Stability of naked singularities in self-similar collapse

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B Sc

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy, is entirely my own work and has not been taken from the work of others, save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed: Thomas Kerter.

I.D. number: 95347232

Date: 20/1/06

This thesis is dedicated to my parents,
Dermot and Denise

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Contents

Abstract

Certain classes of solutions to Einstein's field equations admit singularities from which light can escape, known as 'naked' singularities Such solutions contradict the Cosmic Censorship hypothesis, however they tend to occur in spacetimes with a high degree of symmetry It is thought that naked singularities are artifacts of these symmetries, and would not survive when the symmetry is broken

In particular, a rich source of naked singularities is the class of self-similar spherically symmetric spacetimes It is the purpose of this thesis to test the stability of these solutions and examine if the naked singularity persists

We first consider the propagation of a scalar field on these background spacetimes and then study gauge-invariant perturbations of the metric and matter tensors We exploit the spherical symmetry of the background to decompose the angular part of the perturbation in terms of spherical harmonics Then we perform a Mellin transform of the field to reduce the problem to a set of coupled ordinary differential equations, and seek solutions for the individual modes The asymptotic behaviour of these modes near singular points of the ODE's is used to calculate a set of gauge invariant scalars, and we examine the finiteness of these scalars on the naked singularity's horizons

The background spacetimes we examine are the self-similar null dust (Vaidya) solution, the self-similar timelike dust (Lemaître-Tolman-Bondi) solution, and finally a general self-similar spacetime whose matter content is unspecified save for satisfying the dominant energy condition

In each case examined we find the Cauchy horizon, signalling the presence of a naked singularity, is stable to linear order, a surprising result that suggests naked singularities may arise in physical models of gravitational collapse The second future similarity horizon which follows the Cauchy horizon is unstable, which suggests that the naked singularity is only visible for a finite time

List of Figures

Chapter 1

Introduction

While General Relativity (GR) is today considered the most accurate description of space and time in the large, for the first half of its life an important aspect of GR was brushed under the carpet singularities A singularity is a point in spacetime* at which the gravitational field diverges As GR is a mathematical theory which cannot handle infinities, we say that at a singularity the laws of physics break down However a number of important early solutions to the field equations, such as the Schwarzschild solution, contained singularities, and thus the theory seemed to be predicting its own demise

These singular solutions were not considered physically significant as they occurred in idealized models with a high degree of symmetry, such symmetry was not thought to exist in nature and consequently singularities were not thought likely to exist in nature either However, the singularity theorems of Hawking and Penrose (see Hawking and Ellis [28]) changed this attitude They were able to show that, under some reasonable criteria which we will not go into in great detail, certain classes of collapse would *inevitably* result in a singularity, without making any assumptions about spacetime symmetries Consequently there was a surge of interest in singularities and in particular black holes A black hole is the region which typically surrounds a singularity and is the interior to an event horizon (that surface marking the limit from which light can escape to infinity from the neighbourhood of the singularity), this has the effect of cutting off the singularity from the external universe

Unfortunately the singularity theorems said very little about the nature of the singularity, they merely conclude it must exist In particular, there are some spacetime models which collapse to a singularity that is *not* covered by an event horizon a naked

*This section is meant as an overview or general discussion, we leave to the next section a definition of spacetime, geodesics and other important concepts in GR singularity These solutions represent a point at which all physical laws must break down, and moreover they have the potential to influence the external universe, threatening predictability in physical laws everywhere

The existence of naked singularities is obviously very undesirable, so much so that Penrose hypothesised that in nature 'there exists a "cosmic censor" who forbids the appearance of naked singularities, clothing each in an absolute event horizon' [43] As this hypothesis is the crucial open question which motivates this thesis, we will give it a precise definition (for a detailed discussion see Wald [48])

We will define a manifold to be a space which is locally similar to Euclidean space in that it can be covered by coordinate patches Timelike curves describe the paths of observers who move with a velocity less than that of light, and null curves describe the paths of light rays Therefore the timelike curves through any point must lie between the null curves through that point to the future or past, regions called the future and past null cones respectively

Consider the manifold $\mathcal{M}$ containing the closed achronal set $\mathcal{S}$ (that is, no two points in $\mathcal{S}$ are connected by a timelike curve) We define the future (respectively past) domain of dependence of $\mathcal{S}$, denoted $D^+(\mathcal{S})$ (respectively $D^-(\mathcal{S})$), to be all points $p \in \mathcal{M}$ such that every past (respectively future) causal (i e timelike or null) curve through p intersects $\mathcal{S}$ See Figure 1 1 In this picture, the paths of light rays follow lines of $\pm 45^\circ$ We see that for every point in $D^+(\mathcal{S})$, for example, the past null cone of that point must fully intersect $\mathcal{S}$

