of the Solution

We assume that we have an underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Let $(X_t)_{t\geq 0}$ model the log-price of the asset at time t, and be a continuous time stochastic process adapted to the filtration. We assume that it follows a stochastic integro-differential equation (specified below) driven by a standard Brownian motion $(B_t)_{t\geq 0}$. We will need to manipulate stochastic integrals with t-dependent integrands. To do this, recall the semi-martingale decomposition of Berger and Mizel.

Lemma 3.3.1 *Let h be a deterministic function which satisfies*

$$h : D \to \mathbb{R} : (s,t) \mapsto h(s,t),$$

Suppose that $\frac{\partial h}{\partial t}$ exists and is uniformly bounded and $h(\cdot,\cdot)$ is uniformly bounded on all D. Then for every $0 \leq t < \infty$

$$\int_0^t h(s,t)\, dB_s = \int_0^t h(s,s)\, dB_s + \int_0^t \left(\int_0^s \frac{\partial h}{\partial t}(u,s)\, dB_u \right) ds. \tag{3.9}$$

Proof: Use the stochastic Fubini theorem. See Protter [64] and Protter (1985) [63]. ◇

We will later use this to prove that the S.I-D.E (3.1) has a unique variation of parameters solution.

Define $g(s,\cdot)$ to be the (unique) solution of the following system:

$$\begin{aligned} \frac{\partial g}{\partial t}(s,t) &= \lambda(t)\left(g(s,t) - \int_s^t w(u,t) g(s,u)\, du \right) - \beta(t) g(s,t) \qquad t \geq s && (3.10)\\ g(s,s) &= 1, \qquad g(s,t) = 0 \qquad t < s && (3.11) \end{aligned}$$

The fact that there is a unique solution to (3.10), (3.11) can be readily inferred, for example, from Theorem 1.7.1 on p.24-26 of Lakshmikantham and Rao [49]. By this theorem, there is also a unique solution $(e(\cdot))$ to the equation

$$\begin{aligned} e'(t) &= \lambda(t)\left(e(t) - \int_0^t w(u,t)e(u)\,du\right) - \beta(t)\left(e(t) - k(t)\right), && (3.12)\\ e(0) &= X_0. && (3.13)\end{aligned}$$

It is not difficult to prove, using a deterministic variant of a proof we later use, that

$$|g(s,t)| \leq e^{(2M_1+M_2)(t-s)} \tag{3.14}$$

and using (3.10) we can prove

$$\left|\frac{\partial g}{\partial t}(s,t)\right| \leq (2M_1 + M_2)e^{(2M_1+M_2)(t-s)}. \tag{3.15}$$

Therefore, if $T < \infty$ and $0 \leq s \leq t \leq T$, then

$$|g(s,t)| \leq e^{(2M_1+M_2)T} \tag{3.16}$$

and

$$\left|\frac{\partial g}{\partial t}(s,t)\right| \leq (2M_1 + M_2)e^{(2M_1+M_2)T}. \tag{3.17}$$

Therefore, both g and $\frac{\partial g}{\partial t}$ are uniformly bounded on compacts. Then we will prove

Theorem 3.3.1 *Under the above hypotheses, (3.1) has a unique, continuous, variation of parameters solution on every interval* $[0,T]$ *given by*

$$X_t = e(t) + \int_0^t \sigma(s)g(s,t)\,dB_s, \tag{3.18}$$

where $g(\cdot,\cdot)$ *is given by (3.10) and (3.11), and* $e(\cdot)$ *is given by (3.12) and (3.13). Moreover,*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} X_t^2\right] < \infty. \tag{3.19}$$

Remark 3.3.1

The existence and uniqueness of solutions to linear stochastic integro-differential equations is covered in Vespri [77]. However, a separate proof of existence and uniqueness is necessary in this case owing to the singular behaviour of $w(\cdot,\cdot)$ as $t \downarrow 0$ (cf. equation (3.4)).

Proof of Theorem 3.3.1: We will temporarily assume existence and uniqueness have been demonstrated, together with (3.19). These facts will be established in Proposition 3.4.1. To prove the representation in (3.18), define

$$Y_t = \int_0^t \sigma(s)g(s,t)\, dB_s, \tag{3.20}$$

and notice by Lemma 3.3.1 and the definition and properties of g that

$$Y_t = \int_0^t \sigma(s)\, dB_s + \int_{s=0}^t \int_{u=0}^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\, dB_u\, ds.$$

Plainly, the process Y is a.s. continuous on $(0,T]$ for any $T>0$. Using the definition of g and the stochastic Fubini theorem we obtain for all $s>0$

$$\int_{u=0}^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\, dB_u = \lambda(s)\left(Y_s - \int_{v=0}^s w(v,s)Y_v\, dv\right) - \beta(s)Y_s. \tag{3.21}$$

(For $s>0$ and any v such that $0\le v\le s$, $w(v,s)$ is finite, so

$$\int_{u=0}^s \int_{v=u}^s \sigma(u)w(v,s)g(u,v)\, dv\, dB_u = \int_{v=0}^s \int_{u=0}^v \sigma(u)w(v,s)g(u,v)\, dB_u\, dv,$$

since the boundedness of the integrand and the stochastic Fubini Theorem allow us to reverse the order of integration. However, since $\lim_{s\downarrow 0}\inf_{0\le v\le s} w(v,s) = +\infty$, we cannot conclude (3.21) for $s=0$.) We can prove, using an argument similar to that of Proposition 6.3.1 that

$$\lim_{s\downarrow 0} Y_s = 0 \quad a.s.$$

so

$$\lim_{s\downarrow 0}\int_{u=0}^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\, dB_u = \lim_{s\downarrow 0}\lambda(s)\left(Y_s - \int_{v=0}^s w(v,s)Y_v\, dv\right) - \beta(s)Y_s = 0 \quad a.s.$$

Therefore, Y_t satisfies

$$Y_t = \int_0^t \sigma(s)\, dB_s + \int_0^t \lambda(s)\left(Y_s - \int_0^s w(v,s)Y_v\, dv\right) - \beta(s)Y_s\, ds.$$

From the definition of e we must have

$$X_t = x(0) + \int_0^t \sigma(s)\, dB_s + \int_0^t \lambda(s)\left(X_s - \int_0^s w(v,s)X_v\, dv\right) - \beta(s)(X_s - k(s))\, ds, \tag{3.22}$$

proving the assertion. ⋄

3.4 Proof of Proposition 3.4.1

The proof of existence and uniqueness is relatively standard, and follows via Picard–Lindelöf iteration. However, since we must also include the influence of the past in our proof, the argument is more cumbersome than is the case in the comparable proof for stochastic differential equations. We proceed using a troika of lemmata.

Lemma 3.4.1 *Let $T < \infty$ and suppose*

$$\mathcal{E} = \{(X_t)_{0 \leq t \leq T} : X_0 \textit{ known, } X \textit{ is cns., } \mathcal{F}_t \textit{ measurable and } \mathbb{E}\left[\sup_{0\leq t\leq T} X_t^2\right] < \infty.\}.$$

Then (3.1) has a unique solution in $\mathcal{E}$.

Proof: Let $\lceil x \rceil$ denote the smallest integer greater than or equal to x. Define

$$N = \lceil T\sqrt{2M_2^2 + 8M_1^2} \rceil + 1.$$

Thus $N \in \mathbb{N}$ and $N \geq 2$. Now define $T_1 = T/N$. Thus

$$\alpha \triangleq T_1\sqrt{2M_2^2 + 8M_1^2} \leq 1 - \frac{1}{N} < 1. \tag{3.23}$$

This constant α will be used in Lemma 3.4.2 to show that a certain functional is a contraction mapping. Now we prove existence and uniqueness in a succession of Banach spaces. Let

$$\mathcal{E}^i = \{(X_t)_{0\leq t\leq (i+1)T_1} : X_0 \text{ known, } X \text{ is cns., } \mathcal{F}_t \text{ measurable and } \mathbb{E}\left[\sup_{0\leq t\leq (i+1)T_1} X_t^2\right] < \infty.\},$$

and

$$\begin{aligned} \mathcal{C}^i = \{(X_t)_{0\leq t\leq (i+1)T_1} &: X_0 \text{ known, } X \text{ is cns., } \mathcal{F}_t \text{ measurable,} \\ &\mathbb{E}\left[\sup_{0\leq t\leq (i+1)T_1} X_t^2\right] < \infty, \text{ and } X_t = X_t^{i-1} \text{ for all } t \in [0, iT_1]\}, \end{aligned}$$

where X^i is the as yet to be proved solution on $\mathcal{C}^i$. Notice that $\mathcal{C}^i \subseteq \mathcal{E}^i$. We can proceed exactly as in the case of stochastic differential equations to prove that there is a unique solution on $\mathcal{E}^0$. Call this solution X^0. Make the induction hypothesis

$$H_i : \text{ There is a unique solution to (3.1), } X^i \in \mathcal{E}^i$$

for $i \geq 0$. Obviously, H_0 is true, and H_{N-1} is the statement of lemma. It is the subject of Lemma 3.4.2 to show that under H_{i-1} there is a unique solution to (3.1) in $\mathcal{C}^i$. We defer this proof temporarily. By reference to Lemma 3.4.2, we readily see that all solutions of (3.1) in $\mathcal{E}^i$ are in $\mathcal{C}^i$. Now let the unique solution to (3.1) in $\mathcal{C}^i$ be Y^i. Since $\mathcal{C}^i \subseteq \mathcal{E}^i$, we have $Y^i \in \mathcal{E}^i$. Thus there is a solution to (3.1) in $\mathcal{E}^i$. Let Z^i be another solution in $\mathcal{E}^i$. But then $Z^i \in \mathcal{C}^i$, so by Lemma 3.4.2, $Y^i = Z^i$. Therefore Y^i is the unique solution in $\mathcal{E}^i$. Call it X^i. So given H_{i-1}, we have established H_i, and the lemma is proven. ◇

Lemma 3.4.2 *Under H_{i-1} there is a unique solution to (3.1) in $\mathcal{C}^i$.*

Proof: Define a mapping Φ with domain in $\mathcal{C}^i$. Let

$$\Phi(Z)_t = Z_0 + \int_0^t \lambda(s)\left(Z_s - \int_0^s w(u,s)Z_u\,du\right) - \beta(s)(Z_s - k(s))\,ds + \int_0^t \sigma(s)\,dB_s$$

for $t > 0$ and $\Phi(Z)_t = Z_0$ for $t = 0$. We will want to show

$$\Phi : \mathcal{C}^i \to \mathcal{C}^i,$$

and Φ is a contraction mapping on the Banach space $(\mathcal{C}^i, \|\cdot\|_i)$ where

$$\|X\|_i = \mathbb{E}\left[\sup_{0\le t\le (i+1)T_1} X_t^2\right]^{\frac{1}{2}}.$$

Obviously, if Z is continuous, $\mathcal{F}_t$ measurable and Z_0 known, the same can be said for $\Phi(Z)$. If $X \in \mathcal{C}^i$, then $X_t = X_t^{i-1}$ for all $t \in [0, iT_1]$, so $\Phi(X)_t = \Phi(X^{i-1})_t = X_t^{i-1}$ for all $t \in [0, iT_1]$. If we have $X \in \mathcal{C}^i$ then we will require

$$\mathbb{E}\left[\sup_{0\le t\le (i+1)T_1} \Phi(X)_t^2\right] < \infty. \tag{3.24}$$

Thus if Φ is a contraction on $\mathcal{C}^i$, it follows that there exists a unique $X^i \in \mathcal{C}^i$ such that

$$\Phi(X^i)_t = X_t^i \quad \forall t \in [0, (i+1)T_1].$$

Therefore, X^i solves (3.1) on $[0, (i+1)T_1]$ uniquely in $\mathcal{C}^i$. Take $X, Y \in \mathcal{C}^i$. First, for all $t \in [0, iT_1]$

$$\Phi(X)_t - \Phi(Y)_t = 0 = X_t - Y_t. \tag{3.25}$$

Let $t \in [iT_1, (i+1)T_1]$. Then

$$\Phi(X)_t - \Phi(Y)_t = \int_{iT_1}^t \lambda(s)\left(X_s - Y_s - \int_{iT_1}^s w(u,s)\,(X_u - Y_u)\,du\right)ds + \int_{iT_1}^t -\beta(s)\,(X_s - Y_s)\,ds. \tag{3.26}$$

Standard bounding arguments give for $U_s = X_s - Y_s$

$$\sup_{iT_1\le t\le (i+1)T_1}\left|\int_{iT_1}^t -\beta(s)U_s\,ds\right|^2 \le M_2^2T_1^2 \sup_{iT_1\le t\le (i+1)T_1} |U_t|^2 \tag{3.27}$$

$$\sup_{iT_1\le t\le (i+1)T_1}\left|\int_{iT_1}^t \lambda(s)\left(U_s - \int_{iT_1}^s w(u,s)U_u\,du\right)ds\right|^2 \le 4M_1^2T_1^2 \sup_{iT_1\le t\le (i+1)T_1} |U_t|^2, \tag{3.28}$$

where we use

$$\left|\int_{iT_1}^s w(u,s)U_u\,du\right| \le \sup_{iT_1\le u\le s} |U_u|.$$

Taking the triangle inequality across (3.26), then using $(a+b)^2 \leq 2(a^2+b^2)$ gives, on taking supremums across the resulting inequality

$$\sup_{iT_1 \leq t \leq (i+1)T_1} |\Phi(X)_t - \Phi(Y)_t|^2 \leq \alpha^2 \sup_{iT_1 \leq t \leq (i+1)T_1} |X_t - Y_t|^2$$

where we use (3.27), (3.28). By virtue of (3.25), we have

$$||\Phi(X) - \Phi(Y)|| \leq \alpha ||X - Y||,$$

so by (3.23), Φ is a contraction. We will now show (3.24). By the definition of Φ, by using $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$ we have

$$\begin{aligned} \Phi(X)_t^2 &\leq 4\left(x(0)^2 + \left(\int_0^t \lambda(s)\left(X_s - \int_0^s w(u,s)X_u\,du\right)ds\right)^2\right) \\ &\quad + 4\left(\left(\int_0^t -\beta(s)(X_s - k(s))\,ds\right)^2 + \left(\int_0^t \sigma(s)\,dB_s\right)^2\right), \end{aligned}$$

and so

$$\begin{aligned} \sup_{0\leq t\leq (i+1)T_1} \Phi(X)_t^2 &\leq 4\left(x(0)^2 + \sup_{0\leq t\leq (i+1)T_1}\left(\int_0^t \lambda(s)\left(X_s - \int_0^s w(u,s)X_u\,du\right)ds\right)^2\right) \quad (3.29) \\ &+ 4\left(\sup_{0\leq t\leq (i+1)T_1}\left(\int_0^t \beta(s)(X_s - k(s))\,ds\right)^2 + \sup_{0\leq t\leq (i+1)T_1}\left(\int_0^t \sigma(s)\,dB_s\right)^2\right). \end{aligned}$$

To bound the third term in (3.29), we use Cauchy-Schwarz and $(a+b)^2 \leq 2(a^2+b^2)$ to obtain

$$\begin{aligned} \sup_{0\leq t\leq (i+1)T_1}\left|\int_0^t \beta(s)(X_s - k(s))\,ds\right|^2 &\leq 2M_2^2 \sup_{0\leq t\leq (i+1)T_1} t\int_0^t X_s^2 + k(s)^2\,ds \\ &\leq 2M_2^2\left((i+1)T_1\right)^2\left(\sup_{0\leq t\leq (i+1)T_1} X_t^2 + \sup_{0\leq t\leq (i+1)T_1} k(t)^2\right), \end{aligned}$$

so the expected value of the third term is bounded. It is trivial to use Doob's maximal inequality to bound the expected value of the fourth term in (3.29). To wit:

$$\mathbb{E}\left[\sup_{0\leq t\leq (i+1)T_1}\left(\int_0^t \sigma(s)\,dB_s\right)^2\right] \leq 4\mathbb{E}\left[\left(\int_0^{(i+1)T_1} \sigma(s)\,dB_s\right)^2\right] \leq 4(i+1)T_1 \sup_{0\leq t\leq (i+1)T_1} \sigma(t)^2. \tag{3.30}$$

We bound the second term in the same manner as the third, viz,

$$\begin{aligned} \sup_{0\leq t\leq (i+1)T_1}\left|\int_0^t \lambda(s)\left(X_s - \int_0^s w(u,s)X_u\,du\right)ds\right|^2 &\leq M_1^2 \sup_{0\leq t\leq (i+1)T_1}\left(\int_0^t 2\sup_{0\leq u\leq s}|X_u|\,ds\right)^2 \\ &\leq 4M_1^2\left((i+1)T_1\right)^2 \sup_{0\leq t\leq (i+1)T_1} X_t^2, \quad (3.31) \end{aligned}$$

and this term also has finite expectation. Therefore, by (3.30) and (3.31), all the terms on the right hand side of (3.29) have finite expectation, so (3.24) is true. This is now sufficient to prove the lemma. ◇

Lemma 3.4.3 *If $(X_t)_{0\le t\le T}$ is a solution to (3.1) and is a.s. continuous and $\mathcal{F}_t$ measurable, then*

$$\mathbb{E}\left[\sup_{0\le t\le T} X_t^2\right] < \infty.$$

Proof: Let X be as in the statement of the lemma. Define

$$T_n = \inf\{s \ge 0 : |X_s| > n\}.$$

Since X is a solution of (3.1), it satisfies (3.22). Using the same argument as in (3.29), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le u\le t\wedge T_n} X_u^2\right] \qquad (3.32)\\
\le\ & 4\left(x(0)^2 + \mathbb{E}\left[\sup_{0\le u\le t\wedge T_n}\left(\int_0^u \lambda(s)\left(X_s - \int_0^s w(r,s)X_r\,dr\right)ds\right)^2\right]\right)\\
+\ & 4\left(\mathbb{E}\left[\sup_{0\le u\le t\wedge T_n}\left(\int_0^u \beta(s)(X_s - k(s))\,ds\right)^2\right] + \mathbb{E}\left[\sup_{0\le u\le t\wedge T_n}\left(\int_0^u \sigma(s)\,dB_s\right)^2\right]\right)\\
\le\ & 4\left(x(0)^2 + \mathbb{E}\left[\left(\int_0^{t\wedge T_n} \lambda(s)\left(X_s - \int_0^s w(r,s)X_r\,dr\right)ds\right)^2\right]\right)\\
+\ & 4\left(\mathbb{E}\left[\left(\int_0^{t\wedge T_n} \beta(s)(X_s - k(s))\,ds\right)^2\right] + \mathbb{E}\left[\left(\int_0^{t\wedge T_n} \sigma(s)\,dB_s\right)^2\right]\right).
\end{aligned}$$

The fourth term on the right hand side of (3.32) can be bounded using Doob's maximal inequality, so we have, since $t \in [0,T]$

$$\mathbb{E}\left[\left(\int_0^{t\wedge T_n} \sigma(s)\,dB_s\right)^2\right] \le 4\int_0^T \sigma(s)^2\,ds.$$

The third term can be bounded in the same way as (3.30) to give

$$\mathbb{E}\left[\left(\int_0^{t\wedge T_n} \beta(s)(X_s - k(s))\,ds\right)^2\right] \le M_2^2\left(2T^2\sup_{0\le s\le T} k(s)^2 + 2T\int_0^t \mathbb{E}\left[\sup_{0\le u\le s\wedge T_n}|X_u|^2\right]ds\right),$$

and the second term on the r.h.s. of (3.32) bounded in the same way as (3.31), yielding

$$\mathbb{E}\left[\left(\int_0^{t\wedge T_n} \lambda(s)\left(X_s - \int_0^s w(r,s)X_r\,dr\right)ds\right)^2\right] \le 4TM_1^2\int_0^t \mathbb{E}\left[\sup_{0\le u\le s\wedge T_n}|X_u|^2\right]ds.$$

Define

$$f^n(t) = \mathbb{E}\left[\sup_{0\le u\le t\wedge T_n} X_u^2\right].$$

Then f^n is finite, continuous and non-negative. Using (3.32), we then have

$$f^n(t) \le 4\left(x(0)^2 + 2T^2M_2^2\sup_{0\le s\le T} k(s)^2 + 4T\sup_{0\le s\le T}\sigma(s)^2\right) + \left(16M_1^2T + 8M_2^2T\right)\int_0^t f^n(s)\,ds,$$

so applying Gronwall's inequality, we have

$$f^n(t) \leq 4\left(x(0)^2 + 2T^2M_2^2 \sup_{0\leq s\leq T} k(s)^2 + 4T \sup_{0\leq s\leq T} \sigma(s)^2\right) e^{(16M_1^2T+8M_2^2T)t}.$$

Let

$$K(T) = 4\left(x(0)^2 + 2T^2M_2^2 \sup_{0\leq s\leq T} k(s)^2 + 4T \sup_{0\leq s\leq T} \sigma(s)^2\right) e^{(16M_1^2T+8M_2^2T)T}. \tag{3.33}$$

Thus $f^n(T) \leq K(T) < \infty$. Now, using Fatou's lemma, we have

$$\mathbb{E}\left[\sup_{0\leq u\leq T} |X_u|^2\right] = \mathbb{E}\left[\lim_{n\to\infty} \sup_{0\leq u\leq T\wedge T_n} |X_u|^2\right] \leq \limsup_{n\to\infty} \mathbb{E}\left[\sup_{0\leq u\leq T\wedge T_n} |X_u|^2\right] = \limsup_{n\to\infty} f^n(T) < \infty,$$

which completes the proof. ⋄

Proposition 3.4.1 *If $T < \infty$ there exists a unique continuous solution to (3.1). Moreover, it satisfies*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} X_t^2\right] < \infty.$$

Proof: Follows directly from Lemma 3.4.2 and Lemma 3.4.3. ⋄

3.5 Regularity of the Solution

In common with linear stochastic differential equations, we can show that the solution to (3.1) has a modification which is Hölder continuous of order γ for all $\gamma \in (0, \frac{1}{2})$, but not of order $\frac{1}{2}$ or greater.

Proposition 3.5.1 *If X is the continuous solution of (3.1), it has a modification which is locally Hölder continuous of order $\gamma \in (0, \frac{1}{2})$. Furthermore, X is not locally Hölder continuous of order $\gamma \geq \frac{1}{2}$. In fact we have, for each $t \in [0, T]$ and $\gamma \geq \frac{1}{2}$*

$$\liminf_{s\uparrow t} \left|\frac{X_t - X_s}{(t-s)^\gamma}\right| = \infty \quad a.s. \tag{3.34}$$

Proof: If X is the continuous solution of (3.1), then, for each s, t such that $0 \leq s \leq t \leq T$ and for each $m \in \mathbb{N}$, we have, using (3.22) and $(a + b + c)^{2m} \leq 3^{2m}(a^{2m} + b^{2m} + c^{3m})$

$$\begin{aligned}(X_t - X_s)^{2m} &\leq 3^{2m}\left(\int_s^t \lambda(u)\left(X_u - \int_0^u w(r,u)X_r\,dr\right)du\right)^{2m} \\ &+ \ 3^{2m}\left(\left(\int_s^t \beta(u)(X_u - k(u))\,du\right)^{2m} + \left(\int_s^t \sigma(u)\,dB_u\right)^{2m}\right).\end{aligned} \tag{3.35}$$

The first term on the right hand side can be bounded using the same ideas as in the proof above, together with Jensen's inequality.

$$\begin{aligned}
&\left(\int_s^t \lambda(u)\left(X_u - \int_0^u w(r,u)X_r\,dr\right)du\right)^{2m} && (3.36)\\
\leq\ & M_1^{2m}|t-s|^{2m-1}\int_s^t \left|X_u - \int_0^u w(r,u)X_r\,dr\right|^{2m} du \\
\leq\ & M_1^{2m}|t-s|^{2m-1}\int_s^t 2^{2m}\left(|X_u|^{2m} + \left|\int_0^u w(r,u)X_r\,dr\right|^{2m}\right) du \\
\leq\ & 2.2^{2m}M_1^{2m}|t-s|^{2m-1}\int_s^t \sup_{0\leq r\leq u}|X_r|^{2m}\,du.
\end{aligned}$$

The same strategy for the second term gives

$$\begin{aligned}
&\left(\int_s^t \beta(u)\left(X_u - k(u)\right)du\right)^{2m} && (3.37)\\
\leq\ & 2^{2m}M_2^{2m}|t-s|^{2m-1}\left(|t-s|\sup_{0\leq u\leq T}k(u)^{2m} + \int_s^t \sup_{0\leq r\leq u}|X_r|^{2m}\,du\right).
\end{aligned}$$

Next, notice that since

$$\int_s^t \sigma(u)\,dB_u \sim \mathcal{N}\left(0, \int_s^t \sigma(u)^2\,du\right),$$

we have

$$\mathbb{E}\left[\left(\int_s^t \sigma(u)\,dB_u\right)^{2m}\right] = \frac{(2m)!}{m!2^m}\left(\int_s^t \sigma(u)^2\,du\right)^m \leq \frac{(2m)!}{m!2^m}M_3^{2m}|t-s|^m. \tag{3.38}$$

An argument analagous to that of Lemma 3.4.3 shows that for each $m \in \mathbb{N}$ and $T \in (0,\infty)$ there exists $K_m(T) < \infty$ such that for all $u \in [0,T]$

$$\mathbb{E}\left[\sup_{0\leq r\leq u}|X_r|^{2m}\right] \leq K_m(T). \tag{3.39}$$

Taking expectations across (3.35), and using (3.36), (3.37), (3.38) and (3.39), we have

$$\begin{aligned}
\mathbb{E}\left[(X_t - X_s)^{2m}\right] \leq\ & +3^{2m}\frac{(2m)!}{m!2^m}M_3^{2m}|t-s|^m \\
+\ & 3^{2m}\left(2^{2m}M_2^{2m}\sup_{0\leq u\leq T}k(u)^{2m} + \left(2.2^{2m}M_1^{2m} + 2^{2m}M_2^{2m}\right)K_m(T)\right)|t-s|^{2m}
\end{aligned}$$

and since $|t-s|^m \leq T^m$, if we define

$$C_m(T) = 3^{2m}\left(T^m\left(2^{2m}M_2^{2m}\sup_{0\leq u\leq T}k(u)^{2m} + \left(2.2^{2m}M_1^{2m} + 2^{2m}M_2^{2m}\right)K_m(T)\right) + \frac{(2m)!}{m!2^m}M_3^{2m}\right)$$

then for all $0 \leq s \leq t \leq T$ we have

$$\mathbb{E}\left[(X_t - X_s)^{2m}\right] \leq C_m(T)|t-s|^m.$$

If $\gamma \in (0, \frac{1}{2})$, let $m = \lceil \frac{1}{1-2\gamma} \rceil + 1$, so $(m-1)/2m > \gamma$. Thus by the Kolmogorov-Čentsov theorem, a modification of X is locally Hölder continuous of order γ (see Karatzas and Shreve p.53-55 [41]).

To show (3.34), define

$$Z_u = \lambda(u)\left(X_u - \int_0^u w(r,u)X_r\,dr\right) - \beta(u)(X_u - k(u)),$$

so Z is a.s. continuous. Thus for every $t \in [0,T]$, it follows that

$$\lim_{s \uparrow t} \frac{\int_s^t Z_u\,du}{t-s} = Z_t \quad \text{a.s.}$$

Let $\gamma \in [\frac{1}{2}, 1]$. Then

$$\limsup_{s \uparrow t} \left| \frac{\int_s^t Z_u\,du}{(t-s)^\gamma} \right| = 0 \quad \text{a.s.} \tag{3.40}$$

Then using (3.22), (3.40), the martingale time change theorem (see Karatzas and Shreve [41]) in conjunction with (3.8) and the law of the iterated logarithm, we have

$$\begin{aligned}
\liminf_{s \uparrow t} \left| \frac{X_t - X_s}{(t-s)^\gamma} \right| &= \liminf_{s \uparrow t} \left| \frac{\int_s^t Z_u\,du}{(t-s)^\gamma} + \frac{\int_s^t \sigma(u)\,dB_u}{(t-s)^\gamma} \right| \\
&\geq \liminf_{s \uparrow t} \left(\left| \frac{\int_s^t \sigma(u)\,dB_u}{(t-s)^\gamma} \right| - \left| \frac{\int_s^t Z_u\,du}{(t-s)^\gamma} \right| \right) \\
&\geq \liminf_{s \uparrow t} \left| \frac{\int_s^t \sigma(u)\,dB_u}{(t-s)^\gamma} \right| + \liminf_{s \uparrow t} - \left| \frac{\int_s^t Z_u\,du}{(t-s)^\gamma} \right| \\
&= \liminf_{s \uparrow t} \left| \frac{\int_s^t \sigma(u)\,dB_u}{(t-s)^\gamma} \right| \\
&= \liminf_{s \uparrow t} \left| \frac{\tilde{B}_{\int_s^t \sigma(u)^2\,du}}{(t-s)^\gamma} \right| \\
&= \infty,
\end{aligned}$$

where $\tilde{B}$ is another standard Brownian motion. This completes the proof of (3.34). $\diamond$

Chapter 4

The Efficacy of Technical Analysis and the Possibility of Pricing Options when the Efficient Market Hypothesis is Violated

4.1 Introduction and Motivation

In this chapter we will motivate the study of the linear stochastic integro-differential equation (3.1) viz:

$$dX_t = \lambda(t)\left(X_t - \int_0^t w(s,t)X_s\,ds\right) - \beta(t)\,(X_t - k(t))\;dt + \sigma(t)\,dB_t,$$

by showing how it arises naturally as a model for price evolution in a market in which some agents believe that past prices have an indicative value in the formation of investment strategies, and other agents believe that prices should revert to levels determined by economic fundamentals. Moreover, by postulating that such classes of agents have the ability to influence price, we construct a model of prices which does not satisfy the Efficient Market Hypothesis: past prices do have an influence on the present price. The price process is therefore not Markovian, and much of our effort in this thesis will centre on overcoming some of the technical difficulties that this creates.

In this chapter, we expand upon the properties we imposed on $w(\cdot,\cdot)$ in Chapter 3; by endowing on $w(\cdot,\cdot)$ monotonicity and asymptotic invariance properties, we note that $w(s,t)$ can be ascribed the properties of the weight attached at time t to a price observed at time s, where $s < t$. The chartists then construct a moving average of past prices using this weight, and by comparing the value of this index with the current price, decide whether to buy or sell. Some consequences of this formulation are then outlined: the chartists' behaviour when the market reaches a record high or record low, and the manner in which chartists wait and see.

We then turn our attention to the chartists' ability to elicit information from a time series: essentially, we want to ask how closely their index function (moving average) mimics the time series. It transpires that if the price was to settle down to a constant value, the index function would also settle down to that constant value; if the price was bounded by a certain number, then the index function would be bounded by that number; if the price exhibited a periodic oscillation, the index function would settle down to a periodic function with the same period as the price. Finally, if the price was growing at a trend of less than iterated exponential growth, then the index function identifies this rate of growth precisely. In Chapter 6, by studying the pathwise asymptotics of (3.1), we show that the chartists' ability to track prices accurately leads to the market exhibiting time consistency: its dynamics do not seem to undermine the beliefs of the agents acting in the market, and therefore it is not unlikely that those agents will continue to hold their beliefs.

In the third section of the chapter, we sketch the method we need to mimic the micro–economic argument in Chapter 2: that is to say, the manner in which we convert a discrete time temporal equilibrium model to a continuous time stochastic process, which is the solution of a stochastic differential equation driven by a Brownian motion. In the modified model, in which the past also has a role, we indicate how the weak convergence proof of Kurtz and Protter might be altered to show how a discrete time equilibrium model could give rise to a stochastic integro–differential equation driven by a standard Brownian motion. As this program of research remains temporarily incomplete, we also provide a heuristic develpoment of this integro-differential equation by positing instananeous demands that are consistent with the choices made for demand functions in the discrete time model used in Chapter 2. Some of the model's similarites with the Black–Scholes model are made, as well as noting its suitability for modelling exchange rates.

The fourth section of the chapter shows that it is possible to price and hedge European options in this model, and that small investors doing so need only know the volatility in the market: the existence or behaviour of the chartists and fundamentalists are not revealed to these small investors—in fact, the small investor's portfolio comprises the same number of risky and riskless assets at any instant as would be the case if the price evolved according to Black-Scholes dynamics with the same time dependent volatility as in (3.1), and the same level of time dependent interest rates. This result shows that the Efficient Market Hypothesis does not have to be satisfied for derivatives to be priced; moreover, that investors pricing these derivative products need not know that it is being violated.

The last section of the chapter mentions that in this model there is a positive relationship between the volume of trade in the market and the volatility. We also show that increased fundamentalist confidence reduces the variability of the price process. This is of some interest in the context of credible exchange rate bands.

Throughout the initial sections of the chapter, we assume the price is a continuous time stochastic process, which is almost surely continuous.

4.2 Chartist Behaviour

4.2.1 Chartists' Weighting and Index Functions

Let $D := \{(s,t) \ : \ 0 \leq s \leq t\}$.

Definition 4.2.1 *The function w is a* **chartist weighting function** *if it satisfies the following:*

(i) $w \in \mathbf{C}(D \backslash \{(0,0)\}, (0,\infty))$.

(ii) $\int_0^t w(s,t)\, ds = 1$ *for all* $t > 0$.

(iii) $w(\cdot, t)$ *is non-decreasing.*

(iv) There exists a function $a : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ *such that*

(a) $a(0) < \infty$,

(b) $\lim_{t\to\infty} \sup_{0\le s\le t} \left|\frac{w(s,t)}{a(t-s)} - 1\right| = 0.$

We call such a function $a(\cdot)$ an **invariant weight for** $w(\cdot,\cdot)$.

One immediately obtains

Lemma 4.2.1 *If $a(\cdot)$ is an invariant weight for a chartist weighting function, then:*

(i) $a \in \mathbf{C}(\mathbb{R}^+ \cup \{0\}, \mathbb{R}^+)$.

(ii) $a(\cdot)$ *is non-increasing.*

(iii) $\lim_{t\to\infty} \int_0^t a(s)\,ds = 1$.