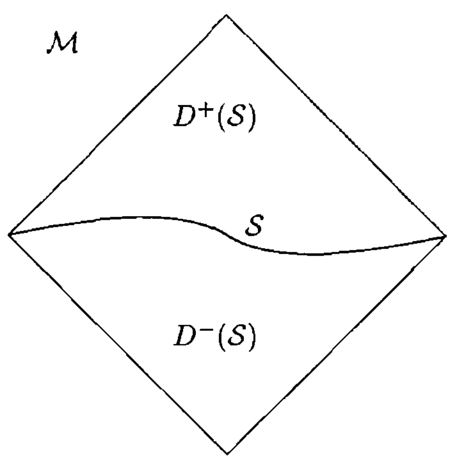


Figure 1 1 The domain of dependence for a closed achronal set $\mathcal{S}$

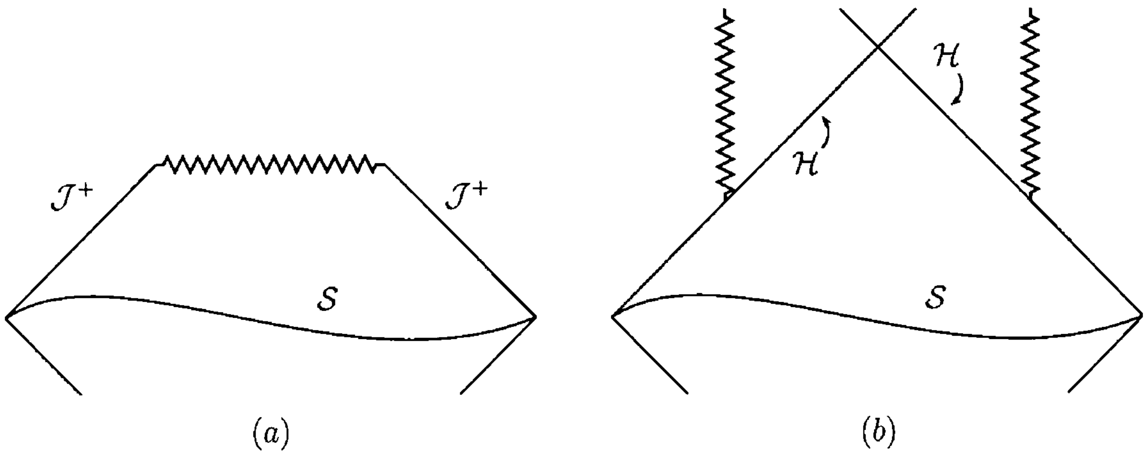


Figure 1 2 Portions of the conformal diagrams for (a) Kruskal-Szekeres and (b) Reissner-Nordstrom spacetimes $\mathcal{S}$ is a closed achronal set The Cauchy horizon, $\mathcal{H}$, and future null infinity, $\mathcal{J}^+$, are also shown

The union of $D^+(\mathcal{S})$ and $D^-(\mathcal{S})$ is called the domain of dependence, $D(\mathcal{S})$ If $D(\mathcal{S}) = \mathcal{M}$, then $\mathcal{S}$ is called a Cauchy surface A spacetime that contains a Cauchy surface is called globally hyperbolic We may formulate Penrose's hypothesis thus

Cosmic Censorship Hypothesis (CCH) (1) All physically reasonable spacetimes are globally hyperbolic

The motivation behind this formulation is that $\mathcal{M}$ may be taken to be spacetime, and $\mathcal{S}$ may typically be taken to be a hypersurface t = constant Due to the nature of the field equations, if we specify Cauchy data on $\mathcal{S}$, the initial data slice, then we will know the solutions to the field equations everywhere in $D(\mathcal{S})$, hence the term Cauchy surface If $D(\mathcal{S}) = \mathcal{M}$, then the solution to Einstein's field equations are known throughout spacetime

On the other hand, if there were a region of spacetime outside of the domain of dependence of $\mathcal{S}$, we would not be able to predict the solutions to the field equations there, based on data specified on $\mathcal{S}$ This breakdown in predictability is exactly what the CCH attempts to rule out The boundary between $D(\mathcal{S})$ and the region beyond is called the Cauchy horizon, $\mathcal{H}$, and it is the presence of a Cauchy horizon which signals a departure from global hyperbolicity Thus we may rephrase the CCH as

Cosmic Censorship Hypothesis (2) No physically reasonable spacetimes contain Cauchy horizons

To be precise, this is the strong formulation of the CCH We should note that there is also a weak version, and we can loosely differentiate between the two thus the strong version of Cosmic Censorship says that no observer may see a singularity, whereas the weak version says that no observer at infinity may see the singularity All the naked singularities in this thesis are globally naked, and thus we will use the strong version as formulated above

In Figure 1 2 (a) we give the conformal diagram for Kruskal-Szekeres spacetime, which is globally hyperbolic and thus satisfies the CCH However, we see from Figure 1 2 (b), the conformal diagram for the Reissner-Nordstrom solution, that this spacetime is *not* globally hyperbolic We see that for the surface $\mathcal{S}$ there is a region of $\mathcal{M}$ outside of $D(\mathcal{S})$, and the corresponding Cauchy horizon is marked with $\mathcal{H}$ The past null cone of an observer in this region would contain the singularity, that is, the singularity is visible to an observer travelling through this region, the singularity is naked Does this imply a failure of the CCH?

The answer is no, for the following reason Chandrasekhar and Hartle [7] have shown that linear perturbations in the metric tensor within $D(\mathcal{S})$ grow without bound as one approaches the Cauchy horizon, in fact the Cauchy horizon itself becomes singular The surface $\mathcal{H}$ in Reissner-Nordstrom spacetime is therefore unstable, to be more precise, $\mathcal{H}$ has a blue-sheet instability, as the wavecrests of light impinging on $\mathcal{H}$ pile up on top of one-another and are infinitely blue-shifted

This is interpreted to mean that an observer attempting to cross the Cauchy horizon will view the entire history of the universe at a glance, a sight of infinite energy which would destroy him or her This prevents an observer from crossing $\mathcal{H}$ and looking on the singularity, and thus the Reissner-Nordstrom solution does not violate the CCH

If the Reissner-Nordstrom solution contains Cauchy horizons, but does not violate the CCH, does this mean we must rephrase the Cosmic Censorship hypothesis? Again no, and the reason is in our understanding of the words "physically reasonable" A physically reasonable spacetime must satisfy a number of criteria, foremost among them being (1) the dominant energy condition and (2) genericity The dominant energy condition places restraints on the stress-energy-momentum tensor and will be discussed in detail in §2 2 The genericity condition is hard to define precisely, however for the purpose of this thesis we will take it to mean that spacetime is not unrealistically symmetric, or that spacetimes which evolve to contain naked singularities must not depend on fine-tuning of initial data The Reissner-Nordstrom spacetime is spherically symmetric and thus we must perturb it away from spherical symmetry to satisfy the "physically reasonable" condition