Proof: Suppose $a(t_1) \le 0$ for some $t_1 \ge 0$. Then by Definition 4.2.1 (iv) (b), for every $\varepsilon \in (0,1)$ there exists $T(\varepsilon) \ge 0$ such that for $t > T(\varepsilon)$

$$\left|\frac{w(t-t_1,t)}{a(t_1)} - 1\right| < \varepsilon.$$

Thus $\frac{w(t-t_1,t)}{a(t_1)} > 1-\varepsilon > 0$. If $a(t_1) < 0$, then $w(t-t_1,t) < 0$, which contradicts Definition 4.2.1 part (i). On the other hand, if $a(t_1) = 0$, from the above we have

$$\frac{w(t-t_1,t)}{a(t_1)} < 1+\varepsilon < 2,$$

which is also incompatible with Definition 4.2.1 part (i). Therefore $a(t) > 0$ for all $t \ge 0$. To show $a(\cdot)$ is non-increasing, set $t_1 < t_2$, and use Definition 4.2.1 part (iii) to obtain

$$\frac{a(t_1)}{a(t_1)} = \frac{\frac{a(t_1)}{w(t-t_1,t)}}{\frac{a(t_2)}{w(t-t_2,t)}} \cdot \frac{w(t-t_1,t)}{w(t-t_2,t)} \ge \frac{\frac{a(t_1)}{w(t-t_1,t)}}{\frac{a(t_2)}{w(t-t_2,t)}}.$$

Taking limits both sides of the inequality, together with Definition 4.2.1 part (iv) (b) and the positivity of $a(\cdot)$ gives $a(t_1) \ge a(t_2)$ as required. To show that $a(\cdot)$ is continuous, observe that for any $t_2 \ge 0$, we have by (ii) and Definition 4.2.1 part (iv) (a)

$$0 < a(t_2) \le a(0) < \infty.$$

Now write

$$\begin{aligned} &a(t_1) - a(t_2) \\ &= a(t_2)\frac{w(t-t_2,t)}{a(t_2)}\left(\frac{a(t_1)}{w(t-t_1,t)} - \frac{a(t_2)}{w(t-t_2,t)}\right) + \frac{a(t_1)}{w(t-t_1,t)}\left(w(t-t_1,t) - w(t-t_2,t)\right). \end{aligned}$$

Let $0 < \varepsilon < 4a(0)$. From Definition 4.2.1 part (iv) (b), there exists $T(\varepsilon) \in \mathbb{R}^+$ such that for all $t > T(\varepsilon)$

$$\left|\frac{w(t-t_i,t)}{a(t_i)} - 1\right| < \frac{\varepsilon}{8a(0)} \qquad \text{for } i = 1,2.$$

Now fix $t = T(\varepsilon) + 1$. It is then easy to show

$$\left|a(t_2)\frac{w(t-t_2,t)}{a(t_2)}\left(\frac{a(t_1)}{w(t-t_1,t)} - \frac{a(t_2)}{w(t-t_2,t)}\right)\right| < \frac{\varepsilon}{2}.$$

Moreover, by the continuity of $w(\cdot,\cdot)$ there exists $\delta(\varepsilon) > 0$ such that $0 < |t_1 - t_2| < \delta(\varepsilon)$ implies

$$|w(t-t_1,t) - w(t-t_2,t)| < \frac{\varepsilon}{4}$$

whence

$$\left|\frac{a(t_1)}{w(t-t_1,t)}\left(w(t-t_1,t) - w(t-t_2,t)\right)\right| < \frac{\varepsilon}{2}.$$

Therefore, for every $\varepsilon \in (0, 4a(0))$, there exists $\delta(\varepsilon) > 0$ such that $0 < |t_1 - t_2| < \delta(\varepsilon)$ implies

$$|a(t_1) - a(t_2)| < \varepsilon,$$

establishing the continuity of $a(\cdot)$. Thus (i) is secured. To establish part (iii), we have by (iv) (b) of the definition that for every $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that for all $t > T(\varepsilon)$

$$\sup_{0 \le s \le t}\left|\frac{w(s,t)}{a(t-s)} - 1\right| < \frac{\varepsilon}{1+\varepsilon}.$$

Then by Definition 4.2.1 part (ii), we have

$$\left|\int_0^t a(s)\,ds - 1\right| = \left|\int_0^t \left(\frac{a(t-s)}{w(s,t)} - 1\right) w(s,t)\,ds\right| \le \sup_{0 \le s \le t}\left|\frac{a(t-s)}{w(s,t)} - 1\right| < \varepsilon,$$

proving part (iii). ◇

We further define a chartist index function

Definition 4.2.2 *Suppose the logarithm of price at time t is given by X_t: if $w(\cdot,\cdot)$ is a chartist weighting function then*

$$a_c(t, X.) = \int_0^t w(s,t) X_s\,ds \tag{4.1}$$

is a **chartist index function**. *If chartists are* **positive feedback traders** *at time t they buy if*

$$a_c(t, X.) < X_t$$

and sell if

$$a_c(t, X_\cdot) > X_t.$$

If, on the other hand, chartists are **negative feedback traders** *at time t they buy if*

$$a_c(t, X_\cdot) > X_t$$

and sell if

$$a_c(t, X_\cdot) < X_t.$$

Remark 4.2.1

We now provide the economic and behavioural motivation behind Definitions 4.2.1 and 4.2.2. Since $w(\cdot, t)$ is non-decreasing for each t (cf., Definition 4.2.1 (iii)), we see that more recent prices make greater contributions to the chartist index function, and hence have greater influence on the chartists' decisions.

Suppose the price does not fluctuate over a period of time. Chartists believe that past patterns in prices are perpetuated; from their standpoint, it would be rational to neither buy nor sell, as they expect the price to remain constant. This perspective is matched by Definitions 4.2.1 and 4.2.2. Let the price be flat on $[0, T]$, so $X_t = X_0$ for all $t \in [0, T]$. Then by Definition 4.2.1 (ii) we have

$$a_c(t, X_\cdot) - X_t = \int_0^t w(s,t) X_s \, ds - X_t = X_0 \left(\int_0^t w(s,t) \, ds - 1 \right) = 0,$$

so chartists neither buy nor sell.

Remark 4.2.2 *If the price is amplified by a factor, the chartists come to the same decision to buy or sell.*

Proof: Let the old price be $(S_t)_{t \geq 0}$, and the new price be $(\gamma S_t)_{t \geq 0}$ for some $\gamma > 0$. Let $X_t^\gamma = \log \gamma S_t$. Then

$$a_c(t, X^\gamma) - X_t^\gamma = \log \gamma \left(1 - \int_0^t w(s,t) \, ds \right) + a_c(t, X_\cdot) - X_t = a_c(t, X_\cdot) - X_t,$$

proving the result, by Definitions 4.2.1 and 4.2.2. $\diamond$

Remark 4.2.3 *If the chartists buy (resp. sell) an exchange rate with price* $(S_t)_{t\geq 0}$*, they sell (resp. buy) an exchange rate with price* $(1/S_t)_{t\geq 0}$.

Proof: Letting $\tilde{S}_t = 1/S_t$, and $\tilde{X}_t = \log \tilde{S}_t$, we see that

$$a_c(t, \tilde{X}.) - \tilde{X}_t = -(a_c(t, X.) - X_t),$$

so that a buy signal for one exchange rate represents a sell signal for the other. ⋄

In other words, if chartists are tracking \$–DM, and believe they should sell dollars, they also believe they should buy Deutschmarks.

Since chartists tend to chase trends, they might be expected to buy at the top and sell at the bottom of the market. The specification we have provided mimics this behaviour. To see this, we make a further definition.

Definition 4.2.3 (Record High,Record Low) *If*

$$X_t = \sup_{s\in[0,t]} X_s \tag{4.2}$$

we say the price reaches a **record high** *at time t, while if*

$$X_t = \inf_{s\in[0,t]} X_s \tag{4.3}$$

we say the price reaches a **record low** *at time t.*

Remark 4.2.4 *If chartists are positive feedback traders, they buy at a record high and sell at a record low; if they are negative feedback traders, they sell at a record high and buy at a record low.*

Proof: Suppose that the chartists are positive feedback traders: the proof in the negative feedback case is identical. Now suppose that X_t is a record high. Then $X_t \geq X_s$ for all $s \leq t$, and using Definition 4.2.1 part (ii) we have

$$a_c(t, X.) - X_t = \int_0^t w(s,t)X_s\,ds - X_t = \int_0^t w(s,t)(X_s - X_t)\,ds \leq 0,$$

so the chartists buy, by Definition 4.2.2. The proof for the record low in the positive feedback case is identical. ⋄

Over the long run, chartists give weight to past prices purely on the basis of the 'age' of the price data. Thus a price observed at time $s < t$ should have a weight at time t dependent asymptotically on $t - s$ alone, $t - s$ being the length of time since the observation. This aspect of chartist behaviour is reflected in Definition 4.2.2 (iv).

The assumption that $w(\cdot, \cdot)$ is continuous, together with Definition 4.2.2, signifies that chartists believe all past prices should be taken into account, and that prices taken at times that are close together should be given similar weight.

A consequence of our model is that chartists are conservative speculators, at least initially. Until a trend has established itself, they will tend to be relatively inactive. Let us make this remark more precise.

Remark 4.2.5 *In the limit as time tends to 0, a chartist has zero net demand.*

Proof: From Definition 4.2.2, we see that the statement of the remark is equivalent to

$$\lim_{t \downarrow 0} X_t - \int_0^t w(s,t) X_s \, ds = 0.$$

Let $\bar{X}_t = X_t - X_0$. Since $t \mapsto X_t(\omega)$ is continuous for almost all $\omega \in \Omega$, $\bar{X}$ is also a.s. continuous. Moreover, $\lim_{t \downarrow 0} \bar{X}_t = 0$, so by Definition 4.2.1 (ii)

$$\begin{aligned} \left| X_t - \int_0^t w(s,t) X_s \, ds \right| &\leq |\bar{X}_t| + \int_0^t w(s,t) |\bar{X}_s| \, ds \\ &\leq |\bar{X}_t| + \max_{0 \leq s \leq t} |\bar{X}_s|. \end{aligned}$$

Taking limits gives the result. $\diamond$

Remark 4.2.6

The "wait and see" attitude that the chartists in this model possess and their lack of reaction to flat prices can also be seen in the behaviour of positive feedback traders in DeLong et al. [19], and of the chartist speculators in the models of DeGrauwe et al. [17].

We also mention that the chartist index function seems to follow trends in the price. In fact, we can show the following:

Remark 4.2.7 *Let $w(s,\cdot)$ be non-increasing, and $T < \infty$. If $X.$ is monotone non-decreasing (resp. monotone non-increasing) on $[0,T]$ then $a_c(\cdot, X.)$ is monotone non-decreasing (resp. monotone non-increasing) on $[0,T]$.*

Proof: Suppose $X.$ is monotone non-decreasing. For all $t \in [0,T]$, define $\tilde{X}_t = X_t - \inf_{0 \le u \le T} X_u$, so $\tilde{X}_t \ge 0$ for all $t \in [0,T]$. Let $t_1 > t_2$; then $w(s,t_2) \ge w(s,t_1)$ and therefore

$$0 \le \int_0^{t_2} (w(s,t_2) - w(s,t_1)\tilde{X}_s \, ds \le \int_0^{t_2} (w(s,t_2) - w(s,t_1)\tilde{X}_{t_2} \, ds. \tag{4.4}$$

Using Definition 4.2.2 and Definition 4.2.1 part (ii), together with (4.4), we obtain

$$\begin{aligned} a_c(t_1, X.) - a_c(t_2, X.) &= a_c(t_1, \tilde{X}.) - a_c(t_2, \tilde{X}.) \\ &= \int_{t_2}^{t_1} w(s,t_1)\tilde{X}_s \, ds - \int_0^{t_2} -(w(s,t_1) - w(s,t_2))\, \tilde{X}_s \, ds \\ &\ge \tilde{X}_{t_2} \int_{t_2}^{t_1} w(s,t_1) \, ds - \tilde{X}_{t_2} \int_0^{t_2} -w(s,t_1) + w(s,t_2) \, ds \\ &= 0, \end{aligned}$$

where the monotonicity of $\tilde{X}$ bounds the first integral. The proof in the case where $X.$ is non-increasing is identical. $\diamond$

4.2.2 Chartists' Tracking Abilities

It is possible to prove that the stylised chartists are able to recognise certain trends and patterns that develop in a time series. If prices settle down to a constant value almost surely, then the chartist index function tends asymptotically to that value. If the price oscillates periodically, the chartist index function tends asymptotically to a periodic function with the same period as the price. Finally, if the price has trend growth of less than iterated exponential rate, the chartist index function then tends asymptotically to the trend growth of the log-price.

Proposition 4.2.1 *Suppose the price settles down over time to a constant value, i.e.,*

$$\lim_{t \to \infty} X_t = x_\infty \quad a.s. \tag{4.5}$$

Then

$$\lim_{t \to \infty} a_c(t, X.) = x_\infty \quad a.s.$$

Proof: Let Ω^* be the measure 1 set for which (4.5) holds. Let $\omega \in \Omega^*$. Then for every $\varepsilon > 0$ there exists $T_1(\varepsilon, \omega) > 0$ such that for all $t > T_1(\varepsilon, \omega)$

$$|X_t(\omega) - x_\infty| < \frac{\varepsilon}{2}. \tag{4.6}$$

It is not difficult to show

$$\lim_{t \to \infty} w(s,t) = 0 \quad \text{for all fixed } s \geq 0, \tag{4.7}$$

so we observe there also exists $T_2(\varepsilon, \omega) > T_1(\varepsilon, \omega)$ which satisfies

$$w(T_1(\varepsilon, \omega), t) < \frac{\varepsilon}{2 \int_0^{T_1(\varepsilon,\omega)} |X_s(\omega) - x_\infty| \, ds} \tag{4.8}$$

for all $t > T_2(\varepsilon, \omega)$. Now let $t > T_2(\varepsilon, \omega)$, and using (4.6), (4.8) and the various properties of $w(\cdot, \cdot)$ we obtain

$$\begin{aligned}
\left| \int_0^t w(s,t) X_s(\omega) \, ds - x_\infty \right| &= \left| \int_0^t w(s,t)(X_s(\omega) - x_\infty) \, ds \right| \\
&\leq \left| \int_0^{T_1} w(s,t)(X_s(\omega) - x_\infty) \, ds \right| + \left| \int_{T_1}^t w(s,t)(X_s - x_\infty) \, ds \right| \\
&\leq w(T_1, t) \int_0^{T_1} |X_s(\omega) - x_\infty| \, ds + \int_{T_1}^t w(s,t) \frac{\varepsilon}{2} \, ds \\
&< \varepsilon,
\end{aligned}$$

which establishes the result. $\diamond$

Returning to the proof of (4.7), we find, using Definition 4.2.1 (i), (iii), that

$$1 = \int_0^t w(u,t) \, du = \int_0^s w(u,t) \, du + \int_s^t w(u,t) \, du \geq 0 + \int_s^t w(s,t) \, du = (t-s) w(s,t).$$

Therefore, $0 < w(s,t) \leq \frac{1}{t-s}$, so (4.7) is true.

The chartists can identify a bounded time series.

Remark 4.2.8 *If $(|X_t|)_{0 \leq t \leq T}$ is bounded by $M \geq 0$, then $(a_c(t, X.))_{0 \leq t \leq T}$ is bounded by M.*

Proof: Omitted. $\diamond$

We now prove the asymptotic periodicity of the chartist index function when the price varies periodically.

Proposition 4.2.2 *Let $T > 0$. Let X_t be T-periodic and continuous. Then there exists a Lipschitz continuous T- periodic function $p(\cdot)$ such that for each $t \in [0,T]$*

$$\lim_{n\to\infty} \int_0^{t+nT} w(s,t+nT)X_s\,ds - p(t) = 0. \tag{4.9}$$

Proof: Let $t \geq 0$. We first show for $n \in \mathbb{N}$ that

$$\lim_{n\to\infty} \int_0^{t+nT} w(s,t+nT)X_s\,ds \qquad \text{exists.} \tag{4.10}$$

We then define $p : \mathbb{R}^+ \cup \{0\} \to \mathbb{R} : t \mapsto p(t)$ by

$$p(t) = \lim_{n\to\infty} \int_0^{t+nT} w(s,t+nT)X_s\,ds \tag{4.11}$$

from which we establish that p is Lipschitz and T-periodic. Evidently (4.10) and (4.11) satisfy (4.9) and the hypotheses on p, thus proving the proposition.

To show (4.10), let $n, m \in \mathbb{N}$, and w.l.o.g. assume $n > m$. Thus, if we can show for each $t \geq 0$ that

$$\lim_{n,m\to\infty} \int_0^{t+nT} w(s,t+nT)X_s\,ds - \int_0^{t+mT} w(s,t+mT)X_s\,ds = 0, \tag{4.12}$$

then (4.10) will follow. Let $M_X = \max_{s\geq 0}|X_s|$. Then

$$\begin{aligned}
&\left|\int_{(n-m)T}^{t+nT} w(s,t+nT)X_s\,ds - \int_0^{t+mT} w(s,t+mT)X_s\,ds\right| \\
&= \left|\int_0^{t+mT} \left(\left(\frac{w(s+(n-m)T,t+nT)}{a(t+mT-s)} - 1\right) + \left(1 - \frac{w(s,t+mT)}{a(t+mT-s)}\right)\right) a(t+mT-s)X_s\,ds\right| \\
&\leq \left(\sup_{0\leq s\leq t+mT}\left|\frac{w(s+(n-m)T,t+nT)}{a(t+mT-s)} - 1\right| + \sup_{0\leq s\leq t+mT}\left|\frac{w(s,t+mT)}{a(t+mT-s)} - 1\right|\right) M_X,
\end{aligned}$$

where we use Lemma 4.2.1 at the last step and the T-periodicity of X throughout. By writing $t^*(n) = t + nT$, we have

$$\sup_{0\leq s\leq t+mT}\left|\frac{w(s+(n-m)T,t+nT)}{a(t+mT-s)} - 1\right| \leq \sup_{0\leq s\leq t^*(n)}\left|\frac{w(s,t^*(n))}{a(t^*(n)-s)} - 1\right|,$$

and

$$\sup_{0\leq s\leq t+mT}\left|\frac{w(s,t+mT)}{a(t+mT-s)} - 1\right| = \sup_{0\leq s\leq t^*(m)}\left|\frac{w(s,t^*(m))}{a(t^*(m)-s)} - 1\right|.$$

From these last two expressions and the previous bound, it follows from Definition 4.2.1 (iv) that

$$\lim_{n,m\to\infty,\, n>m} \int_{(n-m)T}^{t+nT} w(s,t+nT)X_s\,ds - \int_0^{t+mT} w(s,t+mT)X_s\,ds = 0,$$

so by obtaining the bound

$$\left| \int_0^{(n-m)T} w(s,t+nT)X_s\,ds \right| \leq M_X \left(1 + \sup_{0\leq s\leq t^*(n)} \left| \frac{w(s,t^*(n))}{a(t^*(n)-s)} - 1 \right| \right) \int_{t+mT}^{t+nT} a(s)\,ds$$

and taking limits as $n \to \infty$, we prove (4.12). Defining p as in (4.11) we automatically have $p(t+T) = p(t)$. To show Lipschitz continuity, we let, without loss of generality, $t_2 > t_1$ so we can use (4.11) to write

$$p(t_2) - p(t_1) = \lim_{n\to\infty} \int_0^{t_1+nT} (w(s,t_2+nT) - w(s,t_1+nT))X_s\,ds + \int_{t_1+nT}^{t_2+nT} w(s,t_2+nT)X_s\,ds.$$

It is not difficult to prove

$$\limsup_{n\to\infty} \left| \int_{t_1+nT}^{t_2+nT} w(s,t_2+nT)X_s\,ds \right| \leq M_X \int_0^{t_2-t_1} a(s)\,ds.$$

It is also true that

$$\limsup_{n\to\infty} \left| \int_0^{t_1+nT} (w(s,t_2+nT) - w(s,t_1+nT))X_s\,ds \right| \leq M_X \int_0^{t_2-t_1} a(s)\,ds. \tag{4.13}$$

To prove (4.13), we make the following partition:

$$\begin{aligned} w(s,t_2+nT) - w(s,t_1+nT) &= (a(t_2+nT-s) - a(t_1+nT-s)) \\ &+ \left(\left(\frac{w(s,t_2+nT)}{a(t_2+nT-s)} - 1 \right) a(t_2+nT-s) \right) + \left(\left(1 - \frac{w(s,t_1+nT)}{a(t_1+nT-s)} \right) a(t_1+nT-s) \right) \end{aligned}$$

and notice that

$$\int_0^{t_1+nT} |a(t_2+nT-s) - a(t_1+nT-s)|\,ds = \int_0^{t_2-t_1} a(s)\,ds - \int_{t_1+nT}^{t_2+nT} a(s)\,ds.$$

Applying the same arguments as above gives us (4.13). Since $a(\cdot)$ is non-increasing, we conclude that

$$|p(t_2) - p(t_1)| \leq 2M_X a(0)|t_2 - t_1|,$$

proving the proposition. ◇

The above remark amounts to the pointwise convergence of the chartist index function to a periodic function. One can move towards uniform convergence by noting the following:

Remark 4.2.9 *If X_t is a continuous T-periodic function, then*

$$\lim_{t\to\infty} a_c(t+T,X.) - a_c(t,X.) = 0.$$

Sketch of Proof: Notice as above we can bound $|X_t| \le M_X$. Write

$$\frac{w(s+T,t+T)}{w(s,t)} - 1 = \left(\frac{w(s+T,t+T)}{a(t-s)} - 1\right)\frac{a(t-s)}{w(s,t)} + \left(\frac{a(t-s)}{w(s,t)} - 1\right),$$

which is vanishingly small as $t \to \infty$, as is $w(T, t+T)$. The following decomposition, in conjunction with the above observations is sufficient to prove the assertion:

$$a_c(t+T, X_\cdot) - a_c(t, X_\cdot) = \int_0^t \left(\frac{w(s+T,t+T)}{w(s,t)} - 1\right) w(s,t)X_s\, ds + \int_0^T w(s,t+T)X_s\, ds. \ \diamond$$

Let $k_1 \in \mathbf{C}^1[0,\infty)$ be a non-decreasing, strictly positive function which satisfies

$$\lim_{t\to\infty} \frac{k_1'(t)}{k_1(t)} = 0, \tag{4.14}$$

$$\lim_{t\to\infty} k_1(t) = \infty. \tag{4.15}$$

To show that the chartists can track prices under some assumptions on the growth rate, we first prove

Lemma 4.2.2 *Let $w(\cdot,\cdot)$ be a chartist weighting function, and $k_1(\cdot)$ satisfy (4.14) and (4.15). Then*

$$\lim_{t\to\infty} \frac{\int_0^t w(s,t)k_1(s)\, ds}{k_1(t)} = 1. \tag{4.16}$$

Proof: Let $a(\cdot)$ be an invariant weight for $w(\cdot,\cdot)$. Then

$$\begin{aligned}
&\left|\frac{\int_0^t w(s,t)k_1(s)\, ds}{k_1(t)} - 1\right| \\
&\le \left|\frac{1}{k_1(t)}\int_0^t \left(\frac{w(s,t)}{a(t-s)} - 1\right) a(t-s)k_1(s)\, ds\right| + \left|\frac{1}{k_1(t)}\int_0^t a(t-s)k_1(s)\, ds - 1\right| \\
&\le \sup_{0\le s\le t}\left|\frac{w(s,t)}{a(t-s)} - 1\right| \frac{1}{k_1(t)}\int_0^t a(t-s)k_1(s)\, ds + \left|\frac{1}{k_1(t)}\int_0^t a(t-s)k_1(s)\, ds - 1\right|.
\end{aligned}$$

From Definition 4.2.1 (iv) the lemma is established if

$$\lim_{t\to\infty} \frac{1}{k_1(t)}\int_0^t a(t-s)k_1(s)\, ds = 1, \tag{4.17}$$

To show this, notice from Lemma 4.2.1 that for every $\varepsilon > 0$, there exists $T(\varepsilon) \in (0,\infty)$ such that

$$\int_{T(\varepsilon)}^\infty a(s)\, ds < \frac{\varepsilon}{2}. \tag{4.18}$$

Next, note by (4.14), (4.15) that

$$\lim_{t\to\infty} \sup_{0\le s\le T(\varepsilon)} \left|\frac{k_1(t-s)}{k_1(t)} - 1\right| = 0. \tag{4.19}$$

Now let $t > T(\varepsilon)$, so

$$
\begin{aligned}
&\left|\frac{1}{k_1(t)}\int_0^t a(t-s)k_1(s)\,ds - 1\right| \\
\leq\ & \left|\int_0^{t-T(\varepsilon)} a(t-s)\left(\frac{k_1(s)}{k_1(t)}-1\right)ds\right| + \left|\int_{t-T(\varepsilon)}^t a(t-s)\left(\frac{k_1(s)}{k_1(t)}-1\right)ds\right| + \left|\int_0^t a(s)\,ds - 1\right| \\
\leq\ & \sup_{0\leq s\leq t-T(\varepsilon)}\left|\frac{k_1(s)}{k_1(t)}-1\right|\int_0^{t-T(\varepsilon)} a(t-s)\,ds + \int_0^{T(\varepsilon)} a(s)\left|\frac{k_1(t-s)}{k_1(t)}-1\right|ds + \left|\int_0^t a(s)\,ds - 1\right| \\
\leq\ & 2\int_{T(\varepsilon)}^t a(s)\,ds + \sup_{0\leq s\leq T(\varepsilon)}\left|\frac{k_1(t-s)}{k_1(t)}-1\right| + \left|\int_0^t a(s)\,ds - 1\right|,
\end{aligned}
$$

where we used Lemma 4.2.1 (ii) to obtain the second inequality, and the monotonicity of $k_1(\cdot)$ and Lemma 4.2.1 (i), (iii) to obtain the third. Taking the limsup both sides of the inequality, and using (4.18), (4.19) and Lemma 4.2.1 (iii), we get

$$\limsup_{t\to\infty}\left|\frac{1}{k_1(t)}\int_0^t a(t-s)k_1(s)\,ds - 1\right| < \varepsilon.$$

Since ε can be chosen arbitrarily small, we have proved (4.17). $\diamond$

We now may proceed with the proof of

Proposition 4.2.3 *Let $k_1(\cdot)$ satisfy (4.14) and (4.15). Further suppose $(X_t)_{t\geq 0}$ is the log-price and $(a_c(t, X.))_{t\geq 0}$ the associated chartist index function. If*

$$\lim_{t\to\infty}\frac{X_t}{k_1(t)} = 1 \quad a.s.$$

then

$$\lim_{t\to\infty}\frac{a_c(t,X.)}{k_1(t)} = 1 \quad a.s.$$

Proof: Let $w(\cdot,\cdot)$ be the chartist weighting function associated with the chartist index function; let $a(\cdot)$ be an invariant weight for $w(\cdot,\cdot)$. For each ω in the sample space, define

$$\epsilon(t)(\omega) = \left|\frac{X_t(\omega)}{k_1(t)} - 1\right|, \tag{4.20}$$

and

$$\Omega_1 = \{\omega : \lim_{t\to\infty}\epsilon(t)(\omega) = 0\}.$$

Then $\mathbb{P}[\Omega_1] = 1$. For each $\omega \in \Omega_1$, let

$$C_2(\omega) = \sup_{t\geq 0}|\epsilon(t)(\omega)| < \infty,$$

as we recall $\epsilon(\cdot)(\omega)$ is continuous. Let

$$C_1 = \sup_{t\geq 0} \frac{1}{k_1(t)} \int_0^t a(t-s)k_1(s) < \infty,$$

and also $C_1 \geq 1$. Define

$$T_1(\varepsilon)(\omega) = \sup\{t \geq 0 : \epsilon(t)(\omega) = \frac{\varepsilon}{2C_1}\}, \tag{4.21}$$

so that for all $t > T_1(\varepsilon)(\omega)$, one has $\epsilon(t) < \frac{\varepsilon}{2C_1}$. Since $k_1 \to \infty$ as $t \to \infty$ there exists $T_2(\varepsilon)(\omega) \geq 0$ such that for all $t > T_2(\varepsilon)(\omega)$

$$k_1(t) > \frac{2C_2(\omega)C_1}{\varepsilon} k_1(T_1(\varepsilon)(\omega)). \tag{4.22}$$

Now let $t > T(\varepsilon)(\omega) := T_1(\varepsilon)(\omega) \vee T_2(\varepsilon)(\omega)$ and bound as follows:

$$\begin{aligned}
&\left|\frac{1}{k_1(t)} \int_0^t a(t-s)k_1(s)\epsilon(s)(\omega)\,ds\right| \qquad (4.23)\\
&\leq \frac{1}{k_1(t)} \int_0^{T_1(\varepsilon)(\omega)} a(t-s)k_1(s)\epsilon(s)(\omega)\,ds + \frac{1}{k_1(t)} \int_{T_1(\varepsilon)(\omega)}^t a(t-s)k_1(s)\epsilon(s)(\omega)\,ds\\
&\leq \frac{k_1(T_1(\varepsilon)(\omega))}{k_1(t)} \frac{1}{k_1(T_1(\varepsilon)(\omega))} \int_0^{T_1(\varepsilon)(\omega)} a(T_1(\varepsilon)(\omega)-s)k_1(s)\,ds. \sup_{s\geq 0}|\epsilon(s)(\omega)|\\
&\quad + \frac{\varepsilon}{2C_1}\frac{1}{k_1(t)} \int_0^t a(t-s)k_1(s)\,ds,
\end{aligned}$$

where we recall $k_1(\cdot) > 0$ and $a(\cdot)$ is strictly positive and non-increasing. Combining (4.21), (4.22) and (4.23) yields:

$$\left|\frac{1}{k_1(t)} \int_0^t a(t-s)k_1(s)\epsilon(s)(\omega)\,ds\right| \leq \frac{\varepsilon}{2C_1C_2(\omega)} \cdot C_1 \cdot C_2(\omega) + \frac{\varepsilon}{2C_1} \cdot C_1 = \varepsilon.$$

Thus for all $\omega \in \Omega_1$ and for every $\varepsilon > 0$ there exists $T(\varepsilon)(\omega) > 0$ such that for all $t > T(\varepsilon)(\omega)$ we have

$$\left|\frac{1}{k_1(t)} \int_0^t a(t-s)k_1(s)\epsilon(s)(\omega)\,ds\right| < \varepsilon. \tag{4.24}$$

In consequence,

$$\lim_{t\to\infty} \frac{1}{k_1(t)} \int_0^t a(t-s)k_1(s)\epsilon(s)\,ds = 0 \quad \mathbb{P}-\text{a.s.} \tag{4.25}$$

Now we have for $\omega \in \Omega_1$

$$\frac{1}{k_1(t)} \int_0^t w(s,t)X_s(\omega)\,ds - 1 = \frac{1}{k_1(t)} \int_0^t w(s,t)\left(\frac{X_s(\omega)}{k_1(s)} - 1\right) k_1(s)\,ds + \int_0^t \frac{w(s,t)k_1(s)}{k_1(t)}\,ds - 1$$

and

$$\begin{aligned}
&\left|\frac{1}{k_1(t)} \int_0^t w(s,t)\left(\frac{X_s(\omega)}{k_1(s)} - 1\right) k_1(s)\,ds\right|\\
&\leq \frac{1}{k_1(t)} \int_0^t \left|\frac{w(s,t)}{a(t-s)} - 1\right| a(t-s)k_1(s)\epsilon(s)(\omega)\,ds + \frac{1}{k_1(t)} \int_0^t a(t-s)k_1(s)\epsilon(s)(\omega)\,ds\\
&\leq \left(1 + \sup_{0\leq s\leq t}\left|\frac{w(s,t)}{a(t-s)} - 1\right|\right) \cdot \frac{1}{k_1(t)} \int_0^t a(t-s)k_1(s)\epsilon(s)(\omega)\,ds \to 0 \quad \text{as } t\to\infty,
\end{aligned}$$

where we used (4.25) at the last stage. Finally,

$$\begin{aligned}
&\lim_{t\to\infty}\left|\frac{a_c(t,X.)}{k_1(t)}-1\right| \\
&\leq \limsup_{t\to\infty}\left|\frac{1}{k_1(t)}\int_0^t w(s,t)\left(\frac{X_s(\omega)}{k_1(s)}-1\right)k_1(s)\,ds\right| + \lim_{t\to\infty}\frac{1}{k_1(t)}\int_0^t w(s,t)k_1(s)\,ds - 1 = 0,
\end{aligned}$$

on using Lemma 4.2.2. Thus

$$\lim_{t\to\infty}\frac{a_c(t,X.)}{k_1(t)} = 1 \quad \mathbb{P}-a.s.,$$

which completes the proof. ⋄

4.3 Development of Price Evolution

In this section we present a heuristic argument for a continuous time evolution of prices in a market which includes chartist traders together with traders with mean-reverting price expectations, and an incomplete, though well–developed, weak convergence argument, shadowing the development of the price equation in Chapter 2. We nonetheless present the outline of this incomplete argument, starting from a discrete time equilibrium model, and passing to continuous time via weak convergence. We aim to show this does not represent a defiency in our theory, as it seems we can modify the arguments of Kurtz and Protter [47] to cover the case of weak convergence in distribution to stochastic integro-differential equations with non-convolution kernels.

4.3.1 Towards Weak Convergence: from a Discrete to a Continuous Time Equilibrium Model

We reprise verbatim (and specialise according to our requirements) the background and results proved by Kurtz and Protter.

For $n = 1, 2, \ldots$ let $F_n : D_{\mathbb{R}}[0,\infty) \to D_{\mathbb{R}}[0,\infty)$ and let U_n and Y_n be processes with sample paths in $D_{\mathbb{R}}[0,\infty)$ adapted to a filtration $\{\mathcal{F}_t^n\}$. Suppose Y_n is a semi-martingale, and that $F_n(x,t) = F_n(x^t,t)$ where $x^t(\cdot) = x(\cdot\wedge t)$, for all $x \in D_{\mathbb{R}}[0,\infty)$ and $t \geq 0$. Let X_n be adapted to $\{\mathcal{F}_t^n\}$ and satisfy

$$X_n(t) = U_n(t) + \int_0^t F_n(X_n, s-)\,dY_n(s). \tag{4.26}$$

Conditions are thus required under which the solutions of this sequence of equations converges weakly to the solution of the limiting equation

$$X(t) = U(t) + \int_0^t F(X, s-)\, dY(s). \tag{4.27}$$

In other words, we need $(U_n, Y_n, X_n) \Rightarrow (U, Y, X)$. Kurtz and Protter show that to be able to effect the desired convergence, conditions are required under which $(X_n, Y_n) \Rightarrow (X, Y)$ implies $(Y_n, F_n(X_n)) \Rightarrow (Y, F(X))$. This follows if $(x_n, y_n) \to (x, y)$ in $D_{\mathbb{R}\times\mathbb{R}}[0, \infty)$ implies

$$(x_n, y_n, F_n(x_n)) \to (x, y, F(x)) \text{ in } D_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}}[0, \infty).$$

It is possible to prove this under some natural assumptions on F_n and F. Let Λ denote the collection of continuous, strictly increasing functions mapping $[0, \infty)$ onto itself. Let Λ^1 be the subset of absolutely continuous functions in Λ for which $\gamma(\lambda) = ||\log \dot{\lambda}||_\infty$ is finite.

Lemma 4.3.1 *Suppose that $\{F_n\}$ and F satisfy the following conditions*

(i) For each compact subset $\mathcal{H} \in D_{\mathbb{R}}[0, \infty)$ and $t > 0$, $\sup_{x\in\mathcal{H}} \sup_{s\leq t} |F_n(x, s) - F(x, s)| \to 0$.

(ii) For $\{x_n\}$ and x in $D_{\mathbb{R}}[0, \infty)$ and each $t > 0$, $\sup_{s\leq t} |x_n(s) - x(s)| \to 0$ implies

$$\sup_{s\leq t} |F(x_n, s) - F(x, s)| \to 0.$$

(iii) For each compact subset $\mathcal{H} \subset D_{\mathbb{R}}[0, \infty)$ and $t > 0$, there exists a continuous function

$$\omega : [0, \infty) \to [0, \infty)$$

with $\omega(0) = 0$ such that for all $\lambda \in \Lambda^1$, $\sup_{x\in\mathcal{H}} \sup_{s\leq t} |F(x \circ \lambda, s) - F(x, \lambda(s))| \leq \omega(\gamma(\lambda))$.

Then $(x_n, y_n) \to (x, y)$ in the Skorohod topology implies $(x_n, y_n, F_n(x_n)) \to (x, y, F(x))$ in the Skorohod topology.

Proof: See Kurtz and Protter [47] p.1057. $\diamond$

We then have

Proposition 4.3.1 *Suppose that (U_n, Y_n, X_n) satisfies (4.26), that $\{(U_n, Y_n, X_n)\}$ is relatively compact in $D_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}}[0, \infty)$, that $(U_n, Y_n) \Rightarrow (U, Y)$, and Y_n is deterministic. If F and F_n satisfy the conditions in Lemma 4.3.1, then any limit point of the sequence $\{X_n\}$ satisfies (4.27).*

Proof: Combine Proposition 5.1. on p. 1056 of [47] with Lemma 4.3.1. $\diamond$

Remark 4.3.1

Kurtz and Protter note (but do not prove) the following: if $g : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and $h : [0, \infty) \to [0, \infty)$ are continuous, then

$$F(x, t) = g(x(t), t)$$

and

$$F(x, t) = \int_0^t h(t - s) g(x(s), s)\, ds$$

satisfy conditions (i) and (ii) of Lemma 4.3.1. Therefore, under the assumption of convolution kernels, the only outstanding difficulty would be the relative compactness of the sequence of solutions.

Kurtz and Protter obviate this difficulty by localising the result. To this end they let $T_1[0, \infty)$ denote the collection of non-decreasing mappings λ of $[0, \infty)$ onto $[0, \infty)$ such that $\lambda(t + h) - \lambda(t) \leq h$ for all $t, h \geq 0$. We assume there exist mappings

$$G_n, G : D_{\mathbb{R}}[0, \infty) \times T_1[0, \infty) \to D_{\mathbb{R}}[0, \infty)$$

such that $F_n(x) \circ \lambda = G_n(x \circ \lambda, \lambda)$ and $F(x) \circ \lambda = G(x \circ \lambda, \lambda)$ for $(x, \lambda) \in D_{\mathbb{R}}[0, \infty) \times T_1[0, \infty)$. They require the following strengthening of conditions (i), (ii) in Lemma 4.3.1 above.

(a) For each compact subset $\mathcal{H} \in D_{\mathbb{R}}[0, \infty) \times T_1[0, \infty)$ and $t > 0$, $\sup_{(x,\lambda) \in \mathcal{H}} \sup_{s \leq t} |G_n(x, \lambda, s) - G(x, \lambda, s)| \to 0$.

(b) For $\{(x_n, \lambda_n)\} \in D_{\mathbb{R}}[0, \infty) \times T_1[0, \infty)$, $\sup_{s \leq t} |x_n(s) - x(s)| \to 0$ and $\sup_{s \leq t} |\lambda_n(s) - \lambda(s)| \to 0$ for each $t > 0$ implies

$$\sup_{s \leq t} |G(x_n, \lambda_n, s) - G(x, \lambda, s)| \to 0.$$

Kurtz and Protter observe that each of the examples in Remark 4.3.1 has a representation in terms of a G satisfying (b) above and that (a) and (b) imply (i)-(iii) in Lemma 4.3.1.

The authors then prove a result more general than the following theorem; however, we specialise their assertion to suit our purposes

Theorem 4.3.1 *Suppose (U_n, Y_n, X_n) satisfies (4.26), $(U_n, Y_n) \Rightarrow (U, Y)$ in the Skorohod topology, and that Y_n is deterministic. Assume that $\{F_n\}$ and F have representations in terms of $\{G_n\}$ and G satisfying (a), (b) above. If there exists a global solution X of (4.27) and weak local uniqueness holds, then $(U_n, Y_n, X_n) \Rightarrow (U, Y, X)$.*

Proof: See p.1058-1059 in [47]. $\diamond$

We will obviously want the limiting form of the equation to be given by (3.1), and we observe that this equation has a strong unique solution on all compacts $[0, T]$.

As in Chapter 2, we will have $\sigma : [0, \infty) \to \mathbb{R}$ and

$$U_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \sigma(k/n) \zeta_n^k$$

where ζ_n^k are iid with zero mean, unit variance, and

$$U(t) = \int_0^t \sigma(s)\, dB_s,$$

where $\{B_t : t \geq 0\}$ is a standard one-dimensional Brownian motion. Then

$$U_n(t) \xrightarrow{\mathcal{D}} U(t) \text{ as } n \to \infty.$$

If we have $Y(t) = t$ and $Y_n(t) \to Y(t)$, since Y_n is deterministic, the condition $(U_n, Y_n) \Rightarrow (U, Y)$ will be satisfied by a result of Slominski [70] together with Slutsky's theorem. If

$$F(x, t) = \lambda(t) \left(x(t) - \int_0^t w(s, t) x(s)\, ds \right) - \beta(t)(x(t) - k(t)),$$

then F satisfies condition (ii) of Lemma 4.3.1. A discrete time equilibrium argument such as given in Chapter 2 gives rise to a sequence $\{F_n\}$ which together with F satisfies part (i) of Lemma 4.3.1. I feel that a little further analysis is required to prove condition (iii), and from that point, to establish (a) and (b). The proof of this final result would enable us to apply the result of Theorem 4.3.1, and so mimic the presentation of Chapter 2.

Instead of the above outline of a rigorous development of a continuous time equilibrium model, we will present a heuristic argument to motivate our study of (3.1).

4.3.2 Economic Foundations of the Stochastic Integro-differential Equation as a Model for Price Evolution

Suppose the market comprises m_1 chartists and m_2 fundamentalists. As in Chapter 2, assume that both classes of agents have log-linear (instantaneous) demands. Over a small time interval dt, chartist $i = 1, \dots, m_1$ has infinitesimal demand over $(t, t+dt)$ given by

$$\int_t^{t+dt} \tilde{\lambda}_i(s) \left(X_s - \int_0^s w_i(u,s) X_u \, du \right) ds$$

where $\tilde{\lambda}_i(\cdot) \in \mathbf{C}([0,\infty),(0,\infty))$ and $w_i(\cdot,\cdot)$ is a chartist weighting function for each i. A high value of $\tilde{\lambda}_i(t)$ indicates that chartist i is very active at time t: he is confident in his prediction, and is able to back it. If

$$\tilde{\lambda}(t) = \sum_{i=1}^{m_1} \tilde{\lambda}_i(t),$$

and

$$w(s,t) = \frac{1}{\tilde{\lambda}(t)} \sum_{i=1}^{m_1} \tilde{\lambda}_i(t) w_i(s,t), \tag{4.28}$$

then $\tilde{\lambda} \in \mathbf{C}([0,\infty),(0,\infty))$ and the aggregate chartist infinitesimal demand over $(t, t+dt)$ is given by

$$\int_t^{t+dt} \tilde{\lambda}(s) \left(X_s - \int_0^s w(u,s) X_u \, du \right) ds. \tag{4.29}$$

Obviously, $w(\cdot,\cdot)$ in (4.28) satisfies properties (i),(ii),(iii) of Definition 4.2.1. It follows from Proposition 4.2.3 that if $k_1(\cdot)$ satisfies (4.14) and (4.15), and $(X_t)_{t\geq 0}$ satisfies

$$\lim_{t\to\infty} \frac{X_t}{k_1(t)} = 1 \quad \text{a.s.}$$

then

$$\lim_{t\to\infty} \frac{\int_0^t w(s,t) X_s \, ds}{X_t} = 1 \quad \text{a.s.}$$

Moreover, since the space of chartist weighting functions is convex, if $\tilde{\lambda}_i(t) = \lambda_i$ for all $i = 1, \dots, m_1$, then $w(\cdot,\cdot)$ is a chartist weighting function. Furthermore, if all chartist weighting functions share a common invariant weight a, then $w(\cdot,\cdot)$ is a chartist weighting function. To simplify the mathematical analysis in the rest of the thesis we will assume that $w(\cdot,\cdot)$ given by (4.28) is a chartist weighting function.

Over a small time interval dt, fundamentalist $j = 1, \ldots, m_2$ has infinitesimal demand over $(t, t+dt)$ given by

$$\int_t^{t+dt} -\tilde{\beta}_j(s)(X_s - k_s(t))\, ds$$

where $\tilde{\beta}_j \in \mathbf{C}([0,\infty),(0,\infty))$ and $k_j \in \mathbf{C}([0,\infty),\mathbb{R})$ for each $j = 1, \ldots, m_2$. As with the chartists above, a high value of $\tilde{\beta}_j(t)$ signifies a high degree of market participation of fundamentalist j at time t. If

$$\tilde{\beta}(t) = \sum_{j=1}^{m_2} \tilde{\beta}_j(t),$$

and

$$k(t) = \frac{1}{\tilde{\beta}(t)} \sum_{j=1}^{m_2} \tilde{\beta}_j(t) k_j(t),$$

then $\tilde{\beta} \in \mathbf{C}([0,\infty),(0,\infty))$, $k \in \mathbf{C}([0,\infty),\mathbb{R})$ and the aggregate fundamentalist infinitesimal demand over $(t, t+dt)$ is given by

$$\int_t^{t+dt} -\tilde{\beta}(s)\,(X_s - k(s))\; ds. \tag{4.30}$$

We further assume that there is random demand arising from either group of speculator or other unmodelled speculators; as in Chapter 2, we assume that over the time interval $(t, t+dt)$ it has magnitude

$$\int_t^{t+dt} \tilde{\sigma}(s)\, dB_s \tag{4.31}$$

for $\tilde{\sigma} \in \mathbf{C}(([0,\infty),\mathbb{R})$.

Let $D(t, t+dt)$ be total excess demand over the time interval $(t, t+dt)$. Then by (4.29), (4.30) and (4.31), we have

$$\begin{aligned} D(t,t+dt) = & \int_t^{t+dt} \tilde{\lambda}(s) \left(X_s - \int_0^s w(u,s) X_u\, du \right) ds \\ & + \int_t^{t+dt} -\tilde{\beta}(s)\,(X_s - k(s))\; ds + \int_t^{t+dt} \tilde{\sigma}(s)\, dB_s. \end{aligned} \tag{4.32}$$

If we further suppose that price does not change when there is zero total excess demand, and that the log-price of the asset adjusts in a Walrasian manner reacting to excess demand, and linearly related to it, then

$$X_{t+dt} - X_t = \alpha D(t, t+dt),$$

for some $\alpha > 0$. We remark that this sluggish Walrasian price adjustment yields a reasonable approximation to the price formation observed in the experimental asset market paper of Smith, Suchanek and Williams [71]. We then have

$$X_t = X_0 + \int_0^t \lambda(s) \left(X_s - \int_0^s w(u,s) X_u \, du \right) - \beta(s)(X_s - k(s)) \, ds + \int_0^t \sigma(s) \, dB_s, \qquad (4.33)$$

where $\lambda(\cdot) = \alpha\tilde{\lambda}(\cdot), \beta(\cdot) = \alpha\tilde{\beta}(\cdot), \sigma(\cdot) = \alpha\tilde{\sigma}(\cdot)$, and have the same properties as their tilded counterparts. Re-writing (4.33) adopting the usual convention for stochastic differential equations, we obtain (3.1), viz:

$$dX_t = \lambda(t) \left(X_t - \int_0^t w(s,t) X_s \, ds \right) - \beta(t)(X_t - k(t)) \, dt + \sigma(t) \, dB_t. \qquad (4.34)$$

Finally notice that high values of $\lambda(\cdot), \beta(\cdot)$ can be interpreted as arising from high degrees of market participation of the chartists and fundamentalists, respectively.

Remark 4.3.2

Notice once again that the excess demand functions are "log–linear" in the sense that

$$D(t, (\alpha X_s)_{0 \le s \le t}) = \alpha D(t, (X_s)_{0 \le s \le t}),$$

so that they correspond basically to those chosen in Chapter 2. Recall that this type of excess demand function is common in monetary economics. Also, we remark that the price process can be thought of as "Ornstein-Uhlenbeck plus memory", so that it may be thought of as related to the literature referenced in Remark 2.2.1.

Remark 4.3.3

Our micro-economic specifications, together with the analysis of Chapter 3, indicate that there is a unique solution to the price evolution which has the same local regularity as Brownian motion. We will show during the next few chapters that the price process shares several properties with that arising from the Black-Scholes price evolution. The following remark, however, can be made immediately.

We will presently show, in Lemma 4.4.3, that X is an Itô process. Therefore, using Itô's rule and the identity $S_t = e^{X_t}$, we see that

$$S_t = S_0 + \int_0^t K_s \, ds + \int_0^t \sigma(s) S_s \, dB_s,$$

where K is a process which depends on S. In consequence, the price process has the same volatility term as a Black-Scholes price evolution with time dependent volatility $\sigma(\cdot)$.

Remark 4.3.4

The rate at which the chartists discount the past has an important effect on the long run asymptotics of the price. In Chapter 5 we prove a precise result which captures this effect for a particular type of memory structure. However, by a brief study of an extreme case, we gain insight as to the general importance of myopia in stabilising prices.

Suppose $w(s,t) = \delta(t-s)$ where $\delta(\cdot)$ is the Dirac function. Notice that $w(s,t) \geq 0$, $\liminf_{s\uparrow t} w(s,t) > w(s,t)$ for all $s < t$ and $\int_0^t w(s,t)\,ds = 1$ for all $t > 0$, so that w behaves similarly to a chartist weighting function.

For this choice of proxy chartist weighting function, the chartist index function at time t is identically equal to X_t, so chartist demand is zero for all time. In these circumstances, with only fundamentalists trading, X behaves like an Ornstein-Uhlenbeck process centered on $k(t)$, and its asymptotics follow those of $k(\cdot)$, so the price dynamics are quite mild.

If chartists have extrapolative expectations, then the less myopic they are, the more stable the price will be. In the above caricature, the chartists make no (destabilising) contribution when their memory has zero length.

Remark 4.3.5

In reality, the assumption that the chartists use the whole price history is very unrealistic, but since the weight of distant prices declines to zero, this means that, effectively, only the most recent data is necessary. However, I believe that an analysis to the same level of completeness could be given under the assumption that chartists censor the price data i.e., all prices that occur a certain time before the present, T_1 say, could be ignored. To see how one might change the hypotheses, define $a(\cdot)$ so that $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, is non-increasing and

$$\int_0^{T_1} a(s)\,ds = 1,$$

so that a reasonable choice for the chartist index function is

$$a_c(t, X.) = \int_{t-T_1}^{t} a(t-s) X_s \, ds.$$

The price evolution in this case will be a stochastic integro-differential equation with delay. Clearly, we may remove the delay by defining $a(t) = 0$ for all $t > T_1$, but at the cost of the continuity of $a(\cdot)$. However, we may still find the resolvent of the related deterministic equation, so a representation of the solution may be written down. From this step, I feel that most of the previous (and forthcoming) analysis can be re-captured.

In a similar manner, it should be possible to develop an analysis of price dynamics if the chartists were to trade on the basis of two moving averages. To see how one might formulate this, let T_1, T_2 be two distinct positive numbers and for $i = 1, 2$, let $a_i \in \mathbf{C}(\mathbb{R}^+, \mathbb{R}^+)$, be non-increasing and satisfy

$$\int_0^{T_i} a_i(s) \, ds = 1.$$

Thus a chartist buy or sell signal could be determined by the sign of

$$D_t = \int_{t-T_1}^{t} a_1(t-s) X_s \, ds - \int_{t-T_2}^{t} a_2(t-s) X_s \, ds,$$

and so by again supposing that the excess demand is log-linear, net chartist demand over the time interval $[t_1, t_2]$ will be given by $\int_{t_1}^{t_2} \lambda(s) D_s \, ds$.

Remark 4.3.6

The linearity present in (3.1) is crucial as it affords us a tractable model: however, a good case can be made for its adoption above mere tractability, particularly in the case of exchange rates. Suppose that $S_t = e^{X_t}$ gives the value of an exchange rate at time t, so

$$X_t = \log S_t.$$

The process $(S_t^*)_{t \geq 0}$ given by $S_t^* = 1/S_t$ is also an exchange rate. We will prove in Lemma 4.4.3 that $(X_t)_{t \geq 0}$ is an Itô process. Then the process $(X_t^*)_{t \geq 0}$ given by

$$X_t^* = -X_t = \log S_t^*$$

is also an Itô process. Since X_t and X_t^* are both logarithms of exchange rates, the structure of the evolution for both X and X^* should be identical. From Itô's rule, we have

$$dX_t^* = \lambda(t) \left(X_t^* - \int_0^t w(s,t) X_s^* \, ds \right) - \beta(t)(X_t^* - k^*(t)) \, dt - \sigma(t) \, dB_t, \tag{4.35}$$

where $k^*(t) = -k(t)$.

Comparing (4.34) and (4.35), we notice that the chartists contributions have identical structures: this chimes with the notion that they treat different time series with the same analysis; whether the exchange rate is \$-DM or DM-\$ they search for trends using the same methodology.

The fundamentalists contributions in (4.34) and (4.35) have the same structure—reversion towards a mean value which is exogenously determined. In each case, the exogeneous mean (k, k^*) is the logarithm of the fundamental, (F, F^*). If the fundamentalists think fair value for the exchange rate S_t at time t is $F_t = e^{k(t)}$, by implication, they think that the fair value for the exchange rate S_t^* is $F_t^* = 1/F_t = e^{-k(t)} = e^{k^*(t)}$.

The noise terms in (4.34) and (4.35) are of the same order of magnitude—this reflects the fact that both exchange rates should have the same volatility.

Remark 4.3.7

We observe that, although linearity in log-price in the chartists' demand functions can be justified by the remarks in the last two sections, we might equally have chosen linearity in price in the chartists' demand functions.

Suppose that the demand at time t given the price history $(S_u)_{0\leq u\leq t}$ is denoted by $D(t, (S_u)_{0\leq u\leq t})$. Let the number of shares bought at time t be N_t (with negative values of N_t denoting sales), so that

$$D(t, (S_u)_{0\leq u\leq t}) = N_t S_t.$$

Suppose that the price was denominated in a different monetary unit (say centimes instead of francs, or pence instead of pounds). This has the effect of multiplying the price by a scalar, say $\lambda > 0$. However, since there has been no change in the underlying economics, the number of shares bought at time t should still be N_t. Therefore,

$$D(t, (\lambda S_u)_{0\leq u\leq t}) = N_t.\lambda S_t,$$

so

$$D(t, (\lambda S_u)_{0\leq u\leq t}) = \lambda D(t, (S_u)_{0\leq u\leq t}). \tag{4.36}$$

The demand equation given by (4.36) is compatible with demand schedules of the form

$$D(t,(S_u)_{0\le u\le t}) = \lambda(t)S_t + \int_0^t \alpha(s,t)S_s\,ds,$$

but incompatible with the log–linear demand schedules given by (4.32). The outcome of adopting this plausible argument has not been followed through in this thesis, though it merits serious analysis. Instead, we adopt the log–linear specification, due to its suitability in exchange rate modelling: suppose that two currencies are traded at exchange rates $(S_t)_{t\ge 0}$ and $(S_t^*)_{t\ge 0}$, where $S_t^* = 1/S_t$. If these two currencies are traded, purchases in one currency must be matched by sales in the other. Using the notation introduced above, this reads:

$$D(t,(S_u)_{0\le u\le t}) = -D(t,(1/S_u)_{0\le u\le t}),$$

which conforms with the log–linear demand schedule outlined in (4.32), while not in agreement with (4.36).

4.4 The Pricing, Hedging and Replication of Options

In this section, we consider the investment strategy of a small investor (that is to say, an investor whose actions do not have any effect on the price). We assume that such an investor holds a portfolio consisting of shares in a risky asset (with log-price dynamics given by (3.1)) and riskless bonds. Bond values will be assumed to vary according to

$$dS_t^0 = \rho(t)S_t^0\,dt, \tag{4.37}$$

where $S_0^0 = 1$, and $\rho \in \mathrm{C}([0,\infty),\mathbb{R})$ and is bounded.

We prove that a small investor can construct a self-financing strategy which does not depend in any way on his knowledge of the behaviour (or even the existence) of chartists and fundamentalists. In fact, for a Black-Scholes model with bond dynamics as in (4.37), and instananeous volatility $\sigma(\cdot)$, the small investor would hedge in an identical manner.

4.4.1 A Martingale Measure for Discounted Prices

To solve the pricing problems, we firstly must construct a new probability measure under which the discounted price of the risky asset is a martingale.

We first will prove a lemma which allows us to use the Girsanov Theorem to construct the martingale measure.

Lemma 4.4.1 *Let*

$$\gamma_t = \frac{-1}{\sigma(t)}\left(e'(t) - \rho(t) + \int_0^t \sigma(s)\frac{\partial g}{\partial t}(s,t)\,dB_s + \frac{1}{2}\sigma(t)^2\right), \tag{4.38}$$

where $\sigma(\cdot), \lambda(\cdot), \beta(\cdot), e(\cdot)$ and $g(\cdot,\cdot)$ are defined in Chapter 2, Section 1, and $\rho(\cdot)$ is defined in (4.37). Then for any finite $T > 0$

(i) $(\gamma_t)_{0\le t\le T}$ *is adapted.*

(ii) $\int_0^T \gamma_t^2\,dt < \infty$ *a.s.*

(iii) $\mathbb{E}\left[e^{\frac{1}{2}\int_0^T \gamma_t^2\,dt}\right] < \infty.$

Remark 4.4.1

The proof of part (iii) above is needed to satisfy the Novikov condition under which we can conclude that the process $(M_t)_{0\le t\le T}$ defined by

$$M_t = \exp\left(\int_0^t \gamma_s\,dB_s - \frac{1}{2}\int_0^t \gamma_s^2\,ds\right) \tag{4.39}$$

is a martingale.

Proof of Lemma 4.4.1: Part (i) is obvious. Define the following:

$$h(s,t) = \frac{\sigma(s)}{\sigma(t)}\frac{\partial g}{\partial t}(s,t), \tag{4.40}$$

$$q(t) = \frac{-1}{\sigma(t)}\left(e'(t) - \rho(t) + \frac{1}{2}\sigma(t)^2\right), \tag{4.41}$$

$$Y_T = \int_0^T \left(\int_0^t h(s,t)\,dB_s\right)^2 dt. \tag{4.42}$$

If

$$K(T) = \frac{M_3^2}{\underline{\sigma}^2}(2M_1 + M_2)^2 e^{2(2M_1+M_2)T}, \tag{4.43}$$

then by (3.7), (3.8), (3.17) we have

$$\sup_{0\le t\le T}\int_0^t h(s,t)^2\,ds \le K(T). \tag{4.44}$$

Using Fubini's theorem, (4.42), (4.43) and (4.44), we have

$$\mathbb{E}[Y_T] = \int_0^T \int_0^t h(s,t)^2 \, ds \, dt \leq TK(T) < \infty.$$

Since

$$\int_0^T \gamma_t^2 \, dt \leq 2 \int_0^T q(t)^2 \, dt + 2Y_T, \tag{4.45}$$

it follows that $\mathbb{E}\left[\int_0^T \gamma_t^2 \, dt\right] < \infty$, and so (ii) is true.

Furthermore, by (4.45) we have

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_t^2 \, dt\right)\right] \leq \exp\left(\int_0^T q(t)^2 \, dt\right) \mathbb{E}[e^{Y_T}],$$

so part (iii) follows if $\mathbb{E}[e^{Y_T}] < \infty$. Using Jensen's inequality, Fubini's theorem and Proposition 6.1.1, we have for each $n \in \mathbb{N}$

$$\mathbb{E}[Y_T^n] \leq T^{n-1} \int_0^T \frac{(2n)!}{n! 2^n} \left(\int_0^s h(u,s)^2 \, du\right)^n ds,$$

whereupon using (4.44) yields

$$\frac{1}{n!}\mathbb{E}[Y_T^n] \leq \frac{(2n)!}{(n!)^2 2^n} \left(TK(T)\right)^n.$$

Therefore $\sum_{n=0}^{\infty} \frac{1}{n!}\mathbb{E}[Y_T^n] < \infty$, and so $\mathbb{E}[e^{Y_T}] < \infty$, proving part (iii). $\diamond$

Lemma 4.4.2 *There exists a probability measure equivalent to $\mathbb{P}$ for which the discounted price of the risky asset, $\tilde{S}_t = S_t/S_t^0$, is a martingale.*

Proof: By Lemma 4.4.1, the process $(M_t)_{0 \leq t \leq T}$ given by (4.39) is a martingale, so the Girsanov Theorem tells us there exists a probability measure $\mathbb{P}^*$ with

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = M_T \tag{4.46}$$

under which the process $(W_t)_{0 \leq t \leq T}$ defined by

$$W_t = B_t - \int_0^t \gamma_s \, ds \tag{4.47}$$

is a standard Brownian motion. Assume temporarily that $(X_t)_{0 \leq t \leq T}$ is an Itô process. We will prove this in Lemma 4.4.3. One can use the stochastic Fubini theorem, in concert with (3.10), (3.12) and (3.18) to show

$$\lambda(t)\left(X_t - \int_0^t w(s,t) X_s \, ds\right) - \beta(t)(X_t - k(t)) = e'(t) + \int_0^t \sigma(s) \frac{\partial g}{\partial t}(s,t) \, dB_s. \tag{4.48}$$

We remark that the function $f(t,x) = e^x e^{-\int_0^t \rho(s)\,ds}$ is of class $\mathbf{C}^{1,2}$, and $f(t, X_t) = \tilde{S}_t$. Since X is an Itô process, we have

$$\tilde{S}_t = \tilde{S}_0 + \int_0^t \tilde{S}_s . - \rho(s)\,ds + \int_0^t \tilde{S}_s\,dX_s + \frac{1}{2}\int_0^t \tilde{S}_s\,d\langle X, X\rangle_s,$$

on using Itô's Lemma. Since $d\langle X, X\rangle_t = \sigma(t)^2\,dt$, we can use (4.38), (4.47), (4.48) and the invariance of the stochastic integral by change of equivalent probability measure to obtain

$$\tilde{S}_t = \tilde{S}_0 + \int_0^t \sigma(s)\tilde{S}_s\,dW_s. \tag{4.49}$$

Thus $\tilde{S}_t$ is an exponential martingale under $\mathbb{P}^*$. $\diamond$

Lemma 4.4.3 *$(X_t)_{0\le t\le T}$ given by (3.1) is an Itô process.*

Proof: Let $K_t = e'(t) + \int_0^t \sigma(s)\frac{\partial g}{\partial t}(s,t)\,dB_s$, so using Lemma 3.3.1 and (3.18) we have

$$X_t = X_0 + \int_0^t K_s\,ds + \int_0^t \sigma(s)\,dB_s,$$

and since the integrand in the stochastic integral in the expression for K_t is deterministic, $(K_t)_{0\le t\le T}$ is adapted. We also have

$$\mathbb{E}\left[\int_0^T |K_s|\,ds\right] \le \int_0^T \left(e'(s)^2 + \int_0^s \sigma(u)^2\left(\frac{\partial g}{\partial t}(u,s)\right)^2 du\right)^{\frac{1}{2}} ds < \infty,$$

so $\int_0^T |K_s|\,ds < \infty$ a.s.. Since $\sigma(\cdot)$ is continuous, $(X_t)_{0\le t\le T}$ is an Itô process. $\diamond$

4.4.2 Pricing, Replication and Hedging of European Options

In this subsection, we follow Sections 4.3.2 and 4.3.3 in Lamberton and Lapeyre [50]. A European option will be defined by a non-negative $\mathcal{F}_T$-measurable, random variable h. We will later specialise and let $h = f(S_T)$, so in the case of a call $f(x) = (x-K)_+$ and $f(x) = (K-x)_+$ in the case of a put.

Let H_t^0 be the quantity of bonds (with dynamics given by (4.37)) held at time t, and H_t be the quantity of risky asset (with log-price dynamics given by (3.1)) held at time t. We take our definition of a self-financing strategy from Section 4.1 of [50].

We define the admissible strategies:

Definition 4.4.1 *A strategy $\phi = (H_t^0, H_t)_{0\le t\le T}$ is admissible if it is self-financing and if the discounted value $\tilde{V}_t(\phi) = H_t^0 + H_t\tilde{S}_t$ of the corresponding portfolio is, for all t, non-negative and such that $\sup_{0\le t\le T}\tilde{V}_t$ is square integrable under $\mathbb{P}^*$.*

An option is replicable if its payoff at maturity is equal to the final value of an admissible strategy. We require that h is square integrable under $\mathbb{P}^*$ (where $\mathbb{P}^*$ is defined by (4.46)).

The following Proposition mirrors Theorem 4.3.2 in [50].

Proposition 4.4.1 *Suppose log-price dynamics are given by (3.1), and bond dynamics by (4.37), and $\rho(\cdot)$ is defined by (4.37). Then any option defined by a non-negative, $\mathcal{F}_T$-measurable random variable h with $\mathbb{E}^*[h^2] < \infty$, is replicable, and the value at time t of any replicating portfolio is given by*

$$V_t = \mathbb{E}^*\left[e^{-\int_t^T \rho(s)\,ds} h|\mathcal{F}_t\right].$$

Proof: In view of Lemma 4.4.2, we may follow the line of analysis in [50] with time dependent interest rates. $\diamond$

When $h = f(S_T)$, it is possible to express the option value V_t at time t as a function of t and S_t. This is Remark 4.3.3. in [50] with time dependent in place of constant interest rates.

Proposition 4.4.2 *Let $S_t = e^{X_t}$ where X_t is given by (3.1). If $h = f(S_T)$, $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $\mathbb{E}^*[f(S_T)^2] < \infty$ then*

$$V_t = F(t, S_t)$$

where

$$F(t,x) = e^{-\int_t^T \rho(s)\,ds} \int_{-\infty}^{\infty} f\left(xe^{\int_t^T \rho(s)-1/2\sigma(s)^2\,ds} e^y\right) \frac{1}{\sqrt{2\pi}\sqrt{\int_t^T \sigma(s)^2\,ds}} \exp\left(\frac{-y^2}{2\int_t^T \sigma(s)^2\,ds}\right) dy. \tag{4.50}$$

Proof: Under $\mathbb{P}^*$, we have from (4.49)

$$S_T = S_t \exp\left(\int_t^T \rho(s)\,ds + \int_t^T \sigma(s)\,dW_s - \frac{1}{2}\int_t^T \sigma(s)^2\,ds\right),$$

so by letting

$$F(t,x) = \mathbb{E}^* \left[e^{-\int_t^T \rho(s)\,ds} f\left(x \exp\left(\int_t^T \rho(s)\,ds + \int_t^T \sigma(s)\,dW_s - \frac{1}{2}\int_t^T \sigma(s)^2\,ds \right)\right)\right],$$

and noting that S_t is $\mathcal{F}_t$-measurable, and that $\int_t^T \sigma(s)\,dW_s$ is independent of $\mathcal{F}_t$, we have

$$V_t = F(t, S_t).$$

Moreover, remarking that

$$\int_t^T \sigma(s)\,dW_s \sim \mathcal{N}\left(0, \int_t^T \sigma(s)^2\,ds\right),$$

we see that $F(t,x)$ is given by (4.50). $\diamond$

Remark 4.4.2

The value of the portfolio at every time is the same as that for a market where the price dynamics are Black-Scholes with time dependent volatility $\sigma(\cdot)$. Thus, the small investor's replicating portfolio seems to be composed as if the prices in the primary market followed Black-Scholes dynamics. In fact, we will now see that when $h = f(S_T)$, that the small investor has a hedging strategy which is identical to that used when the market is Black-Scholes. In other words, the small investor neither knows, nor needs to know, the microstructure of the market, and will be unable to discern that he is hedging in a non-Black-Scholes (indeed a non–E.M.H.) environment. This adds to evidence that the Black-Scholes option pricing formula is robust to deviations from its hypotheses.