The perturbed spacetime then does not contain a naked singularity, and the CCH is corroborated

There are other possible counter-examples to the CCH, for example the Kerr spacetime [48], certain classes of self-similar perfect fluid [42] and dust [30] solutions, and the self-similar scalar field [8], all of which have a high degree of symmetry Further, there are Cauchy horizons in spacetimes containing colliding plane waves [16], and there may be naked singularities in spacetimes featuring critical collapse [17], which depend on a fine tuning of initial data However, in order to be a strong counter-example to the CCH a naked singularity must be stable As mentioned above, the way to test for stability is to perturb the background spacetime and examine whether the naked singularity persists or not, it does not for Reissner-Nordstrom, will it persist for these other spacetimes?

It is the purpose of this thesis to perform a stability analysis of a class of these possible counter examples to the CCH As self-similar spherically symmetric (4-S) spacetimes are a rich source of possible counter examples to the CCH, we will devote our attention to these spacetimes In particular, and in increasing order of generality, we will consider the self-similar Vaidya (null dust) spacetime, the self-similar Lemaître-Tolman-Bondi (timelike dust) spacetime, and finally we consider a generic 4-S spacetime, that is without specifying the matter content (save for satisfying the dominant energy condition)

The layout of the thesis will be as follows In the remainder of this chapter, we provide a brief overview of some of the important concepts and definitions of General Relativity, and introduce any relevant equations which will be used elsewhere in the thesis We conclude the chapter with some important mathematical definitions and theorems relating to the solutions of singular ordinary differential equations which will be used throughout the thesis

In Chapter 2 we will describe precisely what self-similar and spherically symmetric mean, and what is required for the dominant energy condition to be satisfied We go on to derive the conditions under which a 4-S spacetime collapses to a naked singularity, and then derive the two special solutions we will consider, the self-similar Vaidya and LTB spacetimes

As a starting point in our stability analysis, in Chapter 3 we will consider the propagation of a minimally coupled massless scalar field to a generic 4-S spacetime which admits naked singularities This scalar field is in essence a toy model to familiarise ourselves with the procedures of a perturbation analysis The scalar field is described by a wave equation, and we expect the perturbation to solve wave-like equations Also, the scalar field analysis displays the main features of a perturbation analysis

- decomposition in terms of spherical harmonics,
- using the Mellin transform to reduce the evolution equations to ordinary differential equations for each mode of the field,
- finding the asymptotic behaviour of the solutions for these modes near certain singular points,
- using these solutions to construct certain invariant scalars,
- initial regularity conditions on the axis and past null cone of the origin,
- finally allowing a specified class of solutions to impinge on the Cauchy horizon and possibly beyond

The main results of this analysis have appeared in [40]

The scalar field of Chapter 3 is a toy model In Chapter 4 we describe a more sophisticated approach gauge invariant linear perturbations of the metric and matter tensors, which corresponds to the true perturbation of the spacetime We will motivate the decomposition using spherical harmonics by giving first a description of scalar multipole decompositions from the Newtonian viewpoint Then we describe in detail the perturbation formalism of Gerlach and Sengupta [13, 14], paying particular attention to the issue of gauge invariance

In Chapter 5 we begin the analysis of perturbations of spacetimes We consider Minkowski spacetime, and examine all modes of both even and odd parity perturbations of flat, vacuum spacetime This is necessary as we must be sure no singularities develop in the absence of matter, and for other reasons which we will discuss at the appropriate time

Chapter 6 sees the perturbations of our first non-empty spacetime the self-similar Vaidya solution We choose this background as a starting point since the metric and matter tensors can be written down in a very simple manner, and the condition for the background to admit a naked singularity is especially simple As this is the first analysis of a non-flat spacetime we will go into some detail in this chapter, and examine all perturbation modes of both even and odd parity The main results of this analysis, and that of the preceding chapter, have appeared in [41]

In Chapter 7 we turn our attention to a more realistic model of collapse than the Vaidya solution, the self-similar timelike dust or Lemaître-Tolman-Bondi (LTB) spacetime This solution represents the collapse of a perfect fluid with vanishing pressure, and is used extensively in models of stellar collapse

Finally, in Chapter 8 we consider perturbations of generic 4-S spacetimes which admit naked singularities Unfortunately, in our quest for genericity we find a limitation as we are not specifying the background matter tensor we cannot say anything useful about the perturbed matter tensor Thus to make headway we must consider a vanishing matter perturbation As we will describe, this is not a problem for odd parity perturbations, however there are only trivial solutions in the even parity sector

In Chapter 9 we present our conclusions, and some suggestions for further work

The perturbation formalism of Gerlach and Sengupta which we describe in Chapter 4 is very robust in that it can be applied to any spherically symmetric background Moreover, the formalism has been tailored for the longitudinal gauge, which we will discuss in detail later, which simplifies the matter perturbation terms Thus this formalism has been used by a number of authors in order to describe perturbations of spherically symmetric spacetimes, among them perturbations of critical behaviour in the massless scalar field by Frolov [11, 12] and Gundlach and Martin-Garcia [19], perturbations of timelike dust solutions by Harada et al [22, 23], and perturbations of perfect fluids by Gundlach and Martin-Garcia [20, 18] These analyses (with the exception of Frolov's) primarily rely on numerical evolution of the perturbation equations, there is a gap in the literature with regards to analytic or asymptotic solutions to perturbations of these spacetimes, which this thesis attempts to fill

In broader terms, perturbations of the metric tensor can be thought of as modelling gravitational waves, an important topic in the current scientific community This formalism has been used for exactly that purpose by numerous authors such as Harada et al [22, 23, 24], Sarbach and Tiglio [45], and similar analyses by Nagar and Rezzolla [35]

The central aim of this thesis is to use this formalism to describe perturbations of self-similar collapse There are a number of reasons we are interested in self-similar spacetimes they are a rich source of naked singularities, the ability to define a similarity coordinate can reduce the complexity of the field equations considerably, and finally there are indications that self-similar solutions can act as attractors in the collapse of non-self-similar solutions, for example Harada [21, 26] In fact, Carr and Coley [4] have gone as far as to propose a 'similarity hypothesis', with which they claim that solutions in general relativity will naturally evolve to a self-similar form The stability of the Cauchy horizon in self-similar collapse has been tested at the eikonal level by Waugh and Lake [50], and at the semi-classical level by Harada and Miyamoto [27]

Chapters 1, 2 and 4 are intended to set up the problem for analysis, and thus are not (entirely) the original work of the author Chapter 5 applies the perturbation formalism to Minkowski spacetime, and thus some of the results in this chapter would be well known Chapters 3, 6, 7 and 8 are entirely the work of the author, and represent the first analytic (as in non-numerical) examination of the perturbations of self-similar spacetimes to test the stability of the Cauchy horizon

The main result of this thesis is that in each case we examine, the Cauchy horizon formed in self-similar collapse is stable due to linear perturbations

Throughout this thesis we set $G = c = 1$ unless otherwise stated, and follow the conventions of Wald [48] In Chapters 1, 2 and 3 we use lowercase Latin indices to denote coordinates on the 4-dimensional spacetime manifold, and from Chapter 4 on we use the convention set out in that chapter

1.1 A primer on differential geometry and General Relativity

In this section we give a brief overview of General Relativity and the relevant equations, geometrical objects etc for this thesis

Einstein's theory of General Relativity extends his theory of Special Relativity (SR) to include gravity In SR, the fundamental invariant which all inertial observers (coordinate systems) agree on is no longer the distance between two points in space, but instead the interval between two events An event is a point in space at a moment in time, which leads to considering space and time joined in a four dimensional continuum called spacetime An event in spacetime therefore has coordinates (t, x, y, z), for example, where t is standard time and x, y, z are rectangular coordinates The interval between two nearby events, ds, is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{1 1 1}$$

where dt is the difference in the time coordinates and so on Note the negative sign before dt^2, this means the matrix whose diagonal contains the coefficients of the line element will have trace (signature) $+2$, and one negative eigenvalue plus three positive eigenvalues, by our convention A (four dimensional) spacetime with this property is called Lorentzian

Consider a vector field $\vec{v}$ (i e through each point there is a unique vector) in spacetime We can interpret a vector field as describing a directional derivative of a function, which we write as

$$\vec{v}f = v^a \frac{\partial f}{\partial x^a} = (v^a \partial_a) f,$$

where f is some function and we use the shorthand $\partial_a = \partial/\partial x^a$ Here x^a represents a set of coordinates, and a runs from 0 to 3, for example $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ Throughout this thesis summation over repeated indices is implied

Thus $\vec{v} = v^a \partial_a$, in other words ∂_a forms a basis for all vectors $\vec{v}$, and v^a are the components of the vector $\vec{v}$ in this basis More precisely, v^a is a tangent vector which is defined on the tangent space spanned by the coordinate basis ∂_a

We define an additional vector space, the dual (or cotangent) vector space, whose basis dx^a is the dual to the coordinate basis such that

$$dx^a(\partial_b) = \delta^a_b,$$

where $\delta^a_b = 1$ if $a = b$ and 0 otherwise is known as the Kronecker delta Dual vectors are those defined on the dual vector space as a linear combination of the dual basis, i e $v_a dx^a$ A tensor of type (or rank) (k, l) is an object which takes k dual vectors and l ordinary vectors and returns a function For example, the stress tensor is of type $(0, 2)$, if it takes as its input two vectors pointing in the x-direction, it returns a function the stress/pressure in the x-direction We will use the 'abstract index notation', and denote a tensor of type (k, l) as

$$T^{a_1 a_2 \cdots a_k}{}_{b_1 b_2 \cdots b_l}$$

Furthermore, a tensor of type (k, l) must satisfy the following transformation rule in changing from coordinates x^a to x'^a

$$T^{a'_1 a'_2 \cdots a'_k}{}_{b'_1 b'_2 \cdots b'_l} = \frac{\partial x'^{a_1}}{\partial x^{a_1}} \cdots \frac{\partial x'^{a_k}}{\partial x^{a_k}} \frac{\partial x^{b_1}}{\partial x'^{b_1}} \cdots \frac{\partial x^{b_l}}{\partial x'^{b_l}} T^{a_1 a_2 \cdots a_k}{}_{b_1 b_2 \cdots b_l} \qquad (1\ 1\ 2)$$

Tensors of rank $(0, 1)$ are ordinary vectors, called contravariant, and tensors of rank $(1, 0)$ are dual vectors, called covariant

We introduce the metric as the generalised inner product used to find the lengths of vectors Using the index notation, this is

$$|\vec{v}|^2 = \vec{v} \cdot \vec{v} = g_{ab} v^a v^b$$

More precisely, the metric is a symmetric tensor of type $(0,2)$ The entries in g_{ab} are found from the line element by

$$ds^2 = g_{ab}dx^a dx^b$$

We formally define spacetime as a Riemannian manifold $\mathcal{M}$ (a space which is made up of pieces of Euclidean space) endowed with a metric g_{ab} of Lorentzian signature, and is denoted $(\mathcal{M}, g_{ab})$