Again following [50] verbatim, we prove

Proposition 4.4.3 *Suppose $F(\cdot,\cdot)$ is given by (4.50), and $f \in \mathbf{C}^2$. Let $S_t = e^{X_t}$ where X_t is given by (3.1). Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $\mathbb{E}^*[f(S_T)^2] < \infty$. If*

$$H_t = \frac{\partial F}{\partial x}(t, S_t)$$

and

$$H_t^0 = e^{-\int_0^t \rho(s)\,ds} F(t, S_t) - H_t e^{-\int_0^t \rho(s)\,ds} S_t$$

Then the strategy $\phi = (H_t^0, H_t)_{0 \le t \le T}$ is admissible.

Proof: See p.71-72 in [50]. $\diamond$

Remark 4.4.3

For the European call with strike price K, we have, just as in the standard case

$$F(t,x) = x\Phi(d_1) - Ke^{-\int_t^T \rho(s)\,ds}\Phi(d_2)$$

where

$$d_1(t,x) = \frac{\log(x/K) + \int_t^T \rho(s)\,ds + \frac{1}{2}\int_t^T \sigma(s)^2\,ds}{\sqrt{\int_t^T \sigma(s)^2\,ds}}$$

and

$$d_2(t,x) = d_1(t,x) - \sqrt{\int_t^T \sigma(s)^2\,ds},$$

in which case $H_t = \Phi(d_1(t,S_t))$ and $H_t^0 = e^{-\int_0^t \rho(s)\,ds}F(t,S_t) - H_t e^{-\int_0^t \rho(s)\,ds}S_t$.

4.5 Volatility, Volume and Confidence

4.5.1 Volatility and Volume

In this section, we show that there is a positive relationship between the volatility in the market and the volume of trade. As mentioned in the introduction, some reseachers have examined the relationship between trading volume and volatility and found it to be positive (see Gallant, Rossi and Tauchen [34], Tauchen and Pitts [76] and Frankel and Froot [29]).

Definition 4.5.1 *For X given by (3.1), the* **instantaneous volume of chartist trade** *at time t is given by*

$$V_t^c = \left|\lambda(t)\left(X_t - \int_0^t w(s,t)X_s\,ds\right)\right|, \tag{4.51}$$

and the **instantaneous volume of fundamentalist trade** *at time t is given by*

$$V_t^f = |\beta(t)(X_t - k(t))|, \tag{4.52}$$

and the **total instantaneous volume of trade** *is given by*

$$V_t^T = V_t^c + V_t^f. \tag{4.53}$$

Remark 4.5.1

Although in this setting we cannot calculate the number of shares changing hands directly, we follow Cabrales and Hoshi [10] by allowing the value of outstanding instantaneous demand to be a measure of trading volume, and hence used to prove a volume–volatility relationship.

We define the following order like relation on volatilities:

Definition 4.5.2 *Let* $\mathcal{P} = \{\sigma : \sigma \in \mathbf{C}([0,\infty),(0,\infty)), \inf_{t\geq 0}\sigma(t) > 0\}$. *Then for any* $\sigma_1, \sigma_2 \in \mathcal{P}$ *we say* $\sigma_1 \succ \sigma_2$ *if and only if* $\sigma_1(t) > \sigma_2(t)$ *for all* $t \geq 0$.

We may now prove the proposition linking volatility and volume; more specifically, we show that the expected volume of trade is always greater whenever there is an upward shift in the volatility coefficient. Thus higher volatility seems to cause more active trading. Moreover, this increase in activity is endemic—all traders have greater expected volumes of trade.

Proposition 4.5.1 *Suppose* X^i *satisfies*

$$dX_t^i = \lambda(t)\left(X_t^i - \int_0^t w(s,t)X_s^i\,ds\right) - \beta(t)(X_t^i - k(t))\,dt + \sigma_i(t)\,dB_t,$$

and $X_0^i = x_0$ *for* $i = 1, 2$. *If*

$$\sigma_1 \succ \sigma_2$$

Then $\mathbb{E}[V_t^{c,1}] > \mathbb{E}[V_t^{c,2}]$, $\mathbb{E}[V_t^{f,1}] > \mathbb{E}[V_t^{f,2}]$ *and* $\mathbb{E}[V_t^{T,1}] > \mathbb{E}[V_t^{T,2}]$.

Proof: The result for V_t^T follows from the other two. The proofs are identical in the cases of V_t^c and V_t^f, so we only prove the result for V_t^c. For $i = 1, 2$ let

$$Y_t^i = E(t) + \int_0^t \sigma_i(s)G(s,t)\,dB_s$$

where $E(t) = \lambda(t)\left(e(t) - \int_0^t w(s,t)e(s)\,ds\right)$ and $G(s,t) = \lambda(t)\left(g(s,t) - \int_s^t w(u,t)g(s,u)\,du\right)$. Then by (3.18) and the stochastic Fubini theorem we have

$$V_t^{c,i} = |Y_t^i|$$

for $i = 1, 2$. Let $\Sigma_i(t) = \int_0^t \sigma_i(s)^2 G(s,t)^2\,ds$ for $i = 1, 2$ so that $\Sigma_1(t) > \Sigma_2(t)$ and

$$Y_t^i \sim \mathcal{N}(E(t), \Sigma_i(t)).$$

If $Z \sim \mathcal{N}(\mu, \sigma^2)$, define $f(\mu, \sigma) := \mathbb{E}[|Z|]$. A straightforward calculation shows

$$\frac{\partial f}{\partial \sigma}(\mu, \sigma) = \frac{1}{\sqrt{2\pi}} 2\sigma e^{-\mu^2/2\sigma^2} > 0,$$

so

$$\mathbb{E}[V_t^{c,1}] = f(E(t), \sqrt{\Sigma_1(t)}) > f(E(t), \sqrt{\Sigma_2(t)}) = \mathbb{E}[V_t^{c,2}],$$

which completes the proof. ◇

4.5.2 Confidence in Fundamentals Reduces Variability

We can show when the chartists have mean-reverting price expectations, that if the fundamentalists are more confident in their predictions, then the variance of the log–price is smaller. This obviously reduces the probability of a deviation of any given size from the expected value, so reducing the system's variability. We will later see that this has the effect of reducing the asymptotic large deviations of the log price. Thus, increased fundamentalist confidence makes the system more stable.

We remark that this provides a partial explanation for the success of managed exchange rate systems: if the investors are confident in the central target, there will be an decreased probability of the bands being tested. This, in turn, maintains the investors' confidence in the system, and the bands have a low probability of being tested and so on. Notice again, however, the importance of investors (of both classes) being confident in the central parity. There is empirical evidence that chartists can change from mean–reverting to extrapolative behaviour if prices become volatile (see Andreassen [3]).

In the next chapter, we will see that if chartists have extrapolative expectations and are the dominant players in the market, prices can rapidly form bubbles or crash. Our modelling thus indicates that monetary authorities must convince speculators (and in particular, fundamentalists) of the sustainability of the central target: if they are unable to do so, it is likely the band will be breached, and the target zone lose its credibility.

Proposition 4.5.2 *Suppose X^i satisfies*

$$dX_t^i = \lambda(t) \left(X_t^i - \int_0^t w(s,t) X_s^i \, ds \right) - \beta_i(t)(X_t^i - k(t))\, dt + \sigma(t)\, dB_t,$$

and $X_0^i = x_0$ *for* $i = 1, 2$. *If* $\lambda(\cdot) < 0$ *and*

$$\beta_1(t) \geq \beta_2(t) \textit{ for all } t \geq 0,$$

then

$$Var[X_t^1] \leq Var[X_t^2] \textit{ for all } t \geq 0, \tag{4.54}$$

and moreover for all $p \in \mathbb{N}$

$$\mathbb{E}\left[(X_t^1 - \mathbb{E}[X_t^1])^{2p}\right] \leq \mathbb{E}\left[(X_t^2 - \mathbb{E}[X_t^2])^{2p}\right]. \tag{4.55}$$

Proof: For $i = 1, 2$, define $g_i(\cdot, \cdot)$ by

$$\frac{\partial g_i}{\partial t}(s,t) = \lambda\left(g(s,t) - \int_s^t w(u,t) g_i(s,u)\, du\right) - \beta_i(t) g_i(s,t),$$

where $g_i(s,s) = 1$ and $g_i(s,t) = 0$ for all $s > t$. Then

$$Var[X_t^i] = \int_0^t \sigma(s)^2 g_i(s,t)^2\, ds. \tag{4.56}$$

Since by Proposition 6.1.1, $X_t \sim \mathcal{N}$, (4.55) follows directly from (4.54). If one could show for all $0 \leq s \leq t$ that

$$0 \leq g_1(s,t) \leq g_2(s,t), \tag{4.57}$$

then (4.54) would follow. Notice by the proof of Remark 8.4.1 that $g_1(s,t) \geq 0$ and $g_2(s,t) \geq 0$. Let $\Delta_s(t) = g_1(s,t) - g_2(s,t)$. Then $\Delta_s(s) = 0$. If we let $'$ denote differentiation with respect to t we have

$$\begin{aligned}
\Delta_s'(t) &= \lambda(t)\left(\Delta_s(t) - \int_s^t w(u,t)\Delta_s(u)\, du\right) - \beta_1(t)\Delta_s(t) + (\beta_2(t) - \beta_1(t)) g_2(s,t) \\
&\leq \lambda(t)\left(\Delta_s(t) - \int_s^t w(u,t)\Delta_s(u)\, du\right) - \beta_1(t)\Delta_s(t).
\end{aligned}$$

Since for $s < t$ the equation

$$z'(t) = \lambda(t)\left(z(t) - \int_s^t w(u,t) z(u)\, du\right) - \beta_1(t) z(t)$$

where $z(s) = 0$ has solution $z(t) \equiv 0$ for all $t \geq s$, by the comparison principle (see e.g., p.13 Lakshmikantham and Rao [49]), we have

$$\Delta_s(t) \leq z(t) = 0,$$

for all $t \geq s$. Thus we have proved (4.57), and so the proposition is true. $\diamond$

Chapter 5

Asymptotics of Linear Stochastic Integro-differential Equations with Separable Kernels: Dominance, Bubbles and Crashes

5.1 Introduction

In this chapter, we will completely characterise the asymptotic behaviour of the price evolution under some simplifying assumptions on the structure of the chartist weighting function. In doing so, we hope to motivate the analysis for general chartist weighting functions (see Chapter 5), and extend the analysis in directions which we cannot yet cover for arbitrary weighting functions. To do this we assume that the role of the chartist weighting function $w(\cdot,\cdot)$ will be taken by an invariant weight for $w(\cdot,\cdot)$.

Suppose first that the chartists have extrapolative expectations: we prove that if they have short memories and are confident in their predictions, then the price explodes or crashes, with both outcomes having positive probability. On the other hand, if the chartists have longer memories, and the fundamentalists are very active in the market, then the asymptotic growth rate of the price is equal to the consensus asymptotic growth rate of the fundamentalists— which also characterises the asymptotic growth rate when chartists have mean-reverting expectations. In the second section of this chapter we prove these facts using the theory of linear stochastic differential equations.

In Section 2, we show, under the conditions in which a bubble or crash is possible, that a crash is more likely when the fundamentalists revise their estimates of fair value downwards, or when the initial price is lower; and a bubble is more probable when the fundamentalist revisions are up, and the initial price higher. When the volatility is constant, we link the strength of investors and degree of feedback to the variability in the price.

In the last section, we demonstrate that the rate of increase or decay of the solution of (3.1) can be no greater than exponential, indicating that the bubble/crash mechanism is also present in the general model, under suitable circumstances.

5.1.1 The Simplified Model

We specify a very particular form for the invariant weight which will take the place of the weighting function. The substitution may be justified on the basis that, for large times, the chartists in this modified model behave in the same manner as those in the full model, and in asymptotic analysis, we are concerned in the structure of the solution for large times. Let $\mu > 0$ and define

$$a(t) = \mu e^{-\mu t}, \tag{5.1}$$

so that $a : [0, \infty) \to (0, \infty)$ is an invariant weight for a chartist weighting function. Further suppose that

$$\lambda(t) \equiv \lambda, \qquad \beta(t) \equiv \beta, \tag{5.2}$$

for some positive constants λ and β.

From (5.1), (5.2), the evolution for the log-price thus reads:

$$dX_t = \lambda \left(X_t - \int_0^t a(t-s) X_s \, ds \right) - \beta(X_t - k(t)) \, dt + \sigma(t) \, dB_t. \tag{5.3}$$

The reason for the choice of invariant weight is now transparent: if we let

$$Y_t = \int_0^t a(t-s)X_s\,ds, \tag{5.4}$$

then the 1-dimensional stochastic integro-differential equation can be transformed to a linear stochastic differential equation in two space dimensions (X, Y). To this end, define

$$A = \begin{pmatrix} \lambda-\beta & -\lambda \\ \mu & -\mu \end{pmatrix}, \tag{5.5}$$

$$b(t) = \begin{pmatrix} \beta k(t) \\ 0 \end{pmatrix}, \tag{5.6}$$

$$\Sigma(t) = \begin{pmatrix} \sigma(t) \\ 0 \end{pmatrix}. \tag{5.7}$$

Notice by (5.3), (5.4), (5.5), (5.6), (5.7) that if $Z_t = (X_t, Y_t)^T$, then

$$dZ_t = (AZ_t + b(t))\,dt + \Sigma(t)\,dB_t. \tag{5.8}$$

The analysis of the asymptotics of (5.3) thus reduces to the study of the asymptotics of the linear equation (5.8). This study, and its economic interpretation, is the primary focus of this chapter.

5.1.2 Dominance

We recall from the last chapter that the magnitude of the functions $\lambda(\cdot), \beta(\cdot)$ can be interpreted as the degree of participation in the market of the chartists and fundamentalists respectively, and that large values of these functions represent confident and liquid groups of agents. We therefore associate the case of large values of λ relative to β with a market in which chartists are more active than fundamentalists: the case of large values of β relative to λ is given a contrary association.

As indicated in Remark 4.3.4, the more heavily chartists weigh the near past, the smaller their demand is. Referring to (5.1), we see that large values of μ can be identified with "short" chartist memories or heavy weighting of the recent past, and that this myopia reduces feedback in the model. Thus large values of μ reduces the activity of chartists in the market.

The above comments motivate a definition:

Definition 5.1.1 *We say* **chartists (resp. fundamentalists) are dominant** *whenever*

$$\lambda - \beta - \mu > 0 \text{ (resp. } < 0). \tag{5.9}$$

We now show that the asymptotics of (5.8) are determined by the dominance of one or other group of speculators.

5.2 Asymptotics

5.2.1 Preliminaries

Let I_2 be the 2×2 identity matrix. If we define

$$\Phi'(t) = A\Phi(t), \qquad \Phi(0) = I_2, \tag{5.10}$$

then the solution of (5.8) is given by

$$Z_t = \Phi(t)Z_0 + \Phi(t)\int_0^t \Phi^{-1}(s)b(s)\,ds + \Phi(t)\int_0^t \Phi^{-1}(s)\Sigma(s)\,dB_s, \tag{5.11}$$

where $Z_0 = (X_0, 0)^T$.

We immediately see that the fundamental matrix Φ has components which decay exponentially whenever all the eigenvalues of A have negative real parts; and that some of its components grow exponentially whenever at least one of the eigenvalues of A has a positive real part.

Let α_1, α_2 be the eigenvalues of A; then $\alpha_1 + \alpha_2 = \lambda - \beta - \mu$ and $\alpha_1\alpha_2 = \beta\mu$, so the eigenvalues of A have positive or negative real parts according as the chartists or fundamentalists are dominant.

For simplicity, we will assume in the sequel that $\alpha_1, \alpha_2 \in \mathbb{R}$.

Define the matrix B by

$$B = P^{-1}AP = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \tag{5.12}$$

and let $(P)_{i,j} = p_{ij}$ and $(P^{-1})_{i,j} = \overline{p}_{ij}$ for $i, j = 1, 2$. Let the matrix Ψ be determined according to:

$$\Psi'(t) = B\Psi(t), \qquad \Psi(0) = I_2. \tag{5.13}$$

Let $Z_t = Z_t^1 + Z_t^2$, where Z^1, Z^2 are the deterministic and stochastic components of Z, respectively: further define

$$\tilde{Z}_t = P^{-1} Z_t, \tag{5.14}$$

and denote by $\tilde{Z}^1, \tilde{Z}^2$ the deterministic and stochastic components of $\tilde{Z}$, respectively.

5.2.2 Asymptotics under Fundamentalist and Chartist Dominance

We assume that $k \in \mathbf{C}[0,\infty)$ and

$$\lim_{t\to\infty} k(t) = \infty. \tag{5.15}$$

Let k_1 be a non-decreasing, strictly positive $\mathbf{C}^1$ function that satisfies (4.14) and

$$\lim_{t\to\infty} \frac{k_1(t)}{k(t)} = 1. \tag{5.16}$$

Moreover, we assume that $\sigma(\cdot)$ satisfies (3.7). We now have

Proposition 5.2.1 *Suppose that $(X_t)_{t\geq 0}$ is the solution of (5.3). Let $k(\cdot)$ satisfy (5.15) and (5.16), and $\sigma(\cdot)$ satisfy (3.7). If fundamentalists are dominant and*

$$\liminf_{t\to\infty} \frac{k(t)}{\sqrt{\log t}} = \infty, \tag{5.17}$$

then

$$\lim_{t\to\infty} \frac{X_t}{k(t)} = 1, \quad a.s. \tag{5.18}$$

Proof: Fundamentalist dominance prescribes $\alpha_1, \alpha_2 < 0$. Using (5.12), (5.13), (5.14), we have

$$\tilde{Z}_t^2 = \begin{pmatrix} \overline{p}_{11} e^{\alpha_1 t} \int_0^t \sigma(s) e^{-\alpha_1 s}\, dB_s \\ \overline{p}_{21} e^{\alpha_2 t} \int_0^t \sigma(s) e^{-\alpha_2 s}\, dB_s \end{pmatrix}. \tag{5.19}$$

Using the martingale time change theorem in concert with (5.17), and using the proof of Proposition 2.3.2 as a model, we obtain

$$\lim_{t\to\infty} \frac{1}{k(t)} \tilde{Z}_t^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so by (5.14), we have

$$\lim_{t\to\infty} \frac{1}{k(t)} Z_t^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{a.s.}$$

Analogously, we find

$$\tilde{Z}_t^1 = \begin{pmatrix} \overline{p}_{11} x_0 e^{\alpha_1 t} \\ \overline{p}_{21} x_0 e^{\alpha_2 t} \end{pmatrix} + \begin{pmatrix} \overline{p}_{11} \beta e^{\alpha_1 t} \int_0^t k(s) e^{-\alpha_1 s}\, ds \\ \overline{p}_{21} \beta e^{\alpha_2 t} \int_0^t k(s) e^{-\alpha_2 s}\, ds \end{pmatrix}. \tag{5.20}$$

From (4.14), (5.15), (5.16) and L'Hôpital's Rule we have for $i = 1, 2$

$$\lim_{t\to\infty} \frac{1}{e^{-\alpha_i t} k_1(t)} \int_0^t e^{-\alpha_i s} k(s)\, ds = \frac{-1}{\alpha_i},$$

and noticing that we may write P as

$$P = \begin{pmatrix} \alpha_1 + \mu & \alpha_2 + \mu \\ \mu & \mu \end{pmatrix}, \tag{5.21}$$

we obtain

$$\lim_{t\to\infty} \frac{\langle Z_t^1, e_1 \rangle}{k(t)} = 1.$$

Since $X_t = \langle Z_t^1 + Z_t^2, e_1 \rangle$ we have proved (5.18). ⋄

Therefore, when fundamentalists dominate, the asymptotic growth rate of the price is the consensus growth rate of the fundamentalists. The case of exponential growth in the fundamentals is now easily handled, and we see that the pathwise asymptotics are the same as in the Black-Scholes case.

Corollary 5.2.1 *Suppose $(S_t)_{t\geq 0}$ is the price process, and $\sigma(\cdot)$ satisfies (3.7). Let $k(\cdot)$ be a continuous function such that, for some $\eta > 0$,*

$$\lim_{t\to\infty} \frac{k(t)}{t} = \eta.$$

If fundamentalists are dominant then

$$\lim_{t\to\infty} \frac{1}{t} \log S_t = \eta, \quad a.s.$$

Proof: Just choose $k_1(t) = \eta t + 1$ and apply Proposition 5.2.1. ⋄

When the chartists dominate, the price dynamics are altogether more explosive. A similar result to the following has been proven in [26].

Proposition 5.2.2 *Suppose that $(X_t)_{t\geq 0}$ is the solution of (5.3). Let $k(\cdot)$ be of exponential order strictly less than ε for every $\varepsilon > 0$, and let $\sigma(\cdot)$ satisfy (3.7). If chartists are dominant, then*

$$\lim_{t\to\infty} \frac{X_t}{e^{\alpha_1 \vee \alpha_2 t}} = R_\infty \sim \mathcal{N}\left(\mu_\infty, \sigma_\infty^2\right) \quad a.s. \tag{5.22}$$

where

$$\mu_\infty = \frac{\alpha_1 \vee \alpha_2 + \mu}{\alpha_1 \vee \alpha_2 - \alpha_1 \wedge \alpha_2}\left(x_0 + \int_0^\infty \beta k(s) e^{-\alpha_1 \vee \alpha_2 s}\, ds\right) \quad and \tag{5.23}$$

$$\sigma_\infty^2 = \left(\frac{\alpha_1 \vee \alpha_2 + \mu}{\alpha_1 \vee \alpha_2 - \alpha_1 \wedge \alpha_2}\right)^2 \int_0^\infty \sigma(s)^2 e^{-2\alpha_1 \vee \alpha_2 s}\, ds. \tag{5.24}$$

Moreover

$$\lim_{t\to\infty} \frac{1}{t} \log |X_t| = \alpha_1 \vee \alpha_2 \quad a.s. \tag{5.25}$$

Proof: Notice if chartists dominate then $\alpha_1, \alpha_2 > 0$. Without loss of generality, we will assume $\alpha_1 > \alpha_2$. From (5.20), we have

$$\frac{\langle Z_t^1, e_1\rangle}{e^{\alpha_1 t}} = p_{11}\overline{p}_{11}\left(x_0 + \int_0^t \beta k(s) e^{-\alpha_1 s}\, ds\right) + p_{12} e^{-(\alpha_1-\alpha_2)t}\left(\overline{p}_{21} x_0 + \overline{p}_{21}\int_0^t \beta k(s) e^{-\alpha_2 s}\, ds\right), \tag{5.26}$$

and using (5.19) we have

$$\frac{\langle Z_t^2, e_1\rangle}{e^{\alpha_1 t}} = p_{11} M_t^1 + p_{12} e^{-(\alpha_1-\alpha_2)t} M_t^2, \tag{5.27}$$

where

$$M_t^1 = \int_0^t \overline{p}_{11}\sigma(s) e^{-\alpha_1 s}\, dB_s \tag{5.28}$$

$$M_t^2 = \int_0^t \overline{p}_{21}\sigma(s) e^{-\alpha_2 s}\, dB_s. \tag{5.29}$$

Since for $i = 1, 2$, M_t^i is a right continuous martingale, $M_0^i = 0$ a.s. and

$$\mathbb{E}\left[\left(M_t^i\right)^2\right] < \infty \quad \text{for all } t \geq 0,$$

then $M^i \in \mathcal{M}_2$ and $\{M_t^i; \mathcal{F}_t; 0 \leq t < \infty\}$ is uniformly integrable. Therefore the Martingale Convergence Theorem allows us to conclude that there exists M_∞^i such that

$$\lim_{t\to\infty} M_t^i = M_\infty^i \quad \text{a.s.}, \tag{5.30}$$

where

$$M_\infty^i \sim \mathcal{N}\left(0, \int_0^\infty \overline{p}_{i1}{}^2 \sigma(s)^2 e^{-2\alpha_i s}\, ds\right). \tag{5.31}$$

Combining (5.26), (5.27), (5.30) and (5.31) gives the desired result.

To prove (5.25), let

$$R_t = \frac{\langle Z_t^1 + Z_t^2, e_1\rangle}{e^{\alpha_1 t}}.$$

Then

$$\frac{1}{t}\log|X_t| - \alpha_1 = \frac{1}{t}\log|R_t|,$$

and since $R_t \to R_\infty$ a.s. as $t \to \infty$ and $\mathbb{P}[R_\infty = 0] = 0$, we have

$$\lim_{t\to\infty}\frac{1}{t}\log|R_t| = 0 \quad \text{a.s.. } \diamond$$

Remark 5.2.1

As in the paper of Schweizer and Föllmer, we remark that not only is the rate of decay or growth very extreme, but growth or decay are both possible with positive probability. By (5.22), (5.23) and (5.24) we see that

$$\mathbb{P}[\lim_{t\to\infty} S_t = \infty] = \mathbb{P}[\lim_{t\to\infty} X_t = \infty] = \mathbb{P}[R_\infty > 0] = 1 - \Phi\left(\frac{-\mu_\infty}{\sigma_\infty}\right), \tag{5.32}$$

and

$$\mathbb{P}[\lim_{t\to\infty} S_t = 0] = \Phi\left(\frac{-\mu_\infty}{\sigma_\infty}\right), \tag{5.33}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\,du$. Naturally, no asset price has ever behaved in this manner over long periods of time, but this analysis provides some insights on the psychology of markets "close to" a crisis or bubble. We can easily prove

Proposition 5.2.3 *Let X^i be a solution of (5.3) for $i = 1, 2$, and $S_t^i = \exp(X_t^i)$. Then we have:*

(i) (Monotonicity in fundamentals) Suppose

$$dX_t^i = \lambda\left(X_t^i - \int_0^t a(t-s)X_s^i\,ds\right) - \beta(X_t^i - k^i(t))\,dt + \sigma(s)\,dB_s$$

and $X_0^i = x_0$ for $i = 1, 2$. If $k^1(t) \geq k^2(t)$ then

$$\mathbb{P}\left[\lim_{t\to\infty} S_t^1 = \infty\right] \geq \mathbb{P}\left[\lim_{t\to\infty} S_t^2 = \infty\right].$$

(ii) (Monotonicity in initial prices) If X^1 and X^2 have the same dynamics, and

$$X_0^1 \geq X_0^2,$$

then

$$\mathbb{P}\left[\lim_{t\to\infty} S_t^1 = \infty\right] \geq \mathbb{P}\left[\lim_{t\to\infty} S_t^2 = \infty\right].$$

Proof: Use the monotonicity of Φ together with (5.23), (5.24) and (5.32). $\diamond$

Remark 5.2.2

Proposition 5.2.3 tells us the following: if the fundamentalists are less optimistic about the underlying value of the asset, there is a greater probability of a crash, even if all other factors remain unchanged. Therefore, even though fundamentalists are dominated by chartists, they can precipitate a crisis. This seems to be quite plausible in explaining actual financial crises. A group of speculators lose confidence in their estimates of the fundamentals, (β goes down, so chartists are dominant) and believe they should revise their estimates of the fundamentals downwards ($k^{new}(\cdot) \leq k^{old}(\cdot)$). The chartists, using their ability to track prices, discern this downward revision, and extrapolate prices downwards. More and more speculators make this extrapolation, and the co-incidence of chartist dominance with the effects of feedback cause the price to drop precipitiously (Propostion 5.2.3).

In the same manner, lower starting values of the price make a crash more likely. In this case, the chartists interpret a lower starting value as a sign that future prices will also be low. Their dominance, in conjunction with feedback, produces the crash.

In respect of the preceeding comments, the following remark of Kindleberger [43] quoted in Lux [54] is quite intersting; Lux writes that "in Kindleberger's theory the period of distress preceeds the ultimate crash. For this period" (he now quotes from [43], p.109)

'... a change in expectations from a state of confidence to one lacking confidence in the future is central'.

Remark 5.2.3

In the case where the fundamentalists are dominant, and $\sigma(\cdot) \equiv \sigma$, we have

$$v_\infty(\lambda, \beta, \mu) := \lim_{t\to\infty} Var[X_t] = \frac{1}{2}\sigma^2 \frac{\beta+\mu}{\beta(\beta+\mu-\lambda)}.$$

It is easy to show that $v_\infty(\cdot,\cdot,\cdot)$ is increasing in λ and decreasing in β and μ. We therefore see that increasing fundamentalist activity or confidence reduces volatility, while increasing chartist activity increases volatility, when chartists have trend chasing expectations. Moreover, if the chartists are less myopic, and introduce more feedback from past prices into the present, they increase the instability in the market.

5.3 Limits on the Growth of the Solution of the S.I.D.E.

We show in this section when the kernel has no particular structure that the fastest possible growth or decay rate for the price is iterated exponential. This indicates that Proposition 5.2.2 should have an analogue in the general case.

Proposition 5.3.1 *Suppose for every $\varepsilon > 0$ that $k(\cdot)$ is of exponential order strictly less than ε. Then*

$$\limsup_{t\to\infty} \frac{1}{t}\log|X_t| \le 2M_1 + M_2 \quad a.s., \tag{5.34}$$

where M_1 and M_2 are given by (3.5) and (3.6).

Proof: If $g(\cdot,\cdot)$ is given by (3.10) and (3.11), by Theorem 3.3.1, we have

$$X_t = e(t) + \int_0^t \sigma(s)g(s,t)\,dB_s,$$

and by (3.12), (3.13)

$$e(t) = x(0)g(0,t) + \int_0^t \beta(s)k(s)g(s,t)\,ds.$$

By (3.14) there exists $K_1 \in (0,\infty)$ such that

$$\frac{|e(t)|}{e^{(2M_1+M_2)t}} \le |x(0)| + M_2\int_0^t |k(s)|e^{-(2M_1+M_2)s}\,ds \le K_1,$$

where we used (3.6) to obtain the first bound. Now define

$$H(s,t) = \frac{\sigma(s)}{K_1} \frac{g(s,t)}{e^{(2M_1+M_2)t}}. \tag{5.35}$$

Then, since $\log(1+x) \leq x$ for all $x \geq 0$, we have

$$\begin{aligned} \frac{1}{t}\log|X_t| - (2M_1 + M_2) &\leq \frac{1}{t}\left(\log K_1 + \log\left(1 + \left|\int_0^t H(s,t)\,dB_s\right|\right)\right) \\ &\leq \frac{1}{t}\log K_1 + \frac{1}{t}\left|\int_0^t H(s,t)\,dB_s\right|, \end{aligned}$$

so if we let

$$Y_t = \frac{1}{t}\int_0^t H(s,t)\,dB_s, \tag{5.36}$$

the proposition is true provided

$$\lim_{t\to\infty} Y_t = 0 \quad \text{a.s.} \tag{5.37}$$

We now prove (5.37).

From (3.7), (3.14) and (5.35), if we let $v(t) = \int_0^t H(s,t)^2\,ds$, then

$$v(t) \leq \frac{M_3^2}{K_1^2}\frac{1}{2(2M_1 + M_2)}.$$

By Proposition 6.1.1, $Y_t \sim \mathcal{N}(0, v(t)/t^2)$, so for every $\varepsilon > 0$ and $n \in \mathbb{N}$, we have

$$\mathbb{P}[|Y_n| > \varepsilon] \leq \frac{1}{\varepsilon^2}\frac{M_3^2}{K_1^2}\frac{1}{2(2M_1 + M_2)}\frac{1}{n^2},$$

so

$$\lim_{n\to\infty} Y_n = 0 \quad \text{a.s.} \tag{5.38}$$

By Lemma 3.3.1, (5.35) and (5.36) we have

$$Y_t = \frac{1}{t}\int_0^t H(s,s)\,dB_s + \frac{1}{t}\int_0^t\int_0^s \frac{\partial H}{\partial t}(u,s)\,dB_u\,ds,$$

and obviously $\lim_{t\to\infty}\frac{1}{t}\int_0^t H(s,s)\,dB_s = 0$ a.s. Let

$$Z_t = \frac{1}{t}\int_0^t\int_0^s \frac{\partial H}{\partial t}(u,s)\,dB_u\,ds, \tag{5.39}$$

so (5.37) is true if and only if $\lim_{t\to\infty} Z_t = 0$. However, by (5.38), $\lim_{n\to\infty} Z_n = 0$ a.s., so if we show

$$\lim_{n\to\infty}\sup_{n\leq t\leq n+1}|Z_t - Z_n| = 0,$$

we will have proved (5.37). Next, since

$$Z_t - Z_n = \left(\frac{n}{t} - 1\right) Z_n + \frac{1}{t}\int_n^t \int_0^s \frac{\partial H}{\partial t}(u,s)\, dB_u\, ds,$$

we are done provided

$$\lim_{n\to\infty} \sup_{n\le t\le n+1} \left| \frac{1}{t}\int_n^t \int_0^s \frac{\partial H}{\partial t}(u,s)\, dB_u\, ds \right| = 0 \quad \text{a.s.} \tag{5.40}$$

To prove (5.40), notice that (3.15) gives

$$\left(\frac{\partial H}{\partial t}(s,t)\right)^2 \le \frac{M_3^2(2M_1+M_2)^2}{K_1^2} e^{-2(2M_1+M_2)s}.$$

We prove (5.40) using a Borel-Cantelli argument; bound as follows:

$$\begin{aligned}
&\mathbb{E}\left[\sup_{n\le t\le n+1} \left(\frac{1}{t}\int_n^t \int_0^s \frac{\partial H}{\partial t}(u,s)\, dB_u\, ds\right)^2\right] \\
\le\ & \frac{1}{(n+1)^2}\mathbb{E}\left[\sup_{n\le t\le n+1} (t-n)\int_n^t \int_0^s \left(\frac{\partial H}{\partial t}(u,s)\, dB_u\right)^2 ds\right] \\
\le\ & \frac{1}{(n+1)^2}\mathbb{E}\left[\int_n^{n+1} \int_0^s \left(\frac{\partial H}{\partial t}(u,s)\, dB_u\right)^2 ds\right] \\
=\ & \frac{1}{(n+1)^2}\int_n^{n+1} \int_0^s \frac{\partial H}{\partial t}(u,s)^2\, du\, ds \\
\le\ & \frac{1}{(n+1)^2}\frac{M_3^2(2M_1+M_2)}{2K_1^2}.
\end{aligned}$$

Using Chebyshev's inequality along with the Borel-Cantelli Lemma allows us to conclude that (5.40) is true, proving the Proposition. ◇

This proof motivates another question: is there an analogue of Proposition 5.2.1 for general chartist weighting functions? The answer, fortunately, is yes, when $\beta(\cdot)$ is a bit bigger than $2\lambda(\cdot)$. The proof of this assertion is the subject of the next chapter.