Due to the Lorentzian nature of spacetime, there are three possibilities for the lengths of vectors positive, zero and negative We call these vectors spacelike, null and timelike respectively Timelike vectors are tangent to the path of inertial (freely-falling) observers whose velocity is less than that of light, null vectors are tangent to the paths of light rays, and spacelike vectors everything else If the inner product of two vectors is zero they are orthogonal, and in this sense null vectors are self-orthogonal If the inner product of two null vectors is < 0, then one is ingoing (it approaches the axis as we move forwards in time), the other outgoing, and both point into the future or past, if their inner product is > 0 then one points into the future and the other into the past

GR builds on this to include gravity in the following way in essence, matter curves spacetime and it is this curvature which we experience as a gravitational field Objects travelling through spacetime follow the straightest paths possible, however because spacetime is curved, the paths the objects follow are also curved, this is why the earth's orbit curls around the sun Information about the curvature of spacetime is contained in the metric, in general the components of g_{ab} will be functions of the coordinates The line element given in (1 1 1) now represents a flat spacetime entirely devoid of matter, named after Einstein's mentor Minkowski

A cornerstone of GR is the idea that all inertial observers will discover the same laws of physics Since each observer will have a preferred coordinate system based on their motion, it is important that a law of physics does not depend on a particular choice of coordinates, i e it should be covariant This is encapsulated in the following principle

Principle of Covariance All physical laws should be written covariantly (in tensor form) to ensure equivalence in different coordinate systems

Tensors are naturally covariant, if we maintain the tensor equation $A_{ab} = B_{ab} + C_{ab}$ is true in one coordinate system, it must be true in all coordinate systems In particular, though a tensor may have different components depending on the coordinate system used to write them down, if a tensor vanishes in one coordinate system it does so in all coordinate systems

However, the ordinary derivative, denoted $\partial_a = \partial/\partial x^a$ or simply $_{,a}$, of a tensor is not covariant, it does not transform according to (1 1 2) Instead, we must construct a covariant derivative, denoted ∇_a or $_{|a}$, using a connection There are many connections to choose from but most important is the metric connection, defined as

$$\Gamma^a{}_{bc} = \tfrac{1}{2} g^{ab} \left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc} \right)$$

Using this we define the covariant derivative of a tensor of type (k, l) as

$$\nabla_c T^{a_1 \quad a_k}{}_{b_1 \quad b_l} = \partial_c T^{a_1 \quad a_k}{}_{b_1 \quad b_l} + \Gamma^{a_1}{}_{dc} T^{d \quad a_k}{}_{b_1 \quad b_l} + \qquad - \Gamma^d{}_{b_1 c} T^{a_1 \quad a_k}{}_{d \quad b_l} -$$

The metric connection is such that $\nabla_c g_{ab} \equiv 0$

As mentioned before, the path of a freely-falling object will deviate from a straight line as it moves through non-flat spacetime Instead it follows a geodesic, the path which is locally of shortest length We find the equation for a geodesic in the following way consider a curve parameterized by u, that is its coordinates are $x^a(u)$, then the tangent vector to the curve is given by $X^a = dx^a/du$ A freely-falling object is non-accelerating, that is $A^b = X^a \nabla_a X^b = 0$, where A^a is the acceleration This is equivalent to

$$\frac{d^2 x^a}{du^2} + \Gamma^a{}_{bc} \frac{dx^a}{du} \frac{dx^b}{du} = 0, \quad \text{and} \quad g_{ab} \frac{dx^a}{du} \frac{dx^b}{du} = \begin{cases} 0 & (null) \\ +1 & (spacelike) \\ -1 & (timelike) \end{cases} \tag{1 1 3}$$

When X^a is timelike the parameter u, often denoted τ, is the proper time, that is, the time interval between two events measured by an observer moving along the curve connecting the two events

We see that when the spacetime is flat, and its metric is given by (1 1 1), the connection vanishes and the geodesic equation is linearly solved, recovering the straight line motion of Newtonian mechanics The reason Newtonian mechanics is such a good approximation is seen in the principle of equivalence, which we give in the following mathematical form

> **Principle of Equivalence** At any point in spacetime, the metric connection can be transformed away

In other words, spacetime is *locally* flat, that is, the spacetime metric can always be written in the form (1 1 1) locally A genuine gravitational field is one in which the connection cannot be transformed away everywhere simultaneously, and to test this we construct from the metric connection the Riemann tensor given by

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed}$$

If the Riemann tensor is non-vanishing then the connection cannot be transformed away everywhere (since the Riemann tensor contains derivatives of the connection)

Contracting the Riemann tensor gives the Ricci tensor, $R_{ab} = R^c{}_{acb} = g^{cd}R_{cadb}$, and contracting the Ricci tensor gives the Ricci scalar, $R = R^a{}_a = g^{ab}R_{ab}$ From these we form the Einstein tensor, defined as

$$G_{ab} \equiv R_{ab} - \tfrac{1}{2} g_{ab} R$$

The Einstein tensor, G_{ab}, contains all the information regarding the geometry of spacetime (up to degrees of freedom) On the other hand, the stress-energy-momentum tensor, T_{ab}, contains information about the matter/energy in the spacetime For example, for a timelike observer with tangent vector v^a, the contraction $T_{ab}v^a v^b$ is the local energy density measured by that timelike observer Einstein put these two concepts together, resulting in the famous (covariant) field equations of General Relativity,

$$G_{ab} = 8\pi T_{ab} \tag{1 1 4}$$

Wheeler has succinctly summed up this relationship between matter and geometry thus "matter tells space how to curve, space tells matter how to move" †

But how does the gravitational field makes itself felt at a distance? The earth curves spacetime around it, but how does the moon know? Gravity, like electromagnetism, propagates as a wave Vibrations in the stuff of spacetime itself travel through the cosmos, generated by the motions of matter In fact, one of the most significant efforts of the international scientific community at the moment is the attempt to detect such gravitational waves