Chapter 6

Pathwise and p^{th} mean Asymptotics of Linear S.I-D.Es

6.1 Introduction

6.1.1 Ouline of the Chapter

Since the seminal papers of Berger and Mizel [6], [7], [8] the properties of stochastic integral and integro-differential equations have attracted the attention of several authors e.g., [63]. In particular, conditions under which such equations have stable solutions, have been studied by Pachpatte [59], in which stability in mean-square and probability have been established. Stability has also been considered by Zan Kan and Zhang [80]. However, since both papers approach very general problems in abstract settings, no stronger convergence results have been achieved.

In this chapter, we study the class of linear stochastic integro-differential equations introduced in Chapter 3 whose structure is intended to reflect the price dynamics in a single asset financial market in which the underlying determinant of the asset's price increases in an exponential-like manner, and in which a portion of the agents trading at any given time use the historical trend of the price as a guide to its future path. In this chapter, we assume that the chartists either have mean-reverting price formations expectations hypotheses, or that they are less dominant than the fundamentalists.

Here, we show that if the chartists use the weighting function endowed upon them in Chapter 4, and the underlying determinant of price increases, for example, exponentially, the price follows the underlying determinants asymptotically. In this, the price mirrors the pathwise asymptotics of the celebrated Black-Scholes equation. Moreover, this result extends present knowledge on pathwise behaviour of stochastic integro-differential equations, as the asymptotic convergence is almost sure. Furthermore, asymptotic convergence is also assured in p^{th} mean, for any $p \in [1, \infty)$.

The variation of parameters solution to the S.I.D.E., allows us to analyse the asymptotic behaviour of X_t by partitioning it into two parts, one of which is completely deterministic, and governed by a deterministic integro-differential equation. Its asymptotics are studied in Section 3, using the converse theory of Liapunov functions. Since it can be shown, for fixed t, that the random component of X_t is normally distributed and can be written as a stochastic integral with adapted, but $t-$ dependent, deterministic integrand, we can employ the theory of stochastic integration for integrals with non-adapted integrands and use the properties of their semimartingale decompositions to show that the random component of X_t is asymptotically negligible relative to its deterministic component. This proof is the subject of Section 6.3, together with the proof of the main theorem. Some generalisations of the main result and remarks as to the time consistency properties that this result indicates, completes the chapter.

6.1.2 Mathematical Preliminaries

We assume $k \in \mathbf{C}[0, \infty)$ satisfies (5.15), and that k_1 is a non-decreasing, strictly positive $\mathbf{C}^1$ function that satisfies (4.14) and (5.16). Moreover, suppose that $\lambda(\cdot)$, $\beta(\cdot)$ and $\sigma(\cdot)$ are in $\mathbf{C}(\mathbb{R}^+ \cup \{0\})$, and satisfy (3.5), (3.6), (3.7) and (3.8).

We assume $w(\cdot, \cdot)$ is a chartist weighting function, and that $a(\cdot)$ is an invariant weight for $w(\cdot, \cdot)$.

Remark 6.1.1

Here $k(\cdot)$ is the logarithm of the fundamental economic process tracked by the fundamentalist traders. Notice that exponential growth in this fundamental process is equivalent to letting $k(t) = \eta t + \nu$ for some $\eta > 0$ and $\nu \in \mathbb{R}$, and so the hypotheses on $k(\cdot), k_1(\cdot)$ are satisfied.

As before, we assume that we have an underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Let $(X_t)_{t\geq 0}$ model the log-price of the asset at time t, and be a continuous time stochastic process adapted to the filtration. We assume that it follows (3.1), viz.,

$$dX_t = \lambda(t)\left(X_t - \int_0^t w(s,t)X_s\,ds\right) - \beta(t)\left(X_t - k(t)\right)\,dt + \sigma(t)\,dB_t, \tag{6.1}$$

$$X_0 = x_0. \tag{6.2}$$

The following result will also be used in the proof of the asymptotic behaviour of $(X_t)_{t\geq 0}$.

Proposition 6.1.1 *Let h be a locally bounded deterministic function which satisfies*

$$h : D \to \mathbb{R} : (s,t) \mapsto h(s,t).$$

Let $t \in [0,T]$. If

$$\int_0^t h(s,t)^2\,ds < \infty$$

and $X_t = \int_0^t h(s,t)\,dB_s$, then

$$X_t \sim \mathcal{N}\left(0, \int_0^t h(s,t)^2\,ds\right). \tag{6.3}$$

Proof: Fix $t \in \mathbb{R}^+$, and recall that Itô integrals with deterministic integrands are normal. ⋄

Further suppose that $e(\cdot)$ is determined by (3.12) and (3.13), and that $g(\cdot,\cdot)$ is given by (3.10) and (3.11). Referring now to Theorem 3.3.1, and in particular to (3.18), we will prove that

$$\lim_{t\to\infty} \frac{X_t}{k(t)} = 1 \quad \mathbb{P}\text{-a.s.}$$

(under some as yet unspecified conditions on $\beta(\cdot)$ and $\lambda(\cdot)$) by proving

$$\lim_{t\to\infty} \frac{e(t)}{k(t)} = 1, \tag{6.4}$$

and then that

$$\lim_{t\to\infty} \frac{1}{k(t)} \int_0^t \sigma(s)g(s,t)\,dB_s = 0 \qquad \mathbb{P}\text{-a.s..} \tag{6.5}$$

The purpose of Section 6.2 is to prove (6.4), while that of Section 6.3 is to prove (6.5), and then present the theorem.

6.2 Asymptotics of Linear Integro-Differential Equations

6.2.1 Notation

Assume for all $t \geq 0$ that

$$\inf_{t \geq 0} \beta(t) - \lambda(t) - \sup_{s \geq 0} |\lambda(s)| := \alpha_1 > 0, \tag{6.6}$$

from which it follows

$$\inf_{t \geq 0} \beta(t) := \alpha_2 > 0. \tag{6.7}$$

Remark 6.2.1

In terms of the economic model, condition (6.7) is equivalent to saying that the fundamentalist speculators do not believe prices should deviate from $k(\cdot)$, while (6.6) indicates that the fundamentalist speculators are dominant in the market. We interpret $\beta(\cdot)$ and $\lambda(\cdot)$ as measures of the strength or confidence of the agents. Large absolute values of these functions are consistent with agents who are confident in their predictions of price, and have the ability to trade heavily on the basis on these predictions.

Recall that $k_1(\cdot)$ is a positive non-decreasing $\mathbf{C}^1$ function which has the same asymptotic behaviour as the fundamentals $k(\cdot)$ in the sense that

$$\begin{aligned} \lim_{t\to\infty} k_1(t) &= \infty, \\ \lim_{t\to\infty} \frac{k(t)}{k_1(t)} &= 1. \end{aligned}$$

Moreover, remember that the growth restrictions on $k(\cdot)$ stipulate that

$$\lim_{t\to\infty} \frac{k_1'(t)}{k_1(t)} = 0. \tag{6.8}$$

Therefore, if α_1, α_2 are as defined in (6.6), (6.7), then by virtue of (6.8)

$$T_1 = \sup\{t \geq 0 \ : \ \left|\frac{k_1'(t)}{k_1(t)}\right| = \frac{\alpha_1 \wedge \alpha_2}{2}\} \tag{6.9}$$

is finite. One can then define

$$\begin{aligned}
K_0 &= \max_{0 \le t \le T_1} k_1(t), \\
K_1 &= \min_{0 \le t \le T_1} |k_1'(t)|, \\
K_2 &= \max\{1, \frac{K_1}{\alpha_1/2} - K_0, \frac{K_1}{\alpha_2/2} - K_0\}.
\end{aligned} \tag{6.10}$$

Lemma 6.2.1 *Let*

$$p_1(t) = \frac{k_1'(t)}{k_1(t) + K_2}, \tag{6.11}$$

where K_2 is defined by (6.10). Then for all $t \ge 0$

(i) $\sup_{t \ge 0}\{\lambda(t) + \sup_{s \ge 0} |\lambda(s)| - \beta(t) - p_1(t)\} < 0$,

(ii) $\sup_{t \ge 0}\{-\beta(t) - p_1(t)\} < 0$.

Proof: (ii) has identical proof to (i). By (6.6), if $|p_1(t)| \le \alpha_1/2$, then (i) is true. We prove this on each of the intervals $[T_1, \infty)$ and $[0, T_1]$ in turn, where T_1 is defined by (6.9).

For $t \ge T_1$, since $K_2 \ge 1$, k_1 non-decreasing and positive, it follows from (6.9) and (6.11) that

$$|p_1(t)| = \frac{k_1'(t)}{k_1(t) + K_2} \le \left|\frac{k_1'(t)}{k_1(t)}\right| \le \frac{\alpha_1}{2}.$$

For $t \in [0, T_1]$, the definitions of K_0 and K_1 and (6.11) yield

$$|p_1(t)| \le \frac{\min_{0 \le t \le T_1} |k_1'(t)|}{\max_{0 \le t \le T_1} k_1(t) + K_2} = \frac{K_1}{K_0 + K_2} \le \frac{\alpha_1}{2},$$

where (6.10) is used to obtain the last bound, proving the desired result. ⋄

Let

$$\alpha(s,t) = \frac{w(s,t)(k_1(s) + K_2)}{k_1(t) + K_2}, \tag{6.12}$$

and for $t > 0$

$$p_2(t) = \lambda(t)\left(1 - \int_0^t \alpha(s,t)\, ds\right) - p_1(t) - \beta(t)\left(1 - \frac{k(t)}{k_1(t) + K_2}\right). \tag{6.13}$$

where $p_1(\cdot)$ is defined by (6.11). Since

$$\lim_{t \downarrow 0} \int_0^t \alpha(s,t)\, ds = 1,$$

if we define

$$p_2(0) = -p_1(0) - \beta(0)\left(1 - \frac{k(0)}{k_1(0) + K_2}\right),$$

then $p_2(\cdot)$ is continuous on $[0, \infty)$.

Define

$$z(t) = \frac{e(t)}{k_1(t) + K_2}, \tag{6.14}$$

where K_2 is given by (6.10). Then $z(\cdot)$ defined in (6.14) above satisfies

$$z'(t) = \lambda(t)\left(z(t) - \int_0^t \alpha(s,t)z(s)\,ds\right) + z(t)\left(-\beta(t) - p_1(t)\right) + p_2(t). \tag{6.15}$$

From Lemma 4.2.2 and (6.12) we have

$$\lim_{t\to\infty} \int_0^t \alpha(s,t)\,ds = 1. \tag{6.16}$$

Using (4.14), (5.16), (3.5), (3.6) and (6.16) we see that

$$\lim_{t\to\infty} p_i(t) = 0, \quad i = 1, 2. \tag{6.17}$$

Define $z_1(\cdot)$ by

$$z_1'(t) = \lambda(t)\left(z_1(t) - \int_0^t \alpha(s,t)z_1(s)\,ds\right) + z_1(t)\left(-\beta(t) - p_1(t)\right). \tag{6.18}$$

We finally observe from (6.12) that

$$\alpha \in \mathbf{C}(D, \mathbb{R}^+), \tag{6.19}$$

and

$$0 \le \int_0^t \alpha(s,t)\,ds \le 1. \tag{6.20}$$

In this section we will prove

Proposition 6.2.1 *Suppose (6.6), (6.7) hold. If $e(\cdot)$ is the solution to (3.12) then*

$$\lim_{t\to\infty} \frac{e(t)}{k(t)} = 1. \tag{6.21}$$

By (5.16), (6.21) is true if

$$\lim_{t\to\infty} z(t) = 0. \tag{6.22}$$

To prove this result, we have recourse to the theory of linear integrodifferential equations.

6.2.2 Some Elements of Integro–Differential Equation Theory

In the following we adopt the notation of Lakshmikantham and Rao (1995) [49] p. 145-157. We specialise and generalise their results liberally. For any $\phi \in \mathbf{C}(\mathbb{R}^+, \mathbb{R})$, define $|\phi|_t = \max_{0 \le s \le t} |\phi(s)|$. Consider the deterministic scalar Volterra equation

$$x'(t) = F(t, x(\cdot)) \tag{6.23}$$

where $F : \mathbb{R}^+ \times \mathbf{C}(\mathbb{R}^+, \mathbb{R}) \to \mathbb{R}$ is continuous and $x(\cdot)$ represents the function x on $[0, t]$ with the values of t always determined by the first co-ordinate of F in (6.23). The solution of (6.23) with initial values (t_0, ϕ) will be denoted by $x(t, t_0, \phi)$, by which we mean $x(t, t_0, \phi) = \phi(t)$ for all $t \le t_0$, where $t_0 \ge 0$ and $\phi : [0, t_0] \to \mathbb{R}$ is a continuous function.

If we define F_i for $i = 1, 2$ by

$$F_1(t, x(\cdot)) = \lambda(t) \left(x(t) - \int_0^t \alpha(s,t) x(s)\, ds \right) + x(t) \left(-\beta(t) - p_1(t) \right), \tag{6.24}$$

and

$$F_2(t, x(\cdot)) = \lambda(t) \left(x(t) - \int_0^t \alpha(s,t) x(s)\, ds \right) + x(t) \left(-\beta(t) - p_1(t) \right) + p_2(t), \tag{6.25}$$

then F_i for $i = 1, 2$ satisfy the hypothesis on F after (6.23), where $\alpha(\cdot, \cdot), p_1(\cdot), p_2(\cdot)$ are as defined by (6.12), (6.11) and (6.13) above.

Let $W(t, \phi) : \mathbb{R}^+ \times \mathbf{C}(\mathbb{R}^+, \mathbb{R}) \to \mathbb{R}$ be a continuous functional satisfying the property of local Lipschitz continuity in ϕ.

Definition 6.2.1 *The derivative of $W(t, \phi)$ with respect to (6.23) is defined by*

$$W'_{(6.23)}(t, \phi) = \limsup_{h \to 0+} \frac{W(t + h, \phi^*) - W(t, \phi)}{h} \tag{6.26}$$

where

$$\phi^*(s) = \begin{cases} \phi(s) & \text{on } 0 \le s \le t, \\ \phi(s) + F(t, \phi)(s - t) & \text{on } t \le s \le t + h. \end{cases}$$

For future reference, we define the homogeneous evolution

$$x'(t) = F_1(t, x(\cdot)). \tag{6.27}$$

We also clarify the notion of exponential asymptotic stability:

Definition 6.2.2 *The zero solution of (6.27) is exponentially asymptotically stable (ExAS) if there exist positive constants* κ, K *such that*

$$|x(t, t_0, \phi)| \leq K e^{-\kappa(t-t_0)} |\phi|_{t_0}, \qquad \text{for all } t_0 \geq 0 \text{ and } t \geq t_0.$$

Let $t \geq 0$, $\phi \in \mathbf{C}([0,t], \mathbb{R})$ and $p(t,\phi) = \sup_{t-r \leq s \leq t} |\phi(s)|$, where $r \geq 0$ and $\phi(s) = \phi(0)$ for $s \leq 0$. We then have the following theorem (see Lakshmikantham and Rao [49] p.148).

Theorem 6.2.1 *Suppose the zero solution of (6.27) is exponentially asymptotically stable. Then there exists a continuous functional* $W(t,\phi)$ *defined for* $t \geq 0$ *and* $\phi \in \mathbf{C}([0,t], \mathbb{R})$, $\kappa > 0$, *and* $K > 0$ *such that*

(i) $p(t,\phi) \leq W(t,\phi) \leq K|\phi|_t$;

(ii) $|W(t,\phi) - W(t,\psi)| \leq K|\phi - \psi|_t$ *for* $\phi, \psi \in \mathbf{C}([0,t], \mathbb{R})$;

(iii) $|W'_{(6.24)}(t,\phi)| \leq -\kappa W(t,\phi)$;

Proof: By setting

$$W(t,\phi) = \sup_{t_0 \geq 0} p(t + t_0, x(\cdot, t, \phi)) e^{\kappa t_0},$$

we see that $W(\cdot,\cdot)$ satisfies (i)-(iii) in the statement of the proof above. The line of proof is analagous to that given in Yoshizawa (1966) [79]. ⋄

Moreover, the following is true

Proposition 6.2.2 *The zero solution of a linear deterministic integro-differential equation is exponentially asymptotically stable if and only if it is uniformly asymptotically stable.*

Proof: Combine the analysis of Theorem 2.2.2 on p.54–55 in [49] with that contained in the monograph of Yoshizawa [79] p.29. ⋄

6.2.3 Asymptotic Equivalence of $e(\cdot)$ and $k(\cdot)$

The aim of this subsection is to prove, under (6.6) and (6.7) that Proposition 6.2.1 is true. A first step towards achieving this goal is to prove

Lemma 6.2.2 *The zero solution of (6.18) is exponentially asymptotically stable.*

To do this we prove a sequence of subsidiary results. Consider the homogeneous equation

$$x'(t) = \lambda(t)\left(x(t) - \int_0^t w(s,t)x(s)\,ds\right) - (\beta(t) + p_1(t))x(t). \tag{6.28}$$

Then

Lemma 6.2.3 *The solutions of (6.18), (6.28) are uniformly bounded.*

Proof: The proofs are identical: we prove the lemma for (6.18). Let $M(t) = \max_{0\le s\le t}|z_1(s)|$. Suppose $z_1(t_1) = M(t_1)$. If $\lambda(t_1) > 0$ then

$$z_1'(t_1) \le (2\lambda(t_1) - \beta(t_1) - p_1(t_1))M(t_1) < 0, \tag{6.29}$$

by Lemma 6.2.1, (6.19) and (6.20). Conversely, if $\lambda(t_1) < 0$, then

$$z_1'(t_1) \le (-\beta(t_1) - p_1(t_1))M(t_1) < 0,$$

so $z_1'(t_1) < 0$ whenever $z_1(t_1) = \max_{0\le s\le t_1}|z_1(s)|$. In the same manner, one can show that $z_1'(t_1) > 0$ whenever $z_1(t_1) = -\max_{0\le s\le t_1}|z_1(s)|$. In other, words, when $z_1(\cdot)$ achieves its running maximum, it decreases: and when $z_1(\cdot)$ achieves its running minimum, it increases. Thus for all $t \ge 0$,

$$|z_1(t)| \le |z_1(0)|,$$

or, if $z_1(t) = \phi(t)$ for all $t \le t_0$, we have

$$|z_1(t, t_0, \phi)| \le |\phi|_{t_0}.$$

so the Lemma is proven. ◇

Next we have

Lemma 6.2.4 *The zero solution of*

$$x'(t) = \lambda(t)\left(x(t) - \int_0^t a(t-s)x(s)\,ds\right) - (\beta(t) + p_1(t))x(t) \tag{6.30}$$

is exponentially asymptotically stable.

Proof: The following definition is standard: for $m \in \mathbf{C}(\mathbb{R}^+, \mathbb{R})$ we have

$$D_- m(t) = \liminf_{h \to 0^-} \frac{m(t+h) - m(t)}{h}.$$

If $x(\cdot)$ is the solution to (6.30), then

$$\begin{aligned} D_-|x(t)| &\leq (\lambda(t) - \beta(t) - p_1(t))|x(t)| + |\lambda(t)| \int_0^t a(t-s)|x(s)|\,ds \\ &\leq (\lambda(t) - \beta(t) - p_1(t))|x(t)| + M_1 \int_0^t a(t-s)|x(s)|\,ds, \end{aligned}$$

where we use (3.5). Let $z(\cdot)$ be the maximal solution of

$$z'(t) = (\lambda(t) - \beta(t) - p_1(t))z(t) + M_1 \int_0^t a(t-s)z(s)\,ds \tag{6.31}$$

with $z(t) = |\phi(t)|$ for all $t \in [0, t_0]$. If $x(t, t_0, \phi)$ denotes the solution of (6.30) with $x(t) = \phi(t)$ for $t \in [0, t_0]$, then

$$|x(t, t_0, \phi)| \leq z(t, t_0, |\phi|) \tag{6.32}$$

by the comparison principle (see e.g., p.13 Lakshmikantham and Rao [49]). We notice from Theorem 2.2.2 on p.154 of [49] that if the solution to (6.31) is $\mathbf{L}^1(\mathbb{R}^+)$, then the zero solution of (6.31) is ExAS. Taking this deduction in conjunction with (6.32) proves the Lemma. A trivial modification to the analysis on p.127 of [49] and (6.6) show that (6.31) is $\mathbf{L}^1(\mathbb{R}^+)$. ◇

Now we show

Lemma 6.2.5 *The zero solution of (6.28) is exponentially asymptotically stable.*

Proof: Let $x(\cdot)$ be a solution of (6.28),

$$g(t, x(\cdot)) = \lambda(t) \int_0^t \left(\frac{w(s,t)}{a(t-s)} - 1\right) a(t-s)x(s)\,ds. \tag{6.33}$$

and

$$h_1(t) = \sup_{0 \leq s \leq t} \left|\frac{w(s,t)}{a(t-s)} - 1\right|. \tag{6.34}$$

Then we have

$$x'(t) = \lambda(t)\left(x(t) - \int_0^t a(t-s)x(s)\,ds\right) - (\beta(t) + p_1(t))x(t) + g(t, x(\cdot)). \tag{6.35}$$

We also have from Lemma 6.2.3 that

$$|x(t, t_0, \phi)| \leq |\phi|_{t_0} \tag{6.36}$$

By Lemma 6.2.4 the zero solution to (6.30) is ExAS. Therefore, Theorem 6.2.1 allows us to conclude that there exists a continuous functional $W(t, \psi)$ defined for $t \geq 0$ and $\psi \in \mathbf{C}([0,t], \mathbb{R})$, and constants $K > 0$, $\kappa > 0$ such that

(i) $|\psi(t)| \leq W(t, \psi) \leq K|\psi|_t$;

(ii) $|W(t, \psi_1) - W(t, \psi_2)| \leq K|\psi_1 - \psi_2|_t$ for $\psi_1, \psi_2 \in \mathbf{C}([0,t], \mathbb{R})$;

(iii) $W'_{(6.30)}(t, \psi) \leq -\kappa W(t, \psi)$.

Using (i), (ii), (iii) above, along with Lemma 4.2.1, (6.34), (6.36), (6.33) we can show

$$\begin{aligned} W'_{(6.35)}(t, x(\cdot)) &\leq W'_{(6.30)}(t, x(\cdot)) + K|g(t, x(\cdot)| \\ &\leq -\kappa W(t, x(\cdot)) + KM_1 h_1(t)|\phi|_{t_0}. \end{aligned}$$

Integrating across this last inequality over $[t_0, t]$ gives

$$W(t, x(\cdot)) \leq W(t_0, \phi)e^{-\kappa(t-t_0)} + KM_1|\phi|_{t_0}e^{-\kappa t}\int_{t_0}^t e^{\kappa s}h_1(s)\,ds,$$

and using (i) above yields

$$|x(t, t_0, \phi)| \leq K|\phi|_{t_0}e^{-\kappa(t-t_0)} + KM_1|\phi|_{t_0}e^{-\kappa t}\int_{t_0}^t e^{\kappa s}h_1(s)\,ds.$$

Since from (6.36) the zero solution to (6.35) is uniformly stable, the last inequality allows us to conclude that the the zero solution to (6.35) is uniformly asymptotically stable, and hence, by Proposition 6.2.2, it must be ExAS. ⋄

Proof of Lemma 6.2.2: Define

$$h_2(t) = 1 - \int_0^t \alpha(s, t)\,ds. \tag{6.37}$$

Let $z_1(\cdot)$ be a solution of (6.18), and define

$$g_1(t, z_1(\cdot)) = \lambda(t)\int_0^t w(s, t)\left(1 - \frac{k_1(s) + K_2}{k_1(t) + K_2}\right) z_1(s)\,ds. \tag{6.38}$$

We can reformulate (6.18) as

$$z_1'(t) = \lambda(t)\left(z_1(t) - \int_0^t w(s,t)z_1(s)\,ds\right) - (\beta(t) + p_1(t))z_1(t) + g_2(t, z_1(\cdot)). \tag{6.39}$$

Suppose $z_1(t) = \phi(t)$ on $[0, t_0]$. From Lemma 6.2.3, (3.5), (3.4), (6.12), (6.37), (6.38), we have

$$|g_1(t, z_1(\cdot))| \leq M_1|\phi|_{t_0}h_2(t). \tag{6.40}$$

Since, by Lemma 6.2.5 the zero solution of (6.28) is exponentially asymptotically stable, we know by Theorem 6.2.1 that there exists a continuous functional $W(t,\psi)$ defined for $t \geq 0$ and $\psi \in \mathbf{C}([0,t],\mathbb{R})$, and constants $K > 0$, $\kappa > 0$ such that

(i) $|\psi(t)| \leq W(t,\psi) \leq K|\psi|_t$;

(ii) $|W(t,\psi_1) - W(t,\psi_2)| \leq K|\psi_1 - \psi_2|_t$ for $\psi_1, \psi_2 \in \mathbf{C}([0,t],\mathbb{R})$;

(iii) $W'_{(6.28)}(t,\psi) \leq -\kappa W(t,\psi)$.

Using (i), (ii), (iii) above, along with (6.37), (6.38), (6.40) we can show

$$\begin{aligned} W'_{(6.39)}(t, z_1(\cdot)) &\leq W'_{(6.28)}(t, z_1(\cdot)) + K|g(t, z_1(\cdot)| \\ &\leq -\kappa W(t, z_1(\cdot)) + KM_1h_2(t)|\phi|_{t_0}. \end{aligned}$$

Integrating across this inequality from t_0 up to t, and using (i)-(iii) above, we obtain

$$|z_1(t, t_0, \phi)| \leq K|\phi|_{t_0}e^{-\kappa(t-t_0)} + K|\phi|_{t_0}M_1e^{-\kappa t}\int_{t_0}^t e^{\kappa s}h_2(s)\,ds.$$

By (6.16), $\lim_{t\to\infty} h_2(t) = 0$, so the zero solution of (6.18) is uniformly asymptotically stable, and by virtue of Proposition 6.2.2, exponentially asymptotically stable. $\diamond$

We finally prove

Lemma 6.2.6 *If x_2 is the solution to*

$$x_2'(t) = F_2(t, x_2(\cdot)), \tag{6.41}$$

then

$$\lim_{t\to\infty} x_2(t) = 0.$$

This result obviously proves Proposition 6.2.1.

Proof of Lemma 6.2.6: Here we modify the proof of Theorem 3.3.6. in [49]. First, by Lemma 6.2.3 the solution to (6.27) is ExAS. Thus we may write

$$|x_1(t, t_0, \phi)| \leq Ke^{-\kappa(t-t_0)}|\phi|_{t_0}.$$

Moreover, by Theorem 6.2.1, there exists a continuous functional $W(t, \phi)$ satisfying (i)-(iii) in the statement of Theorem 6.2.1 above. If we put $r = 0$, then $p(t, \phi) = |\phi(t)|$. Therefore

$$W(t, \phi) = \sup_{t_0 \geq 0} |x_1(t + t_0, t, \phi)|e^{\kappa t_0}.$$

Let x_2 be the solution of (6.41), we can use Definition 6.2.1 to calculate

$$W'_{(6.41)}(t, x_2(\cdot)) = \limsup_{h \to 0+} \frac{W(t + h, x_2^*) - W(t, x_2)}{h}$$

where we let

$$x_2^*(s) = \begin{cases} x_2(s) & \text{on } 0 \leq s \leq t, \\ x_2(s) + F_1(t, x_2(\cdot))(s - t) + p_2(t)(s - t) & \text{on } t \leq s \leq t + h. \end{cases}$$

and we used $F_2(t, x_2(\cdot)) - F_1(t, x_2(\cdot)) = p_2(t)$. Define

$$\tilde{x}_2^*(s) = \begin{cases} x_2(s) & \text{on } 0 \leq s \leq t, \\ x_2(s) + F_1(t, x_2(\cdot))(s - t) & \text{on } t \leq s \leq t + h. \end{cases}$$

Now, the principle of superposition for linear integro-differential equations yields

$$x_1(t + h + t_0, t + h, x_2^*) = x_1(t + t_0 + h, t + h, x_2^* - \tilde{x}_2^*) + x_1(t + t_0 + h, t + h, \tilde{x}_2^*),$$

and if we define $\tilde{\phi} = x_2^* - \tilde{x}_2^*$ then

$$\tilde{\phi}(s) = \begin{cases} 0 & \text{on } 0 \leq s \leq t, \\ p_2(t)(s - t) & \text{on } t \leq s \leq t + h. \end{cases}$$

This leads to the bounding argument:

$$\begin{aligned} W'_{(6.41)}(t, x_2(\cdot)) &\leq W'_{(6.27)}(t, x_2(\cdot)) + \limsup_{h \to 0+} \frac{1}{h} \sup_{t_0 \geq 0} |x_1(t + h + t_0, t + h, \tilde{\phi})|e^{\kappa t_0} \\ &\leq -\kappa W(t, x_2(\cdot)) + \limsup_{h \to 0+} \frac{1}{h} W(t + h, \tilde{\phi}) \\ &\leq -\kappa W(t, x_2(\cdot)) + \limsup_{h \to 0+} \frac{1}{h} K|\tilde{\phi}|_{t+h} \\ &= -\kappa W(t, x_2(\cdot)) + \limsup_{h \to 0+} \frac{1}{h} K \sup_{t \leq s \leq t+h} |p_2(t)(s - t)| \\ &\leq -\kappa W(t, x_2(\cdot)) + K|p_2(t)|. \end{aligned}$$

Integrating across the last inequality on $[t_0, t]$, we have

$$W(t, x_2(t)) \leq W(t_0, \phi)e^{-\kappa(t-t_0)} + Ke^{-\kappa t}\int_{t_0}^{t} e^{\kappa s}|p_2(s)|\, ds,$$

and using Theorem 6.2.1, we obtain for $t \geq t_0$

$$|x_2(t, t_0, \phi)| \leq K|\phi|_{t_0}e^{-\kappa(t-t_0)} + Ke^{-\kappa t}\int_{t_0}^{t} e^{\kappa s}|p_2(s)|\, ds.$$

The Lemma follows on applying (6.17). ⋄

6.3 Proof of Convergence

In order to proceed with the proof of (6.5), we first require some subsidiary results.

Lemma 6.3.1 *If $g(\cdot,\cdot)$ is given by the solution of (3.10) with initial condition given by (3.11), and (6.6) and (6.7) are true, then for all $t > 0$ we have*

(i) $|g(s,t)| \leq 1 \qquad \forall\ (s,t) \in D,$

(ii) $\left|\frac{\partial g}{\partial t}(s,t)\right| \leq 2M_1 + M_2,$

where M_1, M_2 are defined in (3.5) and (3.6).