We will consider gravitational waves moving through vacuum Thus the matter tensor

†Without meaning to take away from the achievement of Einstein, it is worth noting that he wasn't the first to suggest that space was curved The non-Euclidean geometries of Bolyai and Lobachevsky lead some to suggest that the universe was hyperbolic (negatively curved) or elliptic (positively curved), as opposed to Euclidean (flat) The essence of this idea lives on in the Friedmann-Robertson-Walker cosmological models

vanishes, and the field equations tell you that therefore the Ricci tensor also vanishes So what is left? Consider a field of particles following timelike geodesics with unit tangent vectors $v^a = dx^a/d\tau = x^a$, with $v^a v_a = -1$, separated by displacement vectors δx^a, with $v^a \delta x_a = 0$ As a wave of gravitational radiation passes through, the paths of the particles will deviate according to the equation of geodesic deviation

$$\delta x^a = R^a{}_{bcd} v^b v^c \delta x^d \tag{1 1 5}$$

The Riemann tensor has twenty independent components, split between ten in the Ricci tensor and the remaining ten in the Weyl tensor C_{abcd}, in the following manner

$$R_{abcd} = C_{abcd} + g_{a[c}R_{d]b} + R_{a[c}g_{d]b} - \tfrac{1}{3}Rg_{a[c}\,g_{d]b},$$

where $X_{[ab]} = \frac{1}{2}\left(X_{ab} - X_{ba}\right)$ is the antisymmetric part of a tensor Thus in vacuum we can replace the Riemann tensor in (1 1 5) with the Weyl tensor

We can decompose the Weyl tensor by constructing a null tetrad (i e a group of four null vectors) To do this we take v^a and three spacelike vectors, s^a, t_1^a and t_2^a, and combine them to form four null vectors

$$\begin{aligned} \ell^a &= v^a + s^a, & \qquad w^a &= \tfrac{1}{\sqrt{2}}(t_1^a + i t_2^a), \\ n^a &= v^a - s^a, & \qquad w^{*a} &= \tfrac{1}{\sqrt{2}}(t_1^a - i t_2^a), \end{aligned}$$

since $\ell^a \ell_a = (v^a + s^a)(v_a + s_a) = v^a v_a + 2v^a s_a + s^a s_a = -1 + 0 + 1 = 0$, and so on Here $*$ means complex conjugation All the information in the Weyl tensor is contained in the five Newman-Penrose Weyl scalars, which are found by contractions of the Weyl tensor with elements of the null tetrad,

$$\begin{aligned} \Psi_0 &= C_{abcd} l^a w^b l^c w^d, \\ \Psi_1 &= C_{abcd} l^a w^b l^c n^d, \\ \Psi_2 &= C_{abcd} l^a w^b n^c w^{*d}, \\ \Psi_3 &= C_{abcd} l^a n^b w^{*c} n^d, \\ \Psi_4 &= C_{abcd} n^a w^{*b} n^c w^{*d} \end{aligned}$$

We may give a physical meaning to each of these scalars by considering the Petrov classification of gravitational fields

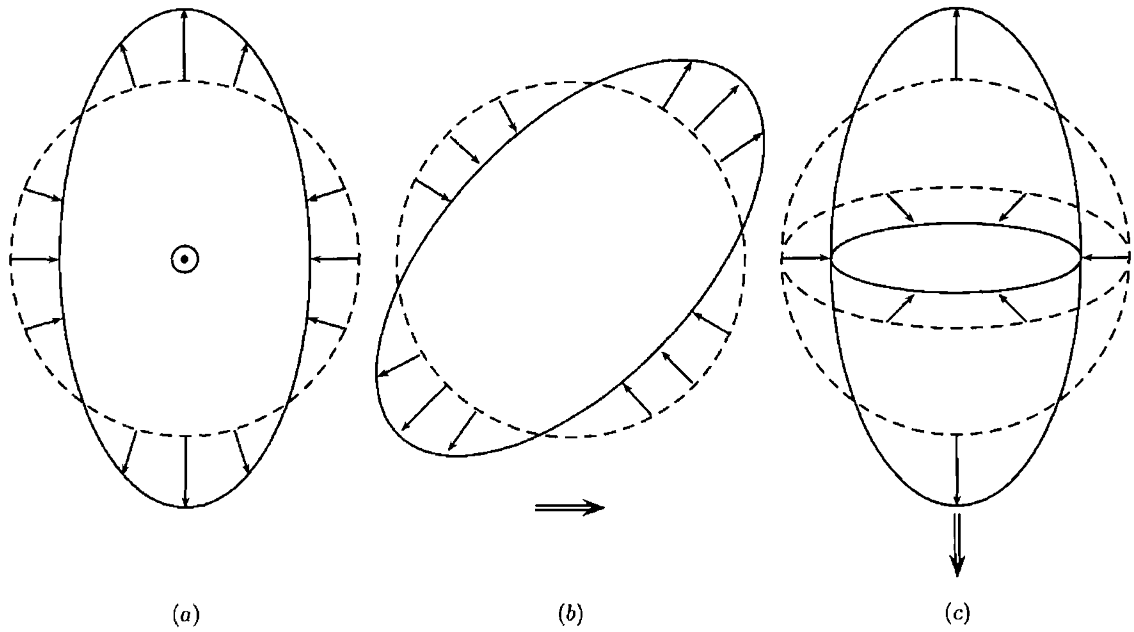


Figure 1 3 The effects of certain types of fields on a cloud of particles (a) A type N field causes movement in a plane perpendicular to the wave direction, (b) a type III field's effects are also planar however they contain the wave direction, and (c) a type D field will distort a sphere into an ellipsoid The symbols $\odot$ (perpendicular to the page), $\Longrightarrow$ and $\Downarrow$ represent the propagation direction of the wave