Proof: The proof of part (i) is analagous to the proof of Lemma 6.2.3. For (ii), note from Definition 4.2.1 (i), (3.10), that

$$\begin{aligned}
\left|\frac{\partial g}{\partial t}(s,t)\right| &\leq |\lambda(t)|\left(|g(s,t)| + \int_s^t w(u,t)|g(s,u)|\, du\right) + |\beta(t)||g(s,t)| \\
&\leq |\lambda(t)|\left(1 + \int_s^t w(u,t)\, du\right) + |\beta(t)| \\
&\leq 2\max_{t\geq 0}|\lambda(t)| + \max_{t\geq 0}|\beta(t)| \\
&= 2M_1 + M_2,
\end{aligned}$$

where part (i) of the Lemma yields the second inequality, Definition 4.2.1 (i) and (ii) the third and (3.5) and (3.6) final equality. ⋄

For notational simplicity, define

$$h(t) = \sqrt{t}. \tag{6.42}$$

We prove (6.5) under the following constraint: there exists $\gamma > 0$ such that

$$\lim_{t\to\infty} t^{\gamma} \frac{\int_0^t \sigma(s)^2\, ds}{k(t)^2} = 0. \tag{6.43}$$

From (5.16), (6.42) and (6.43) we immediately have

$$\begin{aligned} \lim_{t\to\infty} t^{\frac{\gamma}{2}} \frac{\int_0^{h(t)} \sigma(s)^2\, ds}{k_1(h(t))^2} &= 0, & (6.44)\\ \lim_{t\to\infty} t^{1+\frac{\gamma}{2}} (h(t+1)-h(t))^2 \frac{\int_0^{h(t+1)} \sigma(s)^2\, ds}{k_1(h(t+1))^2} &= 0. & (6.45) \end{aligned}$$

Proposition 6.3.1 *Suppose that there exists $\gamma > 0$ such that (6.43) holds. Let $g(\cdot,\cdot)$ be the solution of (3.10) with initial condition given by (3.11), where (6.6) and (6.7) hold. Then*

$$\lim_{t\to\infty} \frac{1}{k(t)} \int_0^t \sigma(s) g(s,t)\, dB_s = 0 \qquad \mathbb{P}-a.s.. \tag{6.46}$$

Proof: Let

$$Y_t = \frac{1}{k_1(t)} \int_0^t \sigma(s) g(s,t)\, dB_s.$$

Then (6.46) follows if

$$\lim_{t\to\infty} Y_t = 0 \qquad \mathbb{P}-a.s..$$

To prove this, we first show that $\{Y_{h(n)}\}_{i=0}^{\infty}$ converges, where $h(\cdot)$ is defined in (6.42). Then, by using the semi-martingale decomposition furnished by the Berger-Mizel Transformation, we show convergence between the grid points $h(n)$. Let $m = 2\lceil \frac{1}{\gamma} \rceil$. Now for every $\varepsilon > 0$, we have

$$\begin{aligned} \mathbb{P}\left[|Y_t| \geq \varepsilon\right] &\leq \frac{1}{\varepsilon^{2m}} \mathbb{E}\left[{Y_t}^{2m}\right] \\ &= \frac{1}{\varepsilon^{2m}} \frac{(2m)!}{m! 2^m} \left(\frac{\int_0^t (\sigma(s) g(s,t))^2\, ds}{k_1(t)^2} \right)^m \\ &\leq \frac{1}{\varepsilon^{2m}} \frac{(2m)!}{m! 2^m} \left(\frac{\int_0^t \sigma(s)^2\, ds}{k_1(t)^2} \right)^m . \end{aligned}$$

where we used Proposition (6.1.1) to obtain the second equality, and Lemma 6.3.1 (i) to achieve the last bound. We then obviously have

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}\left[|Y_{h(n)}| \geq \varepsilon\right] &\leq \frac{1}{\varepsilon^{2m}} \frac{(2m)!}{m! 2^m} \sum_{n=1}^{\infty} \frac{1}{n^{\gamma m}} \left(n^{\gamma} \frac{\int_0^{h(n)} \sigma(s)^2\, ds}{k_1(h(n))^2} \right)^m \\ &< \infty. \end{aligned}$$

where the choice of m, together with (6.44), ensure that the sum is finite. Thus for $n \in \mathbf{N}$ the Borel-Cantelli Lemma allows us to conclude that

$$\lim_{n\to\infty} Y_{h(n)} = 0 \qquad \mathbb{P}-a.s.. \tag{6.47}$$

To achieve convergence in general, we will need to interpolate between the gridpoints. To do this, notice that $\sigma(\cdot)$ is uniformly bounded as are $g(\cdot,\cdot)$ and $\frac{\partial g}{\partial t}(\cdot,\cdot)$ by Lemma 6.3.1. Thus we may decompose Y_t using Theorem 3.3.1, to wit

$$Y_t = \frac{1}{k_1(t)}\int_0^t \sigma(s)\,dB_s + \frac{1}{k_1(t)}\int_0^t \left(\int_0^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\,dB_u\right)\,ds, \tag{6.48}$$

where we used substituted (3.11) in the Itô integral. Next we need to show

$$\lim_{t\to\infty}\frac{1}{k_1(t)}\int_0^t \sigma(s)\,dB_s = 0 \qquad \mathbb{P}-a.s.. \tag{6.49}$$

If

$$\int_0^\infty \sigma(s)^2\,ds < \infty,$$

then by the Martingale Convergence theorem there exists a random variable M_∞ with finite square variation such that

$$\lim_{t\to\infty}\int_0^t \sigma(s)\,dB_s = M_\infty \qquad \mathbb{P}-a.s..$$

so (6.49) is true. On the other hand, if

$$\int_0^\infty \sigma(s)^2\,ds = \infty,$$

we observe that the Itô integral is a time-changed Brownian motion: (see Karatzas and Shreve p.174 [41]) indeed there exists another Brownian motion $\tilde{B}$ such that

$$\frac{1}{k_1(t)}\int_0^t \sigma(s)\,dB_s = \frac{\tilde{B}_u}{k_1(t(u))} \tag{6.50}$$

where $u = \int_0^{t(u)} \sigma(s)^2\,ds$ and since by (6.43) it follows that

$$\lim_{t\to\infty}\frac{\int_0^t \sigma(s)^2\,ds}{k_1(t)^2}\log\left(\log\left(\int_0^t \sigma(s)^2\,ds\right)\right) = 0.$$

we have, on applying the law of the iterated logarithm to the r.h.s. of (6.50) that (6.49) is true. From (6.47), (6.48), (6.49), if we define

$$Z_t = \frac{1}{k_1(t)}\int_0^t \left(\int_0^s \sigma(u)\frac{\partial g}{\partial t}g(u,s)\,dB_u\right)\,ds, \tag{6.51}$$

then

$$\lim_{n\to\infty} Z_{h(n)} = 0 \qquad \mathbb{P}-a.s.,$$

and (6.46) is true (and hence the proposition) if and only if

$$\lim_{t\to\infty} Z_t = 0 \qquad \mathbb{P}-a.s.. \tag{6.52}$$

We prove (6.52) as follows: for each $t \in \mathbb{R}^+$ there exists $n \in \mathbf{N}$ such that $h(n) \le t < h(n+1)$. Using (6.51), we have

$$Z_t - Z_{h(n)} = \left(\frac{k_1(h(n))}{k_1(t)} - 1\right) Z_{h(n)} + \frac{1}{k_1(t)} \int_{h(n)}^t \left(\int_0^s \sigma(u) \frac{\partial g}{\partial t}(u,s)\, dB_u\right) ds,$$

so (6.52) is equivalent to

$$\lim_{n\to\infty} \sup_{h(n)\le t\le h(n+1)} \left| \frac{1}{k_1(t)} \int_{h(n)}^t \left(\int_0^s \sigma(u) \frac{\partial g}{\partial t}(u,s)\, dB_u\right) ds \right| = 0 \qquad \mathbb{P}-a.s.. \tag{6.53}$$

We now prove (6.53) by another Borel-Cantelli argument.

$$\begin{aligned}
&\mathbb{E}\left[\sup_{h(n)\le t\le h(n+1)} \left(\frac{1}{k_1(t)} \int_{h(n)}^t \left(\int_0^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\, dB_u\right) ds\right)^2\right] \\
\le\ & \frac{1}{k_1(h(n+1))^2} \mathbb{E}\left[\sup_{h(n)\le t\le h(n+1)} \left(\int_{h(n)}^t \left(\int_0^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\, dB_u\right) ds\right)^2\right] \\
\le\ & \frac{1}{k_1(h(n+1))^2} \mathbb{E}\left[\sup_{h(n)\le t\le h(n+1)} (t-h(n)) \int_{h(n)}^t \left(\int_0^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\, dB_u\right)^2 ds\right] \\
\le\ & \frac{h(n+1)-h(n)}{k_1(h(n+1))^2} \mathbb{E}\left[\sup_{h(n)\le t\le h(n+1)} \int_{h(n)}^t \left(\int_0^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\, dB_u\right)^2 ds\right] \\
\le\ & \frac{h(n+1)-h(n)}{k_1(h(n+1))^2} \mathbb{E}\left[\int_{h(n)}^{h(n+1)} \left(\int_0^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\, dB_u\right)^2 ds\right] \\
=\ & \frac{h(n+1)-h(n)}{k_1(h(n+1))^2} \int_{h(n)}^{h(n+1)} \mathbb{E}\left[\left(\int_0^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\, dB_u\right)^2\right] ds \\
=\ & \frac{h(n+1)-h(n)}{k_1(h(n+1))^2} \int_{h(n)}^{h(n+1)} \left(\int_0^s (\sigma(u) g_t(u,s))^2\, du\right) ds \\
\le\ & \frac{(h(n+1)-h(n))^2}{k_1(h(n+1))^2} \cdot (2M_1+M_2)^2 \int_0^{h(n+1)} \sigma(u)^2\, du \\
=\ & \frac{(2M_1+M_2)^2}{n^{1+\frac{\gamma}{2}}} \cdot \left(n^{1+\frac{\gamma}{2}} \frac{(h(n+1)-h(n))^2}{k_1(h(n+1))^2} \int_0^{h(n+1)} \sigma(u)^2\, du\right) \\
<\ & \frac{(2M_1+M_2)^2}{n^{1+\frac{\gamma}{2}}} \text{ for } n \text{ sufficiently large.}
\end{aligned}$$

where we use the monotonicity of $k_1(\cdot)$ to get the first inequality, Cauchy Schwarz to obtain the second, Fubini to get the first equality, Proposition 6.1.1 to obtain the second equality, Lemma 6.3.1

(ii) to achieve the fifth bound, and (6.45) to obtain the final bound. Thus

$$\sum_{n=1}^{\infty} \mathbb{E}\left[\sup_{h(n)\le t\le h(n+1)} \left(\frac{1}{k_1(t)}\int_{h(n)}^{t}\left(\int_0^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\,dB_u\right)ds\right)^2\right] < \infty,$$

and so by Chebyshev's inequality, we have for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\sup_{h(n)\le t\le h(n+1)} \left|\frac{1}{k_1(t)}\int_{h(n)}^{t}\left(\int_0^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\,dB_u\right)ds\right| \ge \varepsilon\right] < \infty.$$

Therefore, by the Borel-Cantelli Lemma we have

$$\lim_{n\to\infty} \sup_{h(n)\le t\le h(n+1)} \left|\frac{1}{k_1(t)}\int_{h(n)}^{t}\left(\int_0^s \sigma(u)\frac{\partial g}{\partial t}(u,s)\,dB_u\right)ds\right| = 0 \qquad \mathbb{P}-a.s..$$

which is precisely (6.53), so the proposition is proven. $\diamond$

We are now in a position to prove the main result.

Theorem 6.3.1 *Let $\lambda(\cdot), \beta(\cdot), \sigma(\cdot)$ be continuous and uniformly bounded and satisfy*

$$\inf_{t\ge 0} \beta(t) - \lambda(t) - \sup_{s\ge 0}|\lambda(s)| > 0.$$

Let $k(\cdot)$ be continuous, $\lim_{t\to\infty} k(t) = \infty$. Suppose there exists a positive, non-decreasing $\mathbf{C}^1$ function, $k_1(\cdot)$, which satisfies

$$\lim_{t\to\infty}\frac{k_1'(t)}{k_1(t)} = 0, \qquad \lim_{t\to\infty}\frac{k(t)}{k_1(t)} = 1.$$

Suppose there exists $\gamma > 0$ such that

$$\lim_{t\to\infty} t^{\gamma}\frac{\int_0^t \sigma(s)^2\,ds}{k(t)^2} = 0.$$

Let X be the solution to (6.1). Then

$$\lim_{t\to\infty}\frac{X_t}{k(t)} = 1,$$

$\mathbb{P}$-a.s. and in p^{th} mean for any $p \in [1,\infty)$.

Proof: We first prove almost sure convergence. One can write, from Theorem 3.3.1, (3.12)

$$\frac{X_t}{k(t)} - 1 = \left(\frac{e(t)}{k(t)} - 1\right) + \frac{1}{k(t)}\int_0^t \sigma(s)g(s,t)\,dB_s, \tag{6.54}$$

By Proposition 6.2.1 the first term on the r.h.s. of (6.54) goes to 0 as $t\to\infty$ and by Proposition 6.3.1 the second term on the r.h.s. of (6.54) goes to 0 $\mathbb{P}-a.s.$ as $t\to\infty$. To prove convergence in p^{th}

mean, if $p \geq 1$, we proceed as follows. Let $r = \lceil \frac{p}{2} \rceil$, so r is a positive integer and $2r \geq p$. Therefore one has

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{X_t}{k(t)} - 1\right)^{2r}\right] &= \mathbb{E}\left[\left(\left(\frac{e(t)}{k(t)} - 1\right) + \frac{1}{k(t)}\int_0^t \sigma(s)g(s,t)\,dB_s\right)^{2r}\right] \\
&\leq 2^{2r-1}\left(\left(\frac{e(t)}{k(t)} - 1\right)^{2r} + \mathbb{E}\left[\left(\frac{1}{k(t)}\int_0^t \sigma(s)g(s,t)\,dB_s\right)^{2r}\right]\right) \\
&\leq 2^{2r-1}\left(\left(\frac{e(t)}{k(t)} - 1\right)^{2r} + \frac{(2r)!}{r!2^r}\left(\frac{\int_0^t (\sigma(s)g(s,t))^2\,ds}{k(t)^2}\right)^r\right) \\
&\leq 2^{2r-1}\left(\left(\frac{e(t)}{k(t)} - 1\right)^{2r} + \frac{(2r)!}{r!2^r}\left(\frac{\int_0^t \sigma(s)^2\,ds}{k(t)^2}\right)^r\right)
\end{aligned}$$

Thus from Proposition 6.2.1 and (6.43), we have

$$\lim_{t\to\infty} \mathbb{E}\left[\left(\frac{X_t}{k(t)} - 1\right)^{2r}\right] = 0,$$

so since $2r \geq p \geq 1$, it follows that

$$\lim_{t\to\infty} \mathbb{E}\left[\left(\frac{X_t}{k(t)} - 1\right)^{p}\right] = 0.$$

Since p was chosen arbitrarily, p^{th} mean convergence has been demonstrated for all $p \in [1,\infty)$. $\diamond$

The same ideas used in the proof of the above theorem may be used to prove

Theorem 6.3.2 *Let $\lambda(\cdot), \beta(\cdot), \sigma(\cdot)$ be continuous and uniformly bounded. Moreover, suppose $\lambda(\cdot) < 0$, and*

$$\inf_{t\geq 0} \beta(t) - \lambda(t) + \inf_{s\geq 0} \lambda(s) > 0.$$

Let $k(\cdot)$ be continuous, $\lim_{t\to\infty} k(t) = \infty$. Suppose there exists a positive, non-decreasing $\mathbf{C}^1$ function, $k_1(\cdot)$, which satisfies

$$\lim_{t\to\infty} \frac{k_1'(t)}{k_1(t)} = 0, \qquad \lim_{t\to\infty} \frac{k(t)}{k_1(t)} = 1.$$

Suppose there exists $\gamma > 0$ such that

$$\lim_{t\to\infty} t^\gamma \frac{\int_0^t \sigma(s)^2\,ds}{k(t)^2} = 0.$$

Let X be the solution to (6.1). Then

$$\lim_{t\to\infty} \frac{X_t}{k(t)} = 1,$$

$\mathbb{P}$-a.s. and in p^{th} mean for any $p \in [1,\infty)$.

Proof: As in Theorem 6.3.1. $\diamond$

Remark 6.3.1

The conditions of Theorem 6.3.2 are in particular satisfied if

$$\inf_{t \geq 0} \beta(t) > 0,$$

whenever $\lambda(\cdot) \equiv \lambda < 0$. Thus we see that if chartists have mean-reverting expectations, they cause the price to follow the consensus growth rate of the fundamentals almost surely.

There is a further useful extension to Theorem 6.3.1.

Theorem 6.3.3 *Suppose $(K_t)_{t \geq 0}$ is a $\mathcal{F}_t$–adapted stochastic process independent of $(B_t)_{t \geq 0}$. Suppose there exists a continuous function $k(\cdot)$ which satisfies*

$$\lim_{t \to \infty} \frac{K_t}{k(t)} = 1 \qquad \mathbb{P} - a.s.$$

along with properties in Theorem 6.3.1. If

$$dX_t = \left(\lambda(t) \left(X_t - \int_0^t w(s,t) X_s \, ds \right) - \beta(t)(X_t - K_t) \right) dt + \sigma(t) \, dB_t \tag{6.55}$$

then

$$\lim_{t \to \infty} \frac{X_t}{K_t} = 1 \qquad \mathbb{P} - a.s..$$

Proof: We apply the argument to the previously deterministic solution to (3.12) on a pathwise basis, so

$$\lim_{t \to \infty} \frac{e(t)}{K_t} = 1 \qquad \mathbb{P} - a.s..$$

The argument used in Proposition 6.3.1 is unaltered by the new hypothesis, as the process $(K_t)_{t \geq 0}$ is independent of $(B_t)_{t \geq 0}$, proving the theorem. $\diamond$

Corollary 6.3.1 *Let K_t satisfy*

$$dK_t = k_1'(t)\,dt + \sigma_1(t)\,dW_t$$

where $(W_t)_{t\geq 0}$ is a $\mathcal{F}_t$–adapted standard Brownian motion independent of $(B_t)_{t\geq 0}$. Let

$$\lim_{t\to\infty} \frac{\int_0^t \sigma_1(s)^2\,ds \log\log\left(\int_0^t \sigma_1(s)^2\,ds\right)}{k_1(t)^2} = 0.$$

If X_t is a solution of (6.55), then

$$\lim_{t\to\infty} \frac{X_t}{K_t} = 1 \qquad \mathbb{P} - a.s..$$

Proof: Follows directly from Theorem 6.3.3. ⋄

Remark 6.3.2

Theorem 6.3.3 means that if the fundamentalists believe that the price should revert to values given by a stochastic process, provided that the process is independent of the price, then the price will follow that process asymptotically. The benefit of this result is that the fundamentalists merely need to observe a time series (or several time series, and then take their weighted average) rather than fabricate some value for the fundamentals.

Remark 6.3.3

We may think of $k(\cdot)$ as representing a fundamentalist index function. To keep consistency with the notation for chartist index functions, let

$$a_f(t, X.) = k(t).$$

Then if the conditions in the above theorems are satisfied, we have

$$\lim_{t\to\infty} \frac{a_f(t, X.)}{X_t} = \lim_{t\to\infty} \frac{a_c(t, X.)}{X_t} = 1 \quad \text{a.s..}$$

This indicates that both groups of speculators correctly gauge the growth rate of the market, asymptotically. It also shows that chartists do not necessarily underperform fundamentalists as investors. The equivalent efficacy (or inefficacy) of technical and security analysts is remarked upon in Malkiel [35], and this result seems to agree with his observation. Recalling from Remark 4.2.4 that

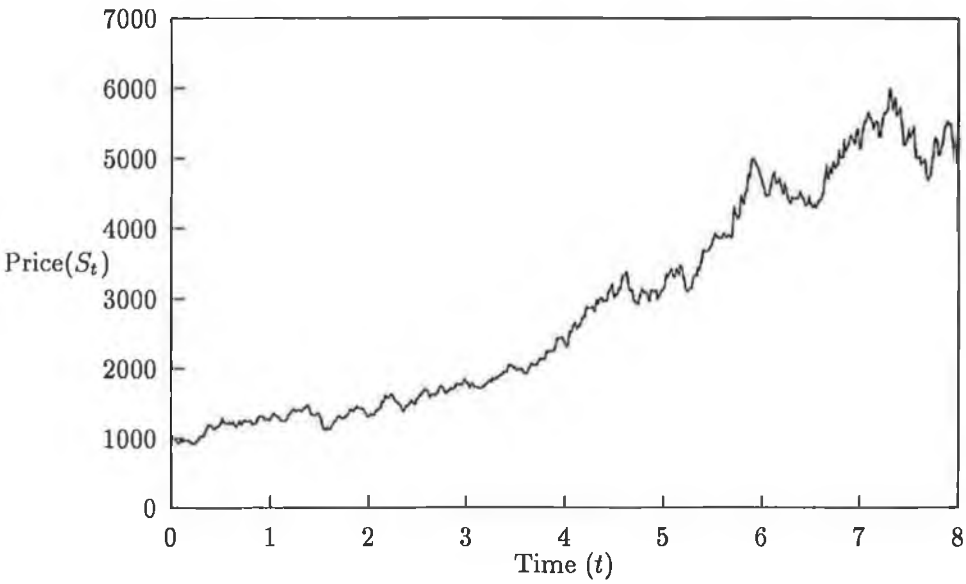

Figure 6.1: A sample path for the Black–Scholes model.

chartists buy at record highs, and that the long run trend in security markets has been upwards, we see that the chartists in our model very often correctly interpret the buy signal their index functions provide. Thus we would expect the chartists to be effective under such conditions.

An interesting consequence of this result is that it suggests an underlying time consistency in the model. Since both groups of speculators have indices which track the market relatively well, it seems plausible that neither group of speculator will be bankrupted by their trading. Thus, since both groups of speculators should survive, the price dynamics do not seem to bring about circumstances which will cause the model to become invalid. However, market conditions in which the model produces crashes or bubbles are not included in this analysis.

Another possible reason for charting is provided by the above theorems: suppose an influential insider trader existed in the market for a particular financial asset. This trader may be assumed to know the fundamental value of the asset at any time, and so will behave like a fundamentalist. We assume that the chartists suspect that such an agent is present in the market; therefore, in the absence of the privileged information which the insider trader has access to, the chartists observe the behaviour of the price path in order to detect that investor's trading pattern. We see in the instance of steadily growing fundamentals that the chartists are able to gain access to the insider trader's information via their index functions. Therefore, in this setting, charting can be seen as a sensible response to limited information. See De Long et. al. [18].

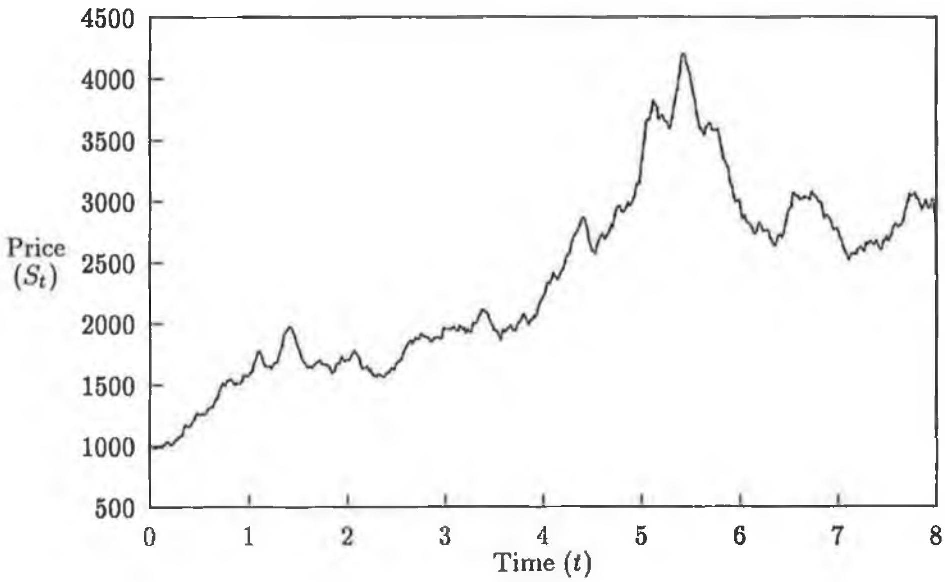

Figure 6.2: Sample path of $S_t = e^{X_t}$ for the model with memory.

The following is a special case of the general result.

Corollary 6.3.2 *Suppose $\lambda(\cdot), \beta(\cdot), \sigma(\cdot)$ are continuous and uniformly bounded, and*

$$\inf_{t\geq 0} \beta(t) - \lambda(t) - \sup_{t\geq 0} |\lambda(s)| > 0.$$

If $k(\cdot)$ is a continuous function such that

$$\lim_{t\to\infty} \frac{k(t)}{t} = \eta,$$

where $\eta > 0$, then

$$\lim_{t\to\infty} \frac{1}{t} \log S_t = \eta,$$

in p^{th} mean and $\mathbb{P}$-a.s..

Proof: Choose $k_1(t) = \eta t + 1$, and the conditions of the Theorem 6.3.1 are satisfied. ⋄

Remark 6.3.4

The pathwise price asymptotics have the same form for the modified and the Black-Scholes models: if prices evolved according to $dS_t = \alpha S_t\, dt + \sigma S_t\, dB_t$, then

$$\lim_{t\to\infty} \frac{1}{t} \log S_t = \alpha - \frac{1}{2}\sigma^2 \quad \mathbb{P}\text{-a.s.},$$

while with $S_t = e^{X_t}$, Corollary 6.3.2 reads

$$\lim_{t \to \infty} \frac{1}{t} \log S_t = \eta \quad \mathbb{P}\text{-a.s.}.$$

Therefore the pathwise price asymptotics for the Black-Scholes model have the same form as those described by Corollary 6.3.2.

Chapter 7

Convergence of the E.D.F. of Periodic Linear S.I-D.Es

7.1 Introduction and Motivation

In this chapter, we show that if we assume that the fundamentalists do not believe the fundamentals change over time (i.e., $k(t) \equiv K^*$), then though neither the price nor the returns converge in distribution, one can prove when the fundamentalists dominate, that the empirical distribution functions of both price and returns converge in a Cesaro sense.

Let the log-price X_t now be governed by

$$dX_t = \lambda(t)\left(X_t - \int_0^t w(s,t)X_s\,ds\right) - \beta(t)\left(X_t - K^*\right)dt + \sigma(t)\,dB_t,$$

with deterministic initial condition given by $X_0 = x_0$, and suppose $\lambda(\cdot), \beta(\cdot)$, and $\sigma(\cdot)$ are either T-periodic or constant and satisfy (6.6). The introduction of periodicity has the same motivation as in Chapter 2.

In the next section, we note how to calculate the covariance of stochastic integrals with t-dependent integrands. We also show that the resolvent of the deterministic integro-differential equation is exponentially bounded, again using the converse theory of Liapunov functions.

In the third section of the chapter, we show that the price does not converge in distribution: however, it is possible to show that the expected value of the empirical distribution function converges to a distribution function with density function given by a continuous superposition of normal densities. This proof is achieved via a mixture of Cauchy sequence and Liapunov function techniques. One can prove a similar result, using the same ideas, for the Δ–returns for any $\Delta > 0$.

In the final section of the chapter, we show (under the proviso that a certain function is not identically constant) that the limiting returns distribution is symmetric, unimodal and has a greater probability of being close to 0 than a normal distribution with the same variance. Furthermore, the limiting returns distribution has heavier tails than a normal distribution with the same variance. More specifically, in the case of the Δ–returns, $r(\Delta;\cdot)$, we show that there exists a T–periodic, Lipschitz continuous, strictly positive function $v_\Delta(\cdot)$ such that

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{t}\int_0^t I_{\{r(\Delta;s)\le x\}}\,ds\right] = F_\Delta(x),$$

where $F_\Delta(\cdot)$ is a distribution function with density $f_\Delta(\cdot)$ given by

$$f_\Delta(x) = \frac{1}{T}\int_0^T \frac{1}{\sqrt{2\pi}\sqrt{v_\Delta(s)}} e^{-\frac{1}{2v_\Delta(s)}x^2}\,ds.$$

7.2 Preliminaries

In this section we introduce the model that we are to study, note some of its properties, and recall some results in the theory of integro-differential equations which we will use later.

Let $e(\cdot)$ solve

$$e(t) = g(0,t)(x_0 - K^*) + K^*. \tag{7.1}$$

One has Theorem 3.3.1 with a different $e(\cdot)$.

Proposition 7.2.1 *Under the above hypotheses, (6.1) has a unique continuous, variation of parameters solution given by*

$$X_t = e(t) + \int_0^t \sigma(s)g(s,t)\,dB_s, \tag{7.2}$$

where $g(\cdot,\cdot)$ is given by (3.10) and (3.11), and $e(\cdot)$ is given by (7.1).

Proof: As Theorem 3.3.1. ◇

Proposition 7.2.2 *Let h_1 be a deterministic function which satisfies*

$$h_1 : D \to \mathbb{R} : (s,t) \mapsto h_1(s,t),$$

and for $D' = \{(s,u)\ :\ 0 < s \le u\}$, h_2 satisfies

$$h_1 : D' \to \mathbb{R} : (s,u) \mapsto h_2(s,u).$$

If

$$\int_0^t h_1(s,t)^2\,ds < \infty,$$

and

$$\int_0^u h_2(s,u)^2\,ds < \infty,$$

then

$$\mathbb{E}\left[\left(\int_0^t h_1(s,t)\,dB_s\right)\left(\int_0^u h_1(s,u)\,dB_s\right)\right] = \int_0^{t\wedge u} h_1(s,t)h_2(s,u)\,ds. \tag{7.3}$$

Proof: Use Riemann sums. ◇

We will also need a modification of a previous result.

Proposition 7.2.3 *Let*

$$x'(t) = \lambda(t)\left(x(t) - \int_0^t w(s,t)x(s)\,ds\right) - \beta(t)x(t). \tag{7.4}$$

Then the zero solution of (7.4) is exponentially asymptotically stable if and only if it is uniformly asymptotically stable.

Proof: Take the analysis of Theorem 2.2.2 on p.54–55 in [49] in conjunction with that contained in Yoshizawa [79] p.29. ◇

Finally, we note that the resolvent decays exponentially.

Proposition 7.2.4 *Suppose that (6.6) holds. There exist constants $K > 0$, $\kappa > 0$ such that the solution of (3.10) with initial conditions (3.11) satisfies*

$$|g(s,t)| \le Ke^{-\kappa(t-s)} \qquad 0 \le s \le t \tag{7.5}$$

To do this, recall

Lemma 7.2.1 *If (6.6) holds, the zero solution of (7.4) is exponentially asymptotically stable.*

Proof: This can be proved in the same manner as Lemma 6.2.5. $\diamond$

By Lemma 7.2.1 there exist t_0- independent constants $K > 0$, $\kappa > 0$ such that

$$|x(t, t_0, \phi)| \leq Ke^{-\kappa(t-t_0)}|\phi|_{t_0} \qquad \forall\, t \geq t_0. \tag{7.6}$$

Now we are in a position to supply the

Proof of Proposition 7.2.4: Let

$$\phi_s(t) = \begin{cases} 0 & \text{for } 0 \leq s < t, \\ 1 & \text{for } s = t. \end{cases}$$

Then $g(s,t) = x(t, s, \phi_s)$ where $x(\cdot,\cdot,\cdot)$ is the solution to (7.4). Therefore

$$|g(s,t)| = |x(t, s, \phi_s)| \leq Ke^{-\kappa(t-s)}|\phi_s|_s = Ke^{-\kappa(t-s)}$$

for all $t \geq s$. $\diamond$

7.3 Non-Convergence of the Transition Density and Convergence of the Empirical Distribution Function

In this section we prove the main results of the chapter: namely, that although neither the price nor the returns converges in distribution, their empirical distribution functions converge. However, more work is required to show the pathwise convergence of the EDFs, the result achieved in Chapter 2. Simulations indicate that pathwise convergence does indeed appear to take place.

We prove that the expected value of the empirical distribution functions converge to distribution functions whose densities are continuous superpositions of Gaussian densities. In view of this, we will ultimately be able to prove that, except in a set of exceptional circumstances, the returns distribution has heavier tails than a normal distribution. In the the following subsection, we will prove the result for the price distribution, while Section 7.3.2 contains the result for the returns distribution.

7.3.1 Price Distribution

Let us make the assumption of the periodicity of the functions $\lambda(\cdot)$, $\beta(\cdot)$ and $\sigma(\cdot)$ explicit: suppose that there exists $T \geq 0$ such that for all $t \geq 0$

$$\lambda(t+T) = \lambda(t) \tag{7.7}$$

$$\beta(t+T) = \beta(t) \tag{7.8}$$

$$\sigma(t+T) = \sigma(t). \tag{7.9}$$

This assumption means that all of the functions $\lambda(\cdot)$, $\beta(\cdot)$, $\sigma(\cdot)$ must be periodic, and must share a common period. Further note that this definition does not preclude two or fewer of the functions from being constant.

We assume, moreover, that $\lambda(\cdot)$, $\beta(\cdot)$, $\sigma(\cdot)$ are continuous and satisfy either

$$\inf_{t\geq 0} \beta(t) - \lambda(t) > \sup_{t\geq 0} |\lambda(t)| \tag{7.10}$$

or, if $\lambda(\cdot) < 0$,

$$\inf_{t\geq 0} \beta(t) - \lambda(t) > -\inf_{t\geq 0} \lambda(t). \tag{7.11}$$

Finally, we assume that (3.8) holds, viz,

$$\inf_{t\geq 0} |\sigma(t)| = \underline{\sigma} > 0. \tag{7.12}$$

For the purposes of Chapters 7 and 8 of this thesis, we call (7.7), (7.8), (7.9), (7.10), (7.11) and (7.12) the usual conditions (UC). We now proceed to prove

Theorem 7.3.1 *Let X_t be the solution of (6.1), and suppose that the usual conditions (UC) hold. Then*

(i) X_t does not converge in distribution as $t \to \infty$.

(ii) There exists a Lipschitz continuous, T-periodic, strictly positive function $v(\cdot)$ such that for all $t \in [0,T]$, if

$$\tilde{X}_t \sim \mathcal{N}(K^*, v(t))$$

then

$$X_{t+nT} \xrightarrow{\mathcal{D}} \tilde{X}_t \qquad \text{as } n \to \infty.$$

(iii)

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{t}\int_0^t I_{\{X_s \le x\}}\, ds\right] = F(x) \tag{7.13}$$

where $F(\cdot)$ is a distribution function with density $f(\cdot)$ given by

$$f(x) = \frac{1}{T}\int_0^T \frac{1}{\sqrt{2\pi}\sqrt{v(s)}} e^{-\frac{1}{2v(s)}(x-K^*)^2}\, ds. \tag{7.14}$$

Proof: We first prove (ii), of which (i) is a trivial consequence. First, notice from Propositions 7.2.1 and 6.1.1 that

$$X_t \sim \mathcal{N}(e(t), \tilde{v}(t)) \tag{7.15}$$

where $\tilde{v}(\cdot)$ satisfies

$$\tilde{v}(t) = \int_0^t \sigma(s)^2 g(s,t)^2\, ds, \tag{7.16}$$

$e(\cdot)$ is given by (3.12) and $g(\cdot,\cdot)$ by (3.10) and (3.11). To prove the desired convergence, first note from (7.1) and (7.5) that

$$\lim_{t\to\infty} e(t) = K^*. \tag{7.17}$$

We now employ a Cauchy sequence argument as in Proposition 4.2.2 to show, for every $t \ge 0$ that

$$\lim_{n\to\infty} \tilde{v}(t+nT) \qquad \text{exists} \tag{7.18}$$

and define

$$v : \mathbb{R}^+ \to \mathbb{R} : t \mapsto v(t) := \lim_{n\to\infty} \tilde{v}(t+nT). \tag{7.19}$$

It then remains to show that $v(\cdot)$ in (7.19) satisfies the conditions in part (ii) of the proposition.

We will first show, for all $t \ge 0$

$$\lim_{n,m\to\infty} \tilde{v}(t+nT) - \tilde{v}(t+mT) = 0 \tag{7.20}$$

which proves (7.18).Without loss of generality, we will assume $n > m$. The following reformulation is immediate

$$\tilde{v}(t+nT) - \tilde{v}(t+mT) = \int_0^{(n-m)T} \sigma(s)^2 g(s,t+nT)^2\, ds \tag{7.21}$$
$$+ \int_0^{t+mT} \sigma(s)^2 \left(g(s+(n-m)T, t+nT)^2 - g(s,t+mT)^2\right)\, ds.$$

Using Proposition 7.2.4, together with (3.7) we have the following bound for the first term on the right hand side of (7.21)

$$\int_0^{(n-m)T} \sigma(s)^2 g(s, t+nT)^2\, ds \le \frac{{M_3}^2 K^2}{2\kappa} e^{-2\kappa t} \left(e^{-2\kappa mT} - e^{-2\kappa nT}\right)$$

so

$$\lim_{n,m\to\infty,\, n>m} \int_0^{(n-m)T} \sigma(s)^2 g(s, t+nT)^2\, ds = 0. \tag{7.22}$$

To tackle the convergence of the second term on the right hand side of (7.21), define for $t \ge s$

$$\Delta_s(n,m;t) = g(s+(n-m)T, t+(n-m)T) - g(s,t), \tag{7.23}$$

so $\Delta_s(n,m;s) = 0$, and $\Delta_s(n,m;t) = 0$ for $t < s$. Using the T–periodicity of $\lambda(\cdot), \beta(\cdot), \sigma(\cdot)$, we have from (7.23) that

$$\Delta_s'(n,m;t) = \lambda(t)\left(\Delta_s(n,m;t) - \int_s^t w(u,t)\Delta_s(n,m;u)\, du\right) - \beta(t)\Delta_s(n,m;t) + f(n,m;s,t), \tag{7.24}$$

where $'$ denotes differentiation with respect to t and

$$f(n,m;s,t) = \lambda(t)\int_s^t g(s+(n-m)T, u+(n-m)T)\left(w(u,(n-m)T, t+(n-m)T) - w(u,t)\right) du. \tag{7.25}$$

If one defines

$$f^*(n,m;t) = M_1\left(\sup_{0\le u\le t}\left|\frac{w(u+(n-m)T, t+(n-m)T)}{a(t-u)} - 1\right| + \sup_{0\le u\le t}\left|\frac{w(u,t)}{a(t-u)} - 1\right|\right)$$

then (3.5), Definition 4.2.1 (iv) and Lemma 6.3.1 (i) enable us to show that

$$|f(n,m;s,t)| \le f^*(n,m;t), \tag{7.26}$$

$$\lim_{t\to\infty} f^*(n,m;t) = 0. \tag{7.27}$$

Observing that the homogeneous part of (7.24) is the differential equation for $g(s,\cdot)$, we can use (7.26) and Proposition 7.2.4 along with a similar line argument to that of Lemma 7.2.1 to show

$$|\Delta_s(n,m;t)| \le Ke^{-\kappa t}\int_s^t e^{\kappa u} f^*(n,m;u)\, du. \tag{7.28}$$

We then have the following bounding argument:

$$
\begin{aligned}
&\left|\int_0^{t+mT} \sigma(s)^2 \left(g(s+(n-m)T, t+(n-m)T)^2 - g(s,t+mT)^2\right) ds\right| \\
&\leq M_3^2 \int_0^{t+mT} \left(|g(s+(n-m)T, t+nT)| + |g(s,t+mT)|\right) |\Delta_s(n,m;t+mT)|\, ds \\
&\leq M_3^2 \int_{s=0}^{t+mT} 2Ke^{-\kappa(t+mT-s)} \left(Ke^{-\kappa(t+mT)} \int_{u=s}^{t+mT} e^{\kappa u} f^*(n,m;u)\, du\right) ds \\
&\leq \frac{2K^2M_3^2}{2\kappa} \left(e^{-2\kappa(t+mT)} \int_0^{t+mT} e^{2\kappa u} f^*(n,m;u)\, du + e^{-2\kappa(t+mT)} \int_0^{t+mT} e^{\kappa u} f^*(n,m;u)\, du\right).
\end{aligned}
$$

where we used (3.7) and (7.23) to obtain the first inequality, Proposition 7.2.4 and (7.28) to obtain the second and Fubini's theorem to obtain the last. By (7.26) and the above we therefore have, on taking limits:

$$
\lim_{n,m\to\infty,\, n>m} \int_0^{t+mT} \sigma(s)^2 \left(g(s+(n-m)T, t+(n-m)T)^2 - g(s,t+mT)^2\right) ds = 0. \tag{7.29}
$$

Equations (7.21), (7.22) and (7.29) now imply (7.20), legitimising the statement of (7.19). We relegate the proof that $v(\cdot)$ defined by (7.19) is positive to the Appendix of this chapter. Evidently, $v(\cdot)$ is T–periodic, so we merely have to prove Lipschitz continuity. To this end, we mimic the line of proof in Proposition 4.2.2. Without loss of generality, let $t_2 > t_1$. Then

$$
\begin{aligned}
v(t_1) - v(t_2) &= \lim_{n\to\infty} - \int_{t_1+nT}^{t_2+nT} \sigma(s)^2 g(s, t_2+nT)^2\, ds \\
&+ \lim_{n\to\infty} \int_0^{t_1+nT} \sigma(s)^2 \left(g(s,t_1+nT)^2 - g(s,t_2+nT)^2\right) ds.
\end{aligned} \tag{7.30}
$$

We bound the second term on the right hand side of (7.30) as follows:

$$
\begin{aligned}
&\left|\int_0^{t_1+nT} \sigma(s)^2 \left(g(s,t_1+nT)^2 - g(s,t_2+nT)^2\right) ds\right| \\
&\leq M_3^2 \int_0^{t_1+nT} \left(Ke^{-\kappa(t_1+nT-s)} + Ke^{-\kappa(t_2+nT-s)}\right) |g(s,t_1+nT) - g(s,t_2+nT)|\, ds \\
&\leq M_3^2 K \left(1 + e^{-\kappa(t_2-t_1)}\right) \int_0^{t_1+nT} e^{-\kappa(t_1+nT-s)} \sup_{0\leq s^*\leq t^*<\infty} \left|\frac{\partial g}{\partial t}(s^*,t^*)\right| |t_2-t_1|\, ds \\
&\leq \frac{2(2M_1+M_2)M_3^2K}{\kappa} |t_2-t_1|.
\end{aligned}
$$

where we used (3.7) and Proposition 7.2.4 to obtain the first bound, and Lemma 6.3.1 (ii) to obtain the third. Using Lemma 6.3.1 (i) also gives us

$$
\left|\int_{t_1+nT}^{t_2+nT} \sigma(s)^2 g(s,t_2+nT)^2\, ds\right| \leq M_3^2 |t_2-t_1|.
$$

Thus, (7.30) and the last two bounds give

$$
|v(t_1) - v(t_2)| \leq M_3^2 \left(1 + \frac{2(2M_1+M_2)K}{\kappa}\right) |t_2-t_1|,
$$

so $v(\cdot)$ satisfies the desired properties.

Finally, for each $t \in [0,T]$, since $X_{t+nT} \sim \mathcal{N}(e(t+nT), \tilde{v}(t+nT))$, by (7.17), (7.19) if $\tilde{X}_t \sim \mathcal{N}(K^*, v(t))$ then

$$X_{t+nT} \xrightarrow{\mathcal{D}} \tilde{X}_t \quad \text{as } n \to \infty,$$

proving part (ii).

Since by (ii), $v(\cdot)$ is continuous, T-periodic and strictly positive, the proof of part (iii) follows from the analysis of Theorem 2.6.1 and Proposition 2.7.2. $\diamond$

7.3.2 Returns Distribution

We recall the standard Definition 2.7.1 and prove

Theorem 7.3.2 *Let $\{r(\Delta;t)\}_{t\geq 0}$ be defined by (2.68), where X_t satisfies (6.1). Then*

(i) $r(\Delta;t)$ does not converge in distribution as $t \to \infty$.

(ii) There exists a Lipschitz continuous, T-periodic, strictly positive function $v_\Delta(\cdot)$ such that for all $t \in [0,T]$, if

$$\tilde{r}(\Delta;t) \sim \mathcal{N}(0, v_\Delta(t))$$

then

$$r(\Delta; t+nT) \xrightarrow{\mathcal{D}} \tilde{r}(\Delta;t) \qquad \text{as } n \to \infty.$$

(iii)

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{t}\int_0^t I_{\{r(\Delta;s)\leq x\}}\, ds\right] = F_\Delta(x) \tag{7.31}$$

where $F_\Delta(\cdot)$ is a distribution function with density $f_\Delta(\cdot)$ given by

$$f_\Delta(x) = \frac{1}{T}\int_0^T \frac{1}{\sqrt{2\pi}\sqrt{v_\Delta(s)}} e^{-\frac{1}{2v_\Delta(s)}x^2}\, ds. \tag{7.32}$$

Proof: (i) is a direct consequence of (ii), and if (ii) is true, (iii) follows in the same manner as in Theorem 7.3.1. We therefore need only prove (ii).

One can use Proposition 7.2.2 to show that

$$r(\Delta; t) \sim \mathcal{N}(e(t+\Delta) - e(t), \tilde{v}_\Delta(t))$$

where

$$\tilde{v}_\Delta(t) = \int_0^t \sigma(s)^2 (g(s, t+\Delta) - g(s,t))^2 \, ds + \int_t^{t+\Delta} \sigma(s)^2 g(s, t+\Delta)^2 \, ds. \tag{7.33}$$

As before we show for any $t \geq 0$ that

$$\lim_{n\to\infty} \tilde{v}_\Delta(t+nT) \qquad \text{exists,} \tag{7.34}$$

and define

$$v_\Delta(t) = \lim_{n\to\infty} \tilde{v}_\Delta(t+nT) \tag{7.35}$$

and show that $v_\Delta(\cdot)$ satisfies the desired properties. Equation (7.34) is true if

$$\lim_{n,m\to\infty,\, n>m} \tilde{v}_\Delta(t+nT) - \tilde{v}_\Delta(t+mT) = 0. \tag{7.36}$$

Partition thus:

$$\begin{aligned}\tilde{v}_\Delta(t+nT) - \tilde{v}_\Delta(t+mT) &= \int_0^{(n-m)T} \sigma(s)^2 (g(s, t+nT+\Delta) - g(s, t+nT))^2 \, ds \qquad (7.37)\\ &+ \int_0^{t+mT} \sigma(s)^2 (g(s+(n-m)T, t+nT+\Delta) - g(s+(n-m)T, t+nT) - g(s, t+mT)\\ &+ g(s, t+mT+\Delta))\,(\Delta_s(n,m; t+mT+\Delta) - \Delta_s(n,m;t+mT))\, ds\end{aligned}$$

The second term on the right hand side of (7.37) can be bounded above using (7.28) and Proposition 7.2.4 by

$$\frac{2K^2 M_3^2 (1+e^{-\kappa\Delta})}{\kappa} \left(2e^{-2\kappa(t+mT)} \int_0^{t+mT} e^{2\kappa u} f^*(n,m;u)\, du + \Delta e^{\kappa\Delta} \sup_{t+mT \leq u \leq t+mT+\Delta} f^*(n,m;u) \right)$$

which tends to 0 as $n, m \to \infty$. To bound the first term on the right hand side of (7.37), write

$$\begin{aligned}&\int_0^{(n-m)T} \sigma(s)^2 (g(s,t+nT+\Delta) - g(s,t+nT))^2 \, ds\\ &\leq \; 2M_3^2 \int_0^{(n-m)T} g(s, t+nT+\Delta)^2 + g(s,t+nT)^2 \, ds,\end{aligned}$$

so employing Proposition 7.2.4 yields

$$\begin{aligned}&\int_0^{(n-m)T} \sigma(s)^2 (g(s,t+nT+\Delta) - g(s,t+nT))^2 \, ds\\ &\leq \; \frac{M_3^2 K^2}{\kappa} (1+e^{-2\kappa\Delta}) \left(e^{-2\kappa(t+mT)} - e^{-2\kappa(t+nT)} \right),\end{aligned}$$

which tends to 0 as $n, m \to \infty$. These limits establish (7.36). The T–periodicity of $v_\Delta(\cdot)$ defined in (7.35) above is clear. To show that it is Lipschitz, let $t_1 < t_2 \le t_1 + \Delta$, and decompose it according to

$$
\begin{aligned}
& v_\Delta(t_2) - v_\Delta(t_1) && (7.38)\\
= \;& \lim_{n\to\infty} \int_{t_2+nT}^{t_2+nT+\Delta} \sigma(s)^2 g(s, t_2 + nT + \Delta)^2\, ds - \int_{t_1+nT}^{t_1+nT+\Delta} \sigma(s)^2 g(s, t_1 + nT + \Delta)^2\, ds && (7.39)\\
+ \;& \int_0^{t_2+nT} \sigma(s)^2 (g(s, t_2 + nT + \Delta) - g(s, t_2 + nT))^2\, ds \\
- \;& \int_0^{t_1+nT} \sigma(s)^2 (g(s, t_1 + nT + \Delta) - g(s, t_1 + nT))^2\, ds.
\end{aligned}
$$

To bound the first term on the right hand side of (7.38), we rearrange it thus

$$
\begin{aligned}
& \int_{t_2+nT}^{t_2+nT+\Delta} \sigma(s)^2 g(s, t_2 + nT + \Delta)^2\, ds - \int_{t_1+nT}^{t_1+nT+\Delta} \sigma(s)^2 g(s, t_1 + nT + \Delta)^2\, ds \\
& \quad = \int_{t_1+nT+\Delta}^{t_2+nT+\Delta} \sigma(s)^2 g(s, t_2 + nT + \Delta)\, ds - \int_{t_1+nT}^{t_2+nT} \sigma(s)^2 g(s, t_1 + nT + \Delta)^2\, ds \\
& \quad + \int_{t_2+nT}^{t_1+nT+\Delta} \sigma(s)^2 (g(s, t_2 + nT + \Delta) - g(s, t_1 + nT + \Delta))^2\, ds.
\end{aligned}
$$

The first two integrals in this rearrangement have absolute values $\le M_3^2(t_2 - t_1)$. The absolute value integrand in the third term is bounded by

$$
M_3^2 \sup_{0 \le s^* \le t^* < \infty} \left| \frac{\partial g^2}{\partial t}(s^*, t^*) \right| (t_2 - t_1),
$$

so the absolute value of the first term on the right hand side of (7.38) is bounded above by

$$
M_3^2 (t_2 - t_1)(2 + 2(2M_1 + M_2)\Delta).
$$

To bound the second term on the right hand side of (7.38), we rearrange it thus

$$
\begin{aligned}
& \int_0^{t_2+nT} \sigma(s)^2 (g(s, t_2 + nT + \Delta) - g(s, t_2 + nT))^2\, ds && (7.40)\\
& \quad - \int_0^{t_1+nT} \sigma(s)^2 (g(s, t_1 + nT + \Delta) - g(s, t_1 + nT))^2\, ds \\
& \quad = \int_{t_1+nT}^{t_2+nT} \sigma(s)^2 (g(s, t_2 + nT + \Delta) - g(s, t_2 + nT))^2\, ds \\
& + \int_0^{t_1+nT} \sigma(s)^2 \left((g(s, t_2 + nT + \Delta) - g(s, t_2 + nT))^2 - (g(s, t_1 + nT + \Delta) - g(s, t_1 + nT))^2 \right) ds.
\end{aligned}
$$

The first term on the right hand side of (7.40) is bounded by $4M_3^2(t_2-t_1)$. Reformulating the second term on the right hand side of (7.40) leads to

$$\begin{aligned}
&\int_0^{t_1+nT} \sigma(s)^2 \left((g(s,t_2+nT+\Delta)-g(s,t_2+nT))^2-(g(s,t_1+nT+\Delta)-g(s,t_1+nT))^2\right) ds \\
&= \int_0^{t_1+nT} \sigma(s)^2 \left((g(s,t_2+nT+\Delta)-g(s,t_1+nT+\Delta))+(g(s,t_1+nT)-g(s,t_2+nT))\right) \\
&\times \quad (g(s,t_2+nT+\Delta)-g(s,t_2+nT)+g(s,t_1+nT+\Delta)-g(s,t_1+nT))\, ds.
\end{aligned}$$

Using Proposition 7.2.4 and the Mean Value Theorem as in (7.38) bounds this quantity by:

$$\frac{2KM_3^2(2M_1+M_2)}{\kappa}\left(1+e^{-\kappa\Delta}\right)\left(1+e^{-\kappa(t_2-t_1)}\right)(t_2-t_1),$$

so therefore

$$|v_\Delta(t_2)-v_\Delta(t_1)| \leq M_3^2\left(6+2(2M_1+M_2)\Delta+\frac{4K(2M_1+M_2)}{\kappa}(1+e^{-\kappa\Delta})\right)(t_2-t_1),$$

establishing Lipschitz continuity. We relegate the proof that $v_\Delta(\cdot)$ is strictly positive to the Appendix of this chapter. ◇

7.4 Properties of the Density Function of the Asymptotic EDF of the Δ-returns

As in Chapter 2, we emphasise several properties of the density functions $f_\Delta(\cdot)$: first, its symmetry and unimodality about 0; second, that the probability of the returns being close to 0 is higher than for a normal density with the same variance; third, that the tail of the distribution is heavier than a normal distribution with the same variance. From this we notice that the price process has the desirable property of being relatively quiescent for periods of time, but also capable of experiencing larger extreme deviations.

Proposition 7.4.1 *Suppose that $v_\Delta(\cdot)$ given by (7.35) is non-constant. Let $F_\Delta(\cdot), f_\Delta(\cdot)$ be as defined in Theorem 7.3.2 Then*

(a) $F_\Delta(\cdot)$ is a symmetric, unimodal distribution with mode at 0.

(b) If $Y \sim \mathcal{N}\left(0, \frac{1}{T}\int_0^T v_\Delta(s)\,ds\right)$, with distribution function $\bar{F}_\Delta(\cdot)$ then

(i) there exists $\underline{a} > 0$ such that for all $|x| \leq \underline{a}$

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{t}\int_0^t I_{\{|r(\Delta;s)|\leq x\}}\,ds\right] > \bar{F}_\Delta(x) - \bar{F}_\Delta(-x). \tag{7.41}$$

(ii) there exists $\bar{a} > \underline{a}$ such that for all $x \geq \bar{a}$

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{1}{t}\int_0^t I_{\{r(\Delta;s)\leq x\}}\,ds\right] > 1 - \bar{F}_\Delta(x). \tag{7.42}$$

(c)

$$\lim_{x\to\infty} \frac{1}{x^2}\log f_\Delta(x) = -\frac{1}{2\max_{0\leq s\leq T} v_\Delta(s)}. \tag{7.43}$$

Proof: Mimic the proof of Proposition 2.7.1. ⋄

7.5 Appendix

We now prove the results outstanding in Theorems 7.3.1, and 7.3.2.

Lemma 7.5.1 *$v(\cdot)$ defined by (7.19) is strictly positive.*

Proof: Let $t \geq 0$. Let ν be the positive solution of

$$\nu^2(2a(0)M_1) + \nu \sup_{t\geq 0}|\beta(t) - \lambda(t)| - 1 = 0, \tag{7.44}$$

and define $C = \frac{1}{\nu}$. By virtue of Definition 4.2.1 (iv) there exists $N_1(t) \in \mathbb{N}$ such that for all $n > N_1(t)$

$$\sup_{0\leq s\leq u, u\geq t+nT-\nu}\left|\frac{w(s,u)}{a(u-s)} - 1\right| < 1. \tag{7.45}$$

From (7.19), there exists $N_2(t) \in \mathbb{N}$ such that for all $n > N_2(t)$

$$\left|v(t) - \int_0^{t+nT} \sigma(s)^2 g(s, t+nT)^2\,ds\right| \leq \frac{1}{2}\frac{\underline{\sigma}^2}{3C} \tag{7.46}$$

where $\underline{\sigma}$ is defined in (3.8). Let $N(t) = \max(N_1(t), N_2(t))$, and let $n > N(t)$. We can then produce the following bound:

$$\begin{aligned}\sup_{t+nT-\nu\leq s\leq t+nT}\int_s^{t+nT} w(s,u)\,du &\leq \nu a(0)\left(1 + \sup_{0\leq s\leq u, u\geq t+nT-\nu}\left|\frac{w(s,u)}{a(u-s)} - 1\right|\right) \\ &\leq 2\nu a(0)\end{aligned} \tag{7.47}$$

where the second inequality follows from (7.45). From [49] p.24–25, the resolvent $g(\cdot,\cdot)$ given by (3.10), (3.11) has the following alternative description:

$$\frac{\partial g}{\partial s}(s,t) = (\beta(s) - \lambda(s))g(s,t) + \int_s^t \lambda(u)w(s,u)g(u,t)\,du. \tag{7.48}$$

Using Lemma 6.3.1 (7.44), (7.47) and (7.48), one can show, for $s \in [t+nT-\nu, t+nT]$ that

$$\left|\frac{\partial g}{\partial s}(s,t)\right| \le C,$$

so from the Mean Value Theorem, one has, for all $s \in [t+nT-\nu, t+nT]$

$$g(s, t+nT) \ge 1 - C(t+nT-s) \ge 0. \tag{7.49}$$

Finally, from (3.8), (7.46) and (7.49), we obtain

$$v(t) > \frac{1}{2}\frac{\underline{\sigma}^2}{3C},$$

which proves the lemma. $\diamond$

Lemma 7.5.2 $v_\Delta(\cdot)$ *defined by (7.35) is strictly positive.*

Proof: Let $t \ge 0$, $C = \sup_{s\ge 0}|\beta(s) - \lambda(s)| + 2M_1\Delta a(0)$. There exists $N_1(t) \in \mathbb{N}$ such that for all $n > N_1(t)$

$$\sup_{0\le s\le u, u\ge t+nT}\left|\frac{w(s,u)}{a(u-s)} - 1\right| < 1.$$

We consider two cases

<u>Case 1:</u> $c\Delta \le 1$

There exists $N_2(t) \in \mathbb{N}$ such that for all $n > N_2(t)$

$$v_\Delta(t) - \left(\int_0^{t+nT} \sigma(s)^2(g(s,t+nT+\Delta) - g(s,t+nT))^2\,ds + \int_{t+nT}^{t+nT+\Delta} \sigma(s)^2 g(s,t+nT+\Delta)^2\,ds\right)$$

$$> -\frac{\Delta\underline{\sigma}^2}{6} \tag{7.50}$$

Let $N(t) = \max(N_1(t), N_2(t))$, and let $n > N(t)$. We then have

$$\sup_{t+nT\le s\le t+nT+\Delta}\int_s^{t+nT+\Delta} w(s,u)\,du \le 2\Delta a(0).$$

Thus for $s \in [t + nT, t + nT + \Delta]$,

$$\left|\frac{\partial g}{\partial s}(s,t)\right| \leq C,$$

and so $g(s, t + nT + \Delta) \geq 1 - \frac{1}{\Delta}(t + nT + \Delta - s)$. The same argument as in Lemma 7.5.1 above yields

$$v_\Delta(t) > \frac{\Delta \sigma^2}{6}.$$

Case 2: $c\Delta > 1$

This time we define $N_2(t)$ by replacing the constant on the right hand side of (7.50) by $\frac{\sigma^2}{6C}$. Once again, $g(s, t + nT + \Delta) \geq 1 - \frac{1}{\Delta}(t + nT + \Delta - s)$ for $s \in [t + nT, t + nT + \Delta]$. Set $T^* = \frac{1}{C}$. Thus

$$v_\Delta(t) > \sigma^2 \int_{t+nT+\Delta-T^*}^{t+nT+\Delta} g(s, t + nT + \Delta)^2 \, ds - \frac{\sigma^2}{6C}$$

so $v_\Delta(t) > \frac{\sigma^2}{6C}$, concluding the proof of the lemma. $\diamond$

Chapter 8

Pathwise Asymptotics of the Extrema of Linear S.I-D.Es

8.1 Introduction and Motivation

In this chapter we consider the almost sure growth rate of the extremes of a linear stochastic integro-differential equation of the form

$$dX_t = \lambda(t)\left(X_t - \int_0^t w(s,t)X_s\,ds\right) - \beta(t)\,(X_t - K^*)\,dt + \sigma(t)\,dB_t \tag{8.1}$$

with deterministic initial condition X_0, periodically oscillating (or constant) functions $\lambda(\cdot), \beta(\cdot)$ and $\sigma(\cdot)$, and asymptotically shift invariant $w(\cdot,\cdot)$.

Exploiting the fact that X_t is Gaussian for all t, one can use the theory of Liapunov functions for deterministic integro-differential equations to show that the resolvent of a related deterministic equation has bounded exponential decay, and hence so does the correlation of X_t. By applying the result of Klass on almost sure fluctuations of maxima of iid random variables, we show that discrete stationary normal sequences with exactly geometric decay in their correlation have the same almost sure fluctuations. It then follows by the normal comparison lemma and its direct consequences that all normal sequences with correlations bounded by geometric decay have large deviations of $O(\sqrt{2\log n})$ almost surely.

We can extend this result to continuous time to achieve a liminf bound:

$$\liminf_{t\to\infty} \frac{\max_{0\le s\le t} X_s}{\sqrt{2\log t}} \ge \sqrt{\max_{0\le s\le T} v(s)} \quad \text{a.s.},$$

where $v(\cdot)$ is a T-periodic function which is the uniform limit of the variance of X_t, $\bar{v}(t)$. The almost sure asymptotic boundedness on the whole real line of

$$\frac{\max_{0\le s\le t} X_s}{\sqrt{2\log t}}$$

has not yet been secured: however, the following "modulo–continuity" result has been attained; for every $h > 0$

$$\limsup_{t\to\infty} \frac{\max_{0\le i\le \lfloor \frac{t}{h}\rfloor} X_{ih}}{\sqrt{2\log t}} \le \sqrt{2\max_{0\le s\le T} v(s)} \quad \text{a.s.},$$

while if there exists $T_1 > 0$ such that $Cov(X_s, X_t) \ge 0$ for all $t, s > T_1$, we have

$$\lim_{t\to\infty} \frac{\max_{0\le i\le \lfloor \frac{t}{h}\rfloor} X_{ih}}{\sqrt{2\log t}} = \sqrt{\max_{0\le s\le T} v(s)} \quad \text{a.s.}$$

While there exists a comprehensive literature relating to the classical and weak limit theory of extremes and sojourns, very few results seem to be available to form a strong limit theory of extremes. In fact, I have only been able to identify a handful of papers, all of which relate to discrete time stochastic processes which are iid. Since we are studying a nonstationary dependent continuous time process, some additional analysis is plainly necessary.

The almost sure asymptotic behaviour of the extremes of discrete time iid processes has been essentially settled by the results of Klass [44], [45], building on the theory pioneered by Barndorff-Nielsen. Both authors concentrated on the normalised liminf of maxima: the treatment of the normalised limsup of the maxima of iid processes follows directly from the following observation: if $X_1, \ldots, X_n$ are a sequence of iid random variables and $M_n = \max_{1\le j\le n} X_j$, then if $\{u_n\}$ is a non-decreasing sequence

$$\mathbb{P}[M_n > u_n \text{ i.o.}] = \mathbb{P}[X_n > u_n \text{ i.o.}] \tag{8.2}$$

and so $\mathbb{P}[M_n > u_n \text{ i.o.}] = 0$ or 1 according as $\sum_{n=1}^{\infty} \mathbb{P}[X > u_n] <$ or $= \infty$.

A readable sketch of the proof of the normalised liminf of maxima is given in Embrechts et al. [22]. The proof is quite technical and its outcome, though not its argument, will be used here.

To prove the desired result, we proceed in three steps, each comprising a section hereafter. First, we observe that we can directly apply the results of Klass to a linear (Gram-Schmidt) transformation of a sequence of stationary normal random variables with exponentially decreasing correlation. By partitioning this sequence into k sequences we can show that the a.s. growth of the partial maxima is identical to that in the iid case. To my knowledge, this is the first a.s. characterisation of the growth rate of maxima of a dependent process.

Next, by using a result which compares the probabilities of maxima of two normal processes when their correlations are ordered in a particular manner, we use the ideas of the first section to show that sequences of jointly normal random variables have maxima of $O(\sqrt{\log n})$. In fact, if the correlations are always non-negative, the a.s. growth of the partial maxima is again identical to that in the iid case.

In the final section, we show that the correlation of the solution of the integro-differential equation conforms with that outlined in the previous section. Starting from a sufficiently large though finite time (in order to discard very low variances), we sample the process at equal intervals that are a certain multiple of T. Aggregrating these sequences, and choosing an aggregate sequence along a set of T-separated times, we can prove the liminf bound. The limsup bound is obtained by sampling the process at multiples of the arbitrary positive number h, and by aggregating these sequences and using the discrete time theory.

We see from the final result that the maxima has order of magnitude $\sqrt{\max_{0\le s\le T} v(s)}\sqrt{2\log t}$ multiplied by a pure number, or, if the correlation is positive after some finite time, is almost surely asymptotic to $\sqrt{\max_{0\le s\le T} v(s)}\sqrt{2\log t}$. Thus, larger values of $\max_{0\le s\le T} v(s)$ lead to larger partial maxima (and by symmetry, smaller partial minima). The analysis in the memory-independent case indicates that higher maximum variance occurs when the market is more uncertain or more likely to engage in trend chasing.

8.2 Fluctuations of Stationary Gaussian Sequences

As usual, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\,du$. The following two results are fundamental to our discussion. First, the Normal Comparison Lemma.

Lemma 8.2.1 *If X_j is a stationary sequence of standard normal variables with $Corr(X_i, X_j) =$*

$r_{|i-j|}$ *and*

$$M_n = \max_{1 \le j \le n} X_j,$$

then

$$|\mathbb{P}[M_n \le u_n] - \Phi(u_n)^n| \le Kn \sum_{j=1}^{n} |r_j| \exp\left(-\frac{-u_n^2}{1+|r_j|}\right), \tag{8.3}$$

where K *depends on* $\delta = \max_{j \ge 1} |r_j| < 1$.

Proof: See Leadbetter et al. p.81-85. [51]. $\diamond$

Particularising the arguments of Klass [44], we have the following in Embrechts et al. [22].

Lemma 8.2.2 *Suppose that* X_j *is a sequence of independent standard normal variables, and let*

$$M_n = \max_{1 \le j \le n} X_j.$$

Then

$$\lim_{n\to\infty} \frac{M_n}{\sqrt{2\log n}} = 1 \quad a.s. \tag{8.4}$$

Proof: This follows from Corollary 3.5.3 in Embrechts et al. [22], and is outlined in Example 3.5.4. over pages 174-176. $\diamond$

Lemma 8.2.3 *Let* X_j *be a stationary sequence of standard normal random variables. Moreover, if* $Cov(X_i, X_j) = \lambda^{|i-j|}$ *for some* $\lambda \in (0,1)$, *then*

$$\limsup_{n\to\infty} \frac{\max_{1\le j\le n} X_j}{\sqrt{2\log n}} \le \sqrt{2} \quad a.s., \tag{8.5}$$

and similarly for $\min_{1\le j\le n} X_j$.

Proof: Let $\alpha > 4$ and define $u_n = \sqrt{\alpha \log n}$. From Mill's estimate we therefore obtain

$$1 - \Phi(u_n) \le \frac{1}{\sqrt{2\pi\alpha}\sqrt{\log n}} \frac{1}{n^{\alpha/2}} \triangleq \psi(n)$$

and so

$$1 - \Phi(u_n)^n \le 1 - (1-\psi(n))^n \le n\psi(n) \tag{8.6}$$

for all $n \geq N_1 = \min\{n \in \mathbb{N} : \psi(n) < 1\}$. Define $r_j = \lambda^j$ and $B_n = Kn\sum_{j=1}^{n} |r_j| exp\left(\frac{-u_n^2}{1+|r_j|}\right)$. From Lemma 8.2.1 we have

$$\mathbb{P}[M_n > u_n] \leq 1 - \Phi(u_n)^n + B_n. \tag{8.7}$$

With the choice of u_n we easily obtain, for some $0 < K_2 < \infty$

$$B_n \leq K\frac{1}{n^{\frac{\alpha}{2}-1}}\sum_{j=1}^{n}|r_j| < \frac{K_2}{n^{\frac{\alpha}{2}-1}}. \tag{8.8}$$

Taking (8.6), (8.7) and (8.8) together we see that for all $n \geq N_1$

$$\mathbb{P}[M_n > \sqrt{\alpha \log n}] \leq \frac{1}{\sqrt{2\pi\alpha}\sqrt{\log n}}\frac{1}{n^{\frac{\alpha}{2}-1}} + K_2\frac{1}{n^{\frac{\alpha}{2}-1}}$$

whence

$$\limsup_{n\to\infty}\frac{M_n}{\sqrt{2\log n}} \leq \sqrt{\frac{\alpha}{2}} \quad \text{a.s..}$$

Taking limits as $\alpha \downarrow 4$ proves the assertion (8.5). By symmetry, the minimum behaves similarly. ⋄

We now prove the principal result in this section.