The Petrov classification groups together spacetimes which share important characteristics in the following way the Weyl tensor can be considered as a 4×4 matrix with 4 eigenvalues The Petrov classification groups spacetimes according to the multiplicity of these eigenvalues Analogously, we could classify spacetimes according to the number and multiplicity of the principal null directions There are five Petrov types (six including conformally flat spacetimes) but only three are of interest with regards to this thesis Petrov type N, type III and type D We note a spacetime may be made up of superpositions of fields of different Petrov types

Szekeres [47] examined the effect of each of these three fields on the cloud of particles mentioned above, see Fig 1 3 For type N fields, Fig 1 3(a), the cloud of particles is distorted only in the plane perpendicular to the direction of the field's propagation Thus type N fields represent pure transverse waves, and moreover it can be shown that for type N fields the Weyl scalars $\Psi_{1,2,3}$ can be set equal to zero Thus we can interpret Ψ_0 and Ψ_4 as the purely transverse wave terms in the ℓ^a and n^a directions respectively

Type III fields again distort a ring of particles in a plane, Fig 1 3(b), only now the plane contains the direction of the field's propagation Thus type III fields represent longitudinal waves, with Ψ_1 and Ψ_3 representing the longitudinal component in the ℓ^a and n^a directions respectively

Finally, for type D fields the effect is no longer planar A sphere of particles is distorted into an ellipsoid whose major axis is in the direction of the field's propagation, Fig 1 3(c) This matches exactly the tidal force experienced by an object falling towards a spherically symmetric source, and in fact the group Petrov type D contains all spherically symmetric spacetimes (and thus all background spacetimes considered in this thesis) In spherically symmetric spacetimes the only non-zero Weyl scalar is Ψ_2, and Szekeres calls this the Coulomb term

1 2 Mathematical preliminaries

1 2 1 The Mellin transform

If we consider the Laplace transform of $f(t)$, a function defined for all $t \geq 0$, given by

$$F(s) = \int_0^\infty e^{-st} f(t) dt,$$

and make a change of dependent variable $t \to \ln z$, the integral becomes

$$\int_{-\infty}^{+\infty} z^{-s} f(z) \frac{dz}{z}$$

This is the Mellin transform, which we formally define as

$$G(s) = \mathbb{M}[g(z)] = \int_0^\infty g(z) z^{-s-1} dz, \tag{1 2 1}$$

with $s \in \mathbb{C}$ (in some literature s is replaced with $-s$) For this transform to exist, there will be a restriction on the allowed values of s, typically to lie in a strip in the complex plane with $\sigma_1 < Re(s) < \sigma_2$ The inverse Mellin transform is given by

$$g(z) = \mathbb{M}^{-1}[G(s)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^s G(s) ds, \tag{1 2 2}$$

where $c \in \mathbb{R}$ is such that $\sigma_1 < c < \sigma_2$ To recover the original function from the Mellin transform, we integrate over the vertical contour in the complex plane of s given by

$Re(s) = c$ We emphasize, as this will be crucial later, that we do not integrate over all values of s in the interval $\sigma_1 < Re(s) < \sigma_2$, only over the vertical contour defined by a specific value of $Re(s)$ in this interval, which we are free to choose We will make use of the following theorem

Theorem 1 2 1 (Mellin Inversion Theorem)
If $g(z)$ is piecewise continuous on the positive real numbers and the integral

$$G(s) = \int_0^\infty z^{-s-1} g(z) dz$$

is absolutely convergent when $\sigma_1 < Re(s) < \sigma_2$, then g is recoverable via the inverse Mellin transform from its Mellin transform G

A series $\sum A_m$ is absolutely convergent if the series $\sum |A_m|$ converges This theorem is particularly useful as it saves us from having to perform the inverse Mellin transform, we merely need to show that $G(s)$ does not diverge in the relevant interval and the theorem guarantees the inverse Mellin transform exists As we will see, the Mellin transform is particularly suited to differential equations arising in self-similar spacetimes, see Chapter 2

The Mellin transform can be extended to functions of many variables, in particular we shall be interested in functions of x (or y), a self-similar coordinate, and r, a coordinate related to the areal radius of a spherically symmetric spacetime Since $0 \leq r < \infty$ covers the entire spacetime, we will take the Mellin transform of such functions, say $g(x, r)$, over r We denote the Mellin transform then as

$$\mathbb{M}[g(x,r)] = G(x,s)$$

The Mellin transform of the x derivatives of $g(x, r)$ can be found by reversing the order of integration and differentiation, as in

$$\mathbb{M}\left[\frac{\partial g(x,r)}{\partial x}\right] = \int_0^\infty \frac{\partial g(x,r)}{\partial x} r^{-s-1} dr = \frac{\partial}{\partial x}\int_0^\infty g(x,r) r^{-s-1} dr = \frac{\partial}{\partial x} G(x,s)$$

The Mellin transform of the r derivatives of $g(x, r)$ can be found using integration by parts by first noting that in the equations governing perturbations of self-similar spacetimes written using coordinates (x, r), the nth r-derivative of any dependent variable $g(x, r)$ is always multiplied by r^n‡ Thus we will be taking the Mellin transform of these