Proposition 8.2.1 *Let $X_j, j = 1, \ldots, n$ be a sequence of standard normal random variables with $Cov(X_i, X_j) = \lambda^{|i-j|}$ for some $\lambda \in (0,1)$. Then*

$$\lim_{n\to\infty}\frac{\max_{1\leq j\leq n} X_j}{\sqrt{2\log n}} = 1 \quad a.s.. \tag{8.9}$$

Proof: Let $k \in \mathbb{N}$ and for each $j = 1, \ldots, k$ define

$$U_n^j = \frac{X_{j+nk} - \lambda^k X_{j+(n-1)k}}{\sqrt{1-\lambda^{2k}}}.$$

Without loss of generality, let $n \neq m$ and notice that $Cov(U_n^j, U_m^j) = 0$, while $Var(U_n^j) = 1$ for all n and j. Thus for each $j = 1, \ldots, k$, $\{U_n^j\}_{n=1}^{\infty}$ is an iid sequence of standard normal random variables. Therefore, using Lemma 8.2.2 we have

$$\lim_{n\to\infty}\frac{\max_{1\leq r\leq n}\left(X_{j+rk} - \lambda^k X_{j+(r-1)k}\right)}{\sqrt{2\log n}}\frac{1}{\sqrt{1-\lambda^{2k}}} = 1 \quad \text{a.s.}$$

for $j = 1, \ldots, k$. Next, define

$$\Omega_j = \{\omega : \lim_{n\to\infty}\frac{\max_{1\leq r\leq n}\left(X_{j+rk}(\omega) - \lambda^k X_{j+(r-1)k}(\omega)\right)}{\sqrt{2\log n}}\frac{1}{\sqrt{1-\lambda^{2k}}} = 1\}$$

and $\Omega = \cap_{j=1}^{k}\Omega_j$, so $\mathbb{P}[\Omega] = 1$. Then for all $\omega \in \Omega$

$$\lim_{n\to\infty} \frac{\max_{1\le r\le n}\left(X_{j+rk}(\omega) - \lambda^k X_{j+(r-1)k}(\omega)\right)}{\sqrt{2\log n}} \frac{1}{\sqrt{1-\lambda^{2k}}} = 1$$

for all $j = 1, \dots, k$ so

$$\lim_{n\to\infty} \frac{\max_{1\le j\le k}\max_{1\le r\le n}\left(X_{j+rk}(\omega) - \lambda^k X_{j+(r-1)k}(\omega)\right)}{\sqrt{2\log n}} \frac{1}{\sqrt{1-\lambda^{2k}}} = 1. \tag{8.10}$$

Therefore

$$\lim_{n\to\infty} \frac{\max_{1\le j\le k}\max_{1\le r\le n}\left(X_{j+rk}(\omega) - \lambda^k X_{j+(r-1)k}(\omega)\right)}{\sqrt{2\log(n+1)k}} \frac{1}{\sqrt{1-\lambda^{2k}}} = 1 \text{ and}$$
$$\lim_{n\to\infty} \frac{\max_{1\le j\le k}\max_{1\le r\le n+1}\left(X_{j+rk}(\omega) - \lambda^k X_{j+(r-1)k}(\omega)\right)}{\sqrt{2\log nk}} \frac{1}{\sqrt{1-\lambda^{2k}}} = 1.$$

Now taking $nk \le N < (n+1)k$, we also have

$$\frac{\max_{1\le j\le k}\max_{1\le r\le n}\left(X_{j+rk} - \lambda^k X_{j+(r-1)k}\right)}{\sqrt{2\log(n+1)k}} \le \frac{\max_{1\le p\le N}\left(X_{p+k} - \lambda^k X_p\right)}{\sqrt{2\log N}} \le \frac{\max_{1\le j\le k}\max_{1\le r\le n+1}\left(X_{j+rk} - \lambda^k X_{j+(r-1)k}\right)}{\sqrt{2\log nk}},$$

so

$$\lim_{N\to\infty} \frac{\max_{1\le p\le N}\left(X_{p+k} - \lambda^k X_p\right)}{\sqrt{2\log N}} \frac{1}{\sqrt{1-\lambda^{2k}}} = 1 \quad \text{a.s.}.$$

For every $\varepsilon > 0$ there exists $k(\varepsilon) \in \mathbb{N}$ such that for all $k > k(\varepsilon)$ we have $|1 - \sqrt{1-\lambda^{2k}}| < \frac{\varepsilon}{2}$ and $\lambda^k < \frac{\varepsilon}{2\sqrt{2}}$. Now fix $k = k(\varepsilon) + 1$. Then

$$\frac{\max_{1\le p\le N} X_{p+k}}{\sqrt{2\log N}} - 1 \le \frac{\max_{1\le p\le N}\left(X_{p+k} - \lambda^k X_p\right)}{\sqrt{2\log N}} \frac{1}{\sqrt{1-\lambda^{2k}}}\sqrt{1-\lambda^{2k}} - 1 + \lambda^k \frac{\max_{1\le p\le N} X_p}{\sqrt{2\log N}}, \tag{8.11}$$

so using Lemma 8.2.3 we have

$$\limsup_{N\to\infty} \frac{\max_{1\le p\le N} X_{p+k}}{\sqrt{2\log N}} - 1 < \varepsilon.$$

Using a decomposition similar to that in (8.11) and Lemma 8.2.3 for the partial minimum, we have

$$\liminf_{N\to\infty} \frac{\max_{1\le p\le N} X_{p+k}}{\sqrt{2\log N}} - 1 > \varepsilon.$$

Thus for all $\varepsilon > 0$ and $k = k(\varepsilon) + 1$ we have

$$\limsup_{N\to\infty} \left|\frac{\max_{1\le p\le N} X_{p+k}}{\sqrt{2\log N}} - 1\right| < \varepsilon. \tag{8.12}$$

Define

$$\begin{aligned} Y_{N,k} &= \frac{\max_{1 \leq p \leq N} X_{p+k}}{\sqrt{2 \log N}}, \\ Y_N &= \frac{\max_{1 \leq p \leq N} X_p}{\sqrt{2 \log N}}, \end{aligned}$$

and the events

$$A_{m,k} = \{\omega : |Y_{m,k}(\omega) - 1| \geq 2\varepsilon\}, \tag{8.13}$$

$$C_m = \{\omega : |Y_m(\omega) - 1| \geq 2\varepsilon\}. \tag{8.14}$$

Observe for any $p \geq N$ that the stationarity of the sequence X implies

$$\mathbb{P}\left[\bigcap_{n=1}^{N} \bigcup_{m=n}^{p} A_{m,k}\right] = \mathbb{P}\left[\bigcap_{n=1}^{N} \bigcup_{m=n}^{p} C_m\right]. \tag{8.15}$$

We will presently prove

$$\mathbb{P}[A_{N,k} \text{ i.o.}] = 0. \tag{8.16}$$

Assume for the moment that this is true. The following argument is valid for both $A_{m,k}$ and C_m. We focus on $A_{m,k}$. Fix $n \in \mathbb{N}$ and define for $p \geq N$

$$B_p = \bigcap_{n=1}^{N} \bigcup_{m=n}^{p} A_{m,k},$$

so $B_p \subseteq B_{p+1}$ for all $p \geq N$. It is also true that

$$\bigcup_{p=N}^{\infty} B_p = \bigcup_{p=N}^{\infty} \bigcap_{n=1}^{N} \bigcup_{m=n}^{p} A_{m,k} = \bigcap_{n=1}^{N} \bigcup_{m=n}^{\infty} A_{m,k},$$

so since B_p is an increasing sequence of events, we have

$$\lim_{p \to \infty} \mathbb{P}[B_p] = \mathbb{P}[\bigcup_{p=N}^{\infty} B_p],$$

so

$$\lim_{p \to \infty} \mathbb{P}\left[\bigcap_{n=1}^{N} \bigcup_{m=n}^{p} A_{m,k}\right] = \mathbb{P}\left[\bigcap_{n=1}^{N} \bigcup_{m=n}^{\infty} A_{m,k}\right]. \tag{8.17}$$

In an identical manner we obtain

$$\lim_{p \to \infty} \mathbb{P}\left[\bigcap_{n=1}^{N} \bigcup_{m=n}^{p} C_m\right] = \mathbb{P}\left[\bigcap_{n=1}^{N} \bigcup_{m=n}^{\infty} C_m\right]. \tag{8.18}$$

Using (8.15), (8.17), (8.18) shows that

$$\mathbb{P}\left[\bigcap_{n=1}^{N} \bigcup_{m=n}^{\infty} C_m\right] = \mathbb{P}\left[\bigcap_{n=1}^{N} \bigcup_{m=n}^{\infty} A_{m,k}\right]$$

and taking limits as $N \to \infty$ in conjunction with (8.16) gives

$$\mathbb{P}[C_N \text{ i.o.}] = \mathbb{P}[A_{N,k} \text{ i.o.}] = 0.$$

Thus for all $\varepsilon > 0$, $\mathbb{P}[|Y_N - 1| \geq 2\varepsilon \text{ i.o.}] = 0$. By the definition of Y_N it follows that

$$\lim_{N\to\infty} \frac{\max_{1\leq p\leq N} X_p}{\sqrt{2\log N}} = 1 \quad \text{a.s.}.$$

We now return to the proof of $\mathbb{P}[A_{N,k} \text{ i.o.}] = 0$. Rewrite equation (8.12) in pointwise form: for every $\omega \in \Omega$ where $\mathbb{P}[\Omega] = 1$ we have

$$\lim_{N\to\infty} \sup_{m\geq N} |Y_{m,k} - 1| < \varepsilon.$$

Thus for each $\omega \in \Omega$ there exists $0 \leq D_k(\omega) < \varepsilon$ such that

$$\lim_{N\to\infty} \sup_{m\geq N} |Y_{m,k} - 1| = D_k(\omega).$$

Returning to definitions, we see that for all $\varepsilon > 0$ and $\omega \in \Omega$ there exists $N(\varepsilon, \omega) \in \mathbb{N}$ such that for all $N > N(\varepsilon, \omega)$

$$\left| \sup_{m\geq N} |Y_{m,k}(\omega) - 1| - D_k(\omega) \right| < \varepsilon$$

so

$$0 \leq |Y_{N,k}(\omega) - 1| \leq \sup_{m\geq N} |Y_{m,k}(\omega) - 1| \leq \left| \sup_{m\geq N} |Y_{m,k}(\omega) - 1| - D_k(\omega) \right| + |D_k(\omega)| < 2\varepsilon.$$

Since $A_{N,k} = \{\omega : |Y_{N,k}(\omega) - 1| \geq 2\varepsilon\}$, it follows that $\mathbb{P}[A_{N,k} \text{ i.o.}] = 0$. $\diamond$

8.3 Normal Sequences with Exponentially Bounded Correlations

In order to determine the almost sure fluctuations of the maxima of the stochastic integro-differential equation, we will need to relax the assumption of precise exponential decay in the correlation over time. In the next section, we prove that the correlation is bounded by an exponentially decaying function. In this section, we will therefore wish to characterise the almost sure fluctuations of normal sequences with exponentially bounded correlation.

Towards this end, we require two further results related to the normal comparison lemma. Firstly, monotonicity in correlation leads to monotonicity in stochastic boundedness in the following sense.

Lemma 8.3.1 *Let X_j, Y_j for $j = 1, \dots, n$ be sequences of standard normal random variables which satisfy*

$$Cov(X_i, X_j) \leq Cov(Y_i, Y_j)$$

for all $i, j = 1, \dots, n$. If $M_n^X = \max_{1\leq j\leq n} X_j$ and $M_n^Y = \max_{1\leq j\leq n} Y_j$, then

$$\mathbb{P}[M_n^X \leq u] \leq \mathbb{P}[M_n^Y \leq u]$$

for all $u \in \mathbb{R}$.

Proof: See Leadbetter et al. p.84. $\diamond$

We will also need a more general version of the normal comparison lemma.

Lemma 8.3.2 *Let $X_1, \dots, X_n$ be jointly normal standardised random variables and define $r_{i,j} = Cov(X_i, X_j)$. If $\delta = \max_{i\neq j} |r_{i,j}| < 1$ then*

$$|\mathbb{P}[M_n \leq u_n] - \Phi(u_n)^n| \leq K \sum_{1\leq i<j\leq n} |r_{i,j}| exp\left(\frac{-u_n^2}{1+|r_{i,j}|}\right)$$

where K is a finite constant, depending on δ.

Proof: See Leadbetter et al. p.84. [51]. $\diamond$

Using this one can prove an analogue of Lemma 8.2.3

Lemma 8.3.3 *Let X_j be a sequence of jointly normal standard random variables satisfying*

$$|Cov(X_i, X_j)| \leq \lambda^{|i-j|}$$

for some $\lambda \in (0, 1)$. Then

$$\limsup_{n\to\infty} \frac{\max_{1\leq j\leq n} X_j}{\sqrt{2\log n}} \leq \sqrt{2} \quad a.s. \tag{8.19}$$

and similarly for $\min_{1\leq j\leq n} X_j$.

Proof: Using Lemma 8.3.2, the proof proceeds in the same manner as that of Lemma 8.2.3. $\diamond$

We are now in a position to prove the main result of this section.

Proposition 8.3.1 *Suppose $X_1, \ldots, X_n$ is a sequence of jointly normal standard random variables satisfying*

$$|Cov(X_i, X_j)| \le \lambda^{|i-j|}$$

for some $\lambda \in (0,1)$. Then

$$1 \le \liminf_{n\to\infty} \frac{\max_{1\le j\le n} X_j}{\sqrt{2\log n}} \le \limsup_{n\to\infty} \frac{\max_{1\le j\le n} X_j}{\sqrt{2\log n}} \le \sqrt{2}. \tag{8.20}$$

Moreover, if $Cov(X_i, X_j) \ge 0$ then

$$\lim_{n\to\infty} \frac{\max_{1<j<n} X_j}{\sqrt{2\log n}} = 1 \quad a.s.. \tag{8.21}$$

Proof: The limsup part of (8.20) follows from Lemma 8.3.3 on letting $\alpha \downarrow 4$. To prove the liminf lower bound, let $Y_1, \ldots, Y_n$ be a sequence of jointly normal standard random variables satisfying $Cov(Y_i, Y_j) = \lambda^{|i-j|}$. Let M_n^X, M_n^Y have the same meaning as in Lemma 8.3.1. Thus for all $\varepsilon > 0$, we have by Lemma 8.3.1:

$$\mathbb{P}[M_n^X \le (1-\varepsilon)\sqrt{2\log n}] \le \mathbb{P}[M_n^Y \le (1-\varepsilon)\sqrt{2\log n}].$$

The proof is thus complete (after taking $\varepsilon \downarrow 0$) by the Borel-Cantelli lemma if we can show

$$\sum_{n=1}^{\infty} \mathbb{P}[M_n^Y \le (1-\varepsilon)\sqrt{2\log n}] < \infty. \tag{8.22}$$

The proof of (8.22) is the subject of Lemma 8.3.5. To prove (8.21), first note the following fact: if

$$\mathbb{P}\left[\bigcup_{j=1}^{n} B_j\right] \le \mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right]$$

and $\mathbb{P}\left[\bigcup_{j=1}^{n} A_j \text{i.o.}\right] = 0$, then

$$\mathbb{P}\left[\bigcup_{j=1}^{n} B_j \text{i.o.}\right] = 0. \tag{8.23}$$

To prove this simply notice that

$$\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}\bigcup_{j=1}^{m} B_j = \bigcap_{n=1}^{\infty}\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} B_j$$

so

$$\mathbb{P}\left[\bigcup_{j=1}^{n} B_j \text{i.o.}\right] = \mathbb{P}\left[\bigcup_{j=1}^{\infty} B_j\right] = \lim_{n\to\infty} \mathbb{P}\left[\bigcup_{j=1}^{n} B_j\right]$$

and analogously for A_j. However, since

$$\lim_{n\to\infty} \mathbb{P}\left[\bigcup_{j=1}^{n} B_j\right] \leq \lim_{n\to\infty} \mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right]$$

we have (8.23). Let $U_1, \dots, U_n$ be a sequence of iid normal standard random variables. Let

$$M_n^U = \max_{1\leq j\leq n} U_j.$$

Then for every $\varepsilon > 0$ we have, by Lemma 8.3.1

$$\mathbb{P}[\cup_{j=1}^n \{X_j > (1+\varepsilon)\sqrt{2\log n}\}] \leq \mathbb{P}[\cup_{j=1}^n \{U_j > (1+\varepsilon)\sqrt{2\log n}\}]$$

Since $\sum_{n=1}^{\infty} \mathbb{P}[U > (1+\varepsilon)\sqrt{2\log n}] < \infty$, by using the independence of $\{U_j > (1+\varepsilon)\sqrt{2\log n}\}$, the Borel-Cantelli lemma, and (8.2), we have

$$\mathbb{P}[M_n^U > (1+\varepsilon)\sqrt{2\log n} \text{ i.o.}] = \mathbb{P}[U_n > (1+\varepsilon)\sqrt{2\log n} \text{ i.o.}] = 0.$$

Putting $A_j = \{U_j > (1+\varepsilon)\sqrt{2\log n}\}$ and $B_j = \{X_j > (1+\varepsilon)\sqrt{2\log n}\}$, we can use (8.23) to prove

$$\mathbb{P}\left[M_n^X > (1+\varepsilon)\sqrt{2\log n} \text{ i.o.}\right] = 0,$$

so taking this in conjunction with (8.22) gives (8.21). $\diamond$

The proof of Lemma 8.3.5 requires a simple preliminary observation.

Lemma 8.3.4 *Suppose X, Y are random variables and $x, y \in \mathbb{R}$. Then*

(i) $\mathbb{P}[X + Y \geq x] \leq \mathbb{P}[X \geq y] + \mathbb{P}[Y \geq x - y]$;

(ii) If $X \geq Y$ a.s. then $\mathbb{P}[X < x] \leq \mathbb{P}[Y < x]$.

Proof:(i) For any $x, y \in \mathbb{R}$, $\mathbb{P}[X + Y < x] \geq \mathbb{P}[\{X < y\} \cap \{Y < x - y\}]$. Thus

$$\begin{aligned}\mathbb{P}[X + Y \geq x] &= 1 - \mathbb{P}[X + Y < x] \leq 1 - \mathbb{P}[\{X < y\} \cap \{Y < x - y\}] \\ &= \mathbb{P}[\{X \geq y\} \cup \{Y \geq x - y\}] \leq \mathbb{P}[X \geq y] + \mathbb{P}[Y \geq x - y].\end{aligned}$$

(ii) Follows from $\{X \leq x\} \subseteq \{Y \leq x\}$. $\diamond$

Lemma 8.3.5 *For the sequence of random variables Y_j, $j = 1, \dots, n$ defined in Proposition 8.3.1 we have, for all $\varepsilon \in (0, 1)$*

$$\sum_{n=1}^{\infty} \mathbb{P}[M_n^Y \leq (1-\varepsilon)\sqrt{2\log n}] < \infty.$$

Proof: For all $\varepsilon \in (0,1)$ there exists $k_1(\varepsilon) \in \mathbb{N}$ such that for all $k > k_1(\varepsilon)$

$$\lambda^k \le \frac{2}{3}\left(\sqrt{9+8\varepsilon-4\varepsilon^2} - 3(1-\varepsilon)\right) \triangleq \alpha(\varepsilon) > 0.$$

Fix $k = k_1(\varepsilon)+1$. Note that $\alpha(\varepsilon)$ is the unique positive root of $f_\varepsilon(x) = -2\varepsilon + \varepsilon^2 + 3x(1-\varepsilon) + \frac{13}{4}x^2$, so we have

$$0 < \frac{1-\varepsilon+\frac{3}{2}\lambda^k}{\sqrt{1-\lambda^{2k}}} < 1. \tag{8.24}$$

Since

$$\frac{\max_{1\le j\le n} Y_{j+k}}{\sqrt{2\log n}} \ge \frac{\max_{1\le j\le n} Y_{j+k} - \lambda^k Y_j}{\sqrt{2\log n}} - \lambda^k \frac{\max_{1\le j\le n} -Y_j}{\sqrt{2\log n}},$$

we use Lemma 8.3.4 part (ii) and then part (i) to bound as follows:

$$\begin{aligned}
&\mathbb{P}\left[\frac{\max_{1\le j\le n} Y_{j+k}}{\sqrt{2\log n}} \le 1-\varepsilon\right] \\
\le\ &\mathbb{P}\left[\frac{\max_{1\le j\le n} Y_{j+k} - \lambda^k Y_j}{\sqrt{2\log n}} - \lambda^k \frac{\max_{1\le j\le n} -Y_j}{\sqrt{2\log n}} \le 1-\varepsilon\right] \\
\le\ &\mathbb{P}\left[\lambda^k \frac{\max_{1\le j\le n} -Y_j}{\sqrt{2\log n}} \ge \frac{3}{2}\lambda^k\right] + \mathbb{P}\left[-\frac{\max_{1\le j\le n} Y_{j+k} - \lambda^k Y_j}{\sqrt{2\log n}} \ge \varepsilon - 1 - \frac{3}{2}\lambda^k\right] \\
=\ &\mathbb{P}\left[\frac{\max_{1\le j\le n} Y_j}{\sqrt{2\log n}} \ge \frac{3}{2}\right] + \Phi\left(\frac{1-\varepsilon+\frac{3}{2}\lambda^k}{\sqrt{1-\lambda^{2k}}}\sqrt{2\log n}\right)^n
\end{aligned}$$

where we also exploit the symmetry of the joint distribution, and the independence of the $\mathcal{N}(0,1)$ random variables

$$\frac{Y_{j+k} - \lambda^k Y_j}{\sqrt{1-\lambda^{2k}}}.$$

To secure the result, we note that the argument of Lemma 8.2.3 immediately yields

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\frac{\max_{1\le j\le n} Y_j}{\sqrt{2\log n}} \ge \frac{3}{2}\right] < \infty. \tag{8.25}$$

Furthermore, by (8.24), Mill's estimate and some analysis, we have

$$\sum_{n=1}^{\infty} \Phi\left(\frac{1-\varepsilon+\frac{3}{2}\lambda^k}{\sqrt{1-\lambda^{2k}}}\sqrt{2\log n}\right)^n < \infty. \tag{8.26}$$

Using the above bound together with (8.25) and (8.26) gives

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\frac{\max_{1\le j\le n} Y_{j+k}}{\sqrt{2\log n}} \le 1-\varepsilon\right] < \infty.$$

The stationarity of the sequence now proves the assertion. ⋄

8.4 Maxima of the Integro-differential Equation

We first need to establish the exponential bound on the correlation. To do this we must first show that $\tilde{v}(\cdot)$ is bounded away from zero.

Lemma 8.4.1 *There exists $T_1, L_1 > 0$ such that for all $t > s > T_1$*

$$|Corr(X_s, X_t)| \leq L_1 e^{-\kappa(t-s)}. \tag{8.27}$$

Proof: We first prove the following: there exists $T_1 > 0$ such that for all $t > T_1$

$$\tilde{v}(t) > \frac{1}{4}\frac{\underline{\sigma}^2}{3C}, \tag{8.28}$$

where C is defined in Lemma 7.5.2, and $\underline{\sigma}$ is given by (3.8). Define $f_n(t_0) = \tilde{v}(t_0 + nT)$ and $f(t_0) = v(t_0)$. We will show there exists $L \in (0, \infty)$ such that for all $t_0, t_1 \in [0, T]$ and all $n \in \mathbb{N}$

$$|f_n(t_0) - f_n(t_1)| \leq L|t_0 - t_1|. \tag{8.29}$$

First of all, we have

$$|f_n(t_0) - f_n(t_1)| \leq \left| \int_{t_0+nT}^{t_1+nT} \sigma(s)^2 g(s, t_1 + nT)^2 \, ds \right|$$
$$+ \left| \int_0^{t_0+nT} \sigma(s)^2 \left(g(s, t_1 + nT)^2 - g(s, t_0 + nT)^2 \right) \, ds \right|.$$

By using the exponential boundedness in the resolvent (Proposition 7.2.4) and the bound on the t-derivative of g (Lemma 6.3.1) one obtains

$$|g(s, t_1 + nT)^2 - g(s, t_0 + nT)^2| \leq K^2 e^{-\kappa(t_1+nT-s)} e^{-\kappa(t_0+nT-s)} (2M_1 + M_2)|t_1 - t_0|.$$

Since $|g|$ is uniformly bounded by 1 (Lemma 6.3.1), we may choose

$$L = M_3^2 \left(1 + \frac{(2M_1 + M_2)K^2}{\kappa} e^{\kappa T} \right)$$

in (8.29). Letting $\mathcal{G} = \{f_n : n \in \mathbb{N}\}$ we see that $\mathcal{G}$ is equicontinuous on $[0, T]$. Moreover, since f_n converges pointwise to f on $[0, T]$, it follows that f_n converges uniformly to f on $[0, T]$ (see e.g. Rudin Q.18 p.156 [65]) i.e.,

$$\lim_{n\to\infty} \sup_{t_0 \in [0,T]} |\tilde{v}(t_0 + nT) - v(t_0)| = 0,$$

so therefore by the T-periodicity of $v(\cdot)$ we have

$$\lim_{t\to\infty} \tilde{v}(t) - v(t) = 0.$$

Thus

$$\liminf_{t\to\infty} \tilde{v}(t) \geq \liminf_{t\to\infty} \tilde{v}(t) - v(t) + \liminf_{t\to\infty} v(t) \geq \frac{1}{2}\frac{\underline{\sigma}^2}{3C},$$

proving (8.28). Using the exponential bound on the resolvent (Proposition 7.2.4), Proposition 7.2.2 and (8.28) leads to choosing

$$L_1 = \frac{K^2 M_3^2}{\frac{1}{2}\frac{\underline{\sigma}^2}{3C}\kappa}$$

proving the assertion. ◇

The proof of this lemma, along with Proposition 8.3.1, provides the material to prove the principal result.

Theorem 8.4.1 *If $(X_t)_{t\geq 0}$ is the solution of (8.1), and the usual conditions apply, then*

$$\liminf_{t\to\infty} \frac{\max_{0\leq s\leq t} X_s}{\sqrt{2\log t}} \geq \sqrt{\max_{0\leq s\leq T} v(s)} \quad a.s., \tag{8.30}$$

while for any $h > 0$, we have

$$\limsup_{t\to\infty} \frac{\max_{0\leq i\leq \lfloor \frac{t}{h} \rfloor} X_{ih}}{\sqrt{2\log t}} \leq \sqrt{2 \max_{0\leq s\leq T} v(s)} \quad a.s. \tag{8.31}$$

We defer the proof of Theorem 8.4.1 for a moment and note the following refinement.

Theorem 8.4.2 *Let $(X_t)_{t\geq 0}$ be the solution of (8.1), and assume the usual conditions. Furthermore, suppose that there exists $T_1 > 0$ such that for all $t, s > T_1$, we have*

$$Cov(X_s, X_t) \geq 0.$$

Then for every $h > 0$

$$\lim_{t\to\infty} \frac{\max_{0\leq i\leq \lfloor \frac{t}{h} \rfloor} X_{ih}}{\sqrt{2\log t}} = \sqrt{\max_{0\leq s\leq T} v(s)} \quad a.s.. \tag{8.32}$$

Proof: Mimic the proof of Theorem 8.4.1 below, using (8.21) in Proposition 8.3.1 instead of Lemma 8.3.3. ◇

Remark 8.4.1 *If $\lambda(\cdot) < 0$ then $Cov(X_s, X_t) \geq 0$ for all $t, s \geq 0$.*

Proof: Define $g_s(t) := g(s,t)$, so

$$D_-|g_s(t)| \leq (\lambda(t) - \beta(t))|g_s(t)| - \lambda(t) \int_s^t w(u,t)|g_s(u)|\, du.$$

If we let $G_s(t) = |g_s(t)|$ then $G_s(s) = 1 = g_s(s)$. Therefore, by the comparison theorem, we have $G_s(t) \leq g_s(t)$ for all $t \geq s$. But this is nothing more than $|g(s,t)| \leq g(s,t)$ for all $t \geq s$. Thus $g(s,t) \geq 0$ and the covariance is non-negative. $\diamond$

Remark 8.4.2

From Theorem 8.4.2, it would obviously be possible to prove (8.32) for exponentially bounded correlation, under the following improvement in Proposition 8.3.1.

Conjecture 8.4.1 *Suppose that $Y_1, \ldots, Y_n$ are jointly normal and standardised and satisfy*

$$Cov(Y_i, Y_j)| \leq \lambda^{|i-j|}$$

for some $\lambda \in (0,1)$. Then

$$\limsup_{n\to\infty} \frac{\max_{1\leq j\leq n} Y_j}{\sqrt{2\log n}} \leq 1 \quad a.s..$$

We now return to the

Proof of Theorem 8.4.1: Let $\alpha \in (0,1)$. Define $N = 1$ if $L_1 \leq 1$, and $N = \lceil \frac{\log L_1 - \log \alpha}{\kappa T} \rceil$ if $L_1 > 1$. Let $n_1 = \lceil \frac{T_1}{T} \rceil$. Further define

$$Y_n^j = \frac{X_{t_0+(nN+j)T+n_1T} - e(t_0 + (nN+j)T + n_1T)}{\sqrt{\bar{v}(t_0 + (nN+j)T + n_1T)}} \qquad j = 0\ldots, N-1,$$

so $Y_n^j \sim \mathcal{N}(0,1)$ and moreover we have without loss of generality for $n > m$

$$|Cov(Y_n^j, Y_m^j)| \leq L_1 e^{-\kappa(n-m)NT} \leq \alpha^{n-m}$$

on using Lemma 8.4.1 and the definition on N. Thus by Proposition 8.3.1, for each $j = 0, \ldots, N-1$

$$\liminf_{n\to\infty} \frac{\max_{0\leq r\leq n-1} \frac{X_{t_0+(rN+j)T+n_1T} - e(t_0+(rN+j)T+n_1T)}{\sqrt{\bar{v}(t_0+(rN+j)T+n_1T)}}}{\sqrt{2\log n}} \geq 1 \quad \text{a.s.},$$

so

$$\liminf_{n\to\infty} \frac{\max_{0\le r\le n-1} \frac{X_{t_0+rT+n_1T}-e(t_0+rT+n_1T)}{\sqrt{\tilde{v}(t_0+rT+n_1T)}}}{\sqrt{2\log n}} \ge 1 \quad \text{a.s..} \tag{8.33}$$

Now choose $t^* \in [0,T]$ so that $v(t^*) = \max_{0\le s\le T} v(s)$, so using (8.33) we have

$$\liminf_{n\to\infty} \frac{\max_{0\le r\le n-1} X_{t^*+rT+n_1T}}{\sqrt{2\log n}} \ge \sqrt{\max_{0\le s\le T} v(s)} \quad \text{a.s..} \tag{8.34}$$

Therefore

$$\begin{aligned}
\liminf_{t\to\infty} \frac{\max_{0\le s\le t} X_s}{\sqrt{2\log t}} &= \liminf_{n\to\infty} \frac{\max_{0\le s\le (n+n_1)T} X_s}{\sqrt{2\log(n+n_1)T}} \\
&= \liminf_{n\to\infty} \frac{\max_{n_1T\le s\le (n+n_1)T} X_s}{\sqrt{2\log n}} \\
&= \liminf_{n\to\infty} \sup_{t_0\in[0,T]} \frac{\max_{0\le r\le n-1} X_{t_0+rT+n_1T}}{\sqrt{2\log n}} \\
&\ge \liminf_{n\to\infty} \frac{\max_{0\le r\le n-1} X_{t^*+rT+n_1T}}{\sqrt{2\log n}} \\
&\ge \sqrt{\max_{0\le s\le T} v(s)} \quad \text{a.s.,}
\end{aligned}$$

proving the liminf bound in (8.30). To tackle (8.31) we adopt a different strategy. Define $M = \lceil \frac{-\log\alpha}{\kappa h} \rceil$ if $L_1 \le 1$: otherwise let $M = \lceil \frac{\log L_1 - \log\alpha}{\kappa h} \rceil$. Further, let $n_2 = \lceil \frac{T_1}{h} \rceil$. For $j = 0, \ldots, M-1$ let

$$Y_n^j = \frac{X_{n_2h+(nM+j)h} - e(n_2h+(nM+j)h)}{\sqrt{\tilde{v}(n_2h+(nM+j)h)}}.$$

Then $Y_n^j \sim \mathcal{N}(0,1)$ and w.l.o.g. for $n > m$ we have

$$|Cov(Y_n^j, Y_m^j)| \le L_1 e^{-\kappa(n-m)Mh} \le \alpha^{n-m},$$

so Proposition 8.3.1 gives us for $j = 0, \ldots, M-1$

$$\limsup_{n\to\infty} \frac{\max_{0\le r\le n} Y_n^j}{\sqrt{2\log n}} \le \sqrt{2} \quad \text{a.s.,}$$

so as in the liminf proof above we readily prove

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2\log n}} \max_{n_2\le i\le n+n_2-1} \frac{X_{ih} - e(ih)}{\tilde{v}(ih)} \le \sqrt{2} \quad \text{a.s.} \tag{8.35}$$

To conclude the argument for the limsup we have

$$
\begin{aligned}
\limsup_{t\to\infty} \frac{\max_{0\le i\le \lfloor \frac{t}{h} \rfloor} X_{ih}}{\sqrt{2\log t}} &= \limsup_{n\to\infty} \frac{\max_{0\le i\le (n+n_2)} X_{ih}}{\sqrt{2\log (n+n_2)h}} \\
&= \limsup_{n\to\infty} \frac{\max_{n_2\le i\le (n+n_2-1)} X_{ih}}{\sqrt{2\log n}} \\
&\le \limsup_{n\to\infty} \frac{\max_{n_2\le i\le (n+n_2-1)} \frac{\sqrt{v(t^*)}}{\sqrt{v(ih)}} X_{ih}}{\sqrt{2\log n}} \\
&= \sqrt{v(t^*)} \cdot \limsup_{n\to\infty} \frac{1}{\sqrt{2\log n}} \max_{n_2\le i\le n+n_2-1} \frac{X_{ih}}{\sqrt{\tilde{v}(ih)}} \\
&\le \sqrt{v(t^*)} \limsup_{n\to\infty} \frac{1}{\sqrt{2\log n}} \max_{n_2\le i\le n+n_2-1} \frac{X_{ih} - e(ih)}{\sqrt{\tilde{v}(ih)}} \\
&\quad + \sqrt{v(t^*)} \limsup_{n\to\infty} \max_{n_2\le i\le n+n_2-1} \frac{e(ih)}{\sqrt{\tilde{v}(ih)}} \\
&\le \sqrt{2}\sqrt{\max_{0\le s\le T} v(s)}
\end{aligned}
$$

where we use (8.35) at the last step. ⋄

Remark 8.4.3

It would appear that the model for prices which we have considered might be more suitable for modelling processes which do not grow secularly. For example, suppose that prices evolve according to the standard Black-Scholes equation

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t,$$

and let $X_t = \log S_t$. The asymptotic deviations from

$$X_t - (\mu - \frac{1}{2}\sigma^2)t$$

are then of order $\sigma\sqrt{2t\log\log t}$.

If on the other hand, X_t is the log–price satisfying the stochastic integro-differential equation, and $k(t) = \eta t + \nu$ for some $\eta > 0$, it should be possible to prove there exists $K_1 > 0$ such that $|e(t) - \eta t| \le K_1$ for all $t \ge 0$. However, from the last theorem, we see that the asymptotic maximal deviations from

$$X_t - \eta t$$

are of order $\sqrt{2\log t}$ times some bounded function. Hence we see that the large deviations are significantly smaller than in the Black-Scholes case. This indicates that we should consider this as a more suitable model for exchange rates, for example, rather than for stocks.

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