‡This is due to the homothetic Killing vector being $t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}$

combinations, for example

$$\mathbb{M}\left[r\frac{\partial g(x,r)}{\partial r}\right] = \int_0^\infty r^{-s}\frac{\partial g}{\partial r}dr = \left[r^{-s}g(x,r)\right]_0^\infty + s\int_0^\infty g(x,r)r^{-s-1}dr = sG(x,s),$$

provided the boundary term in the square brackets vanishes, which places restrictions on s and g For the second derivative, we find

$$\mathbb{M}\left[r^2\frac{\partial^2 g}{\partial r^2}\right] = s(s-1)G(x,s),$$

and in general we obtain

$$\mathbb{M}\left[r^n\frac{\partial^n g}{\partial r^n}\right] = r^{n-s}\frac{\partial^n r^s}{\partial r^n}G(x,s),$$

which suggests we can quickly perform a Mellin transform of an equation by replacing $g(x,r)$ with $r^sG(x,s)$ (this is the analogue to the Laplace transform shortcut of writing $f(x,r)$ as $e^{rs}F(x,s)$) A partial differential equation in (x,r) will therefore be reduced to an ordinary differential equation containing the parameter s and derivatives of $G(x,s)$ w r t x Then the solution for $G(x,s)$ is multiplied by r^s and integrated over a vertical contour in the complex s plane to recover the original function $g(x,r)$ The conditions under which we can perform this integration are set out in Theorem 1 2 1 (the Mellin inversion theorem)

Once we have reduced the partial differential equations to ordinary differential equations (ODE's), we must solve these for the Mellin transformed quantities These equations typically contain a number of singular points corresponding to important surfaces in the spacetime, at which we must find qualitative behaviour of solutions Here our analysis most often falls into one of two classes either a second or higher order ODE in one variable with regular singular points, which we discuss in the next section, §1 3 2, or a first order system of ODE's with regular or irregular singular points, which we discuss in §1 3 3

1 2 2 Frobenius theorem

This theorem is very useful for giving infinite series solutions to ODE's near regular singular points, thus first we must define what a regular singular point is the nth order

ODE in $f(x)$ has a regular singular point at $x = 0$ if the ODE is of the form

$$x^n f^{(n)}(x) + x^{n-1} b_1(x) f^{(n-1)}(x) + \cdots + b_n(x) f(x) = 0, \tag{1 2 3}$$

with each b_i analytic at $x = 0$ Thus we can Taylor expand each b_i about $x = 0$, and we denote such as expansion as

$$b_i(x) = \sum_{m=0}^{\infty} b_{i,m}\, x^m$$

The simple case of each b_i constant is called the Euler equation We consider the point $x = 0$ for simplicity in presentation, an equation with a singular point at $x = x_0$ can be transformed to the above standard form by a simple transformation

The theorem of Frobenius is most familiar for the second order differential equation, which we give here

Theorem 1 2 2 (2nd order Frobenius theorem)
If $f(x)$ solves the equation

$$x^2 f''(x) + x b_1(x) f'(x) + b_2(x) f(x) = 0$$

with b_1, b_2 analytic at $x = 0$, then we define the (2nd order) indicial equation as

$$I_2(\lambda) \equiv \lambda(\lambda - 1) + b_{1,0}\lambda + b_{2,0}$$

Depending on how the roots λ_1, λ_2 of the indicial equation (hereafter indicial exponents) are related to each other, a basis of linearly independent solutions, $f_1(x)$ and $f_2(x)$, around $x = 0$ is defined thus

Case 1 $\lambda_1 - \lambda_2 \notin \mathbb{Z}$

$$f_1(x) = \sum_{m=0}^{\infty} A_m\, x^{m+\lambda_1}, \qquad f_2(x) = \sum_{m=0}^{\infty} B_m\, x^{m+\lambda_2}$$

Case 2 $\lambda_1 = \lambda_2$

$$f_1(x) = \sum_{m=0}^{\infty} A_m\, x^{m+\lambda_1}, \qquad f_2(x) = f_1(x) \ln x + \sum_{m=1}^{\infty} B_m\, x^{m+\lambda_1}$$

Case 3 $\lambda_1 - \lambda_2 = \mathbf{n} \in \mathbb{Z}$

$$f_1(x) = \sum_{m=0}^{\infty} A_m\, x^{m+\lambda_1}, \qquad f_2(x) = k f_1(x) \ln x + \sum_{m=0}^{\infty} B_m\, x^{m+\lambda_2},$$

where $\lambda_1 > \lambda_2$ and k may or may not be zero

$A_0 = B_0 = 1$ and A_m and B_m are determined by recurrence relations involving the coefficients of the expansions of the b_i

Note that in Case 2 there must be a logarithmic term in the second solution, however in Case 3 there may or may not be We will later give explicitly how to calculate the constant k

It will be important to generalize this theorem to nth order ODE's in later sections, in particular the case where there are a number of indicial exponents differing by integers To elucidate this fully, we will next describe in detail how to derive the constants corresponding to k above for the 3rd order ODE with indicial exponents $0, 1, 2$, as this is of crucial importance to the scenario that arises in Chapter 7 Then, for completeness, we will give a general theorem due to Littlefield and Desai [32]

Consider the 3rd order ODE in $f(x)$,

$$L[f] = x^3 f^{(3)}(x) + x^2 b_1(x) f''(x) + x\, b_2(x) f'(x) + b_3(x) f(x) = 0, \tag{1 2 4}$$

where the b_i are analytic near $x = 0$ We propose an infinite series solution near $x = 0$ of the form

$$\phi = \sum_{m=0}^{\infty} A_m(\lambda) x^{m+\lambda}$$

Subbing this solution in we find

$$\begin{aligned} L[\phi] = & \sum_{m=0}^{\infty} A_m (m+\lambda)(m+\lambda-1)(m+\lambda-2) x^{m+\lambda} \\ & + b_1 \sum_{m=0}^{\infty} A_m (m+\lambda)(m+\lambda-1) x^{m+\lambda} + b_2 \sum_{m=0}^{\infty} A_m (m+\lambda) x^{m+\lambda} + b_3 \sum_{m=0}^{\infty} A_m x^{m+\lambda} \end{aligned}$$

For this to be an actual solution we require the coefficient of each power of x to be zero, the coefficient of x^λ is

$$A_0\left[\lambda(\lambda-1)(\lambda-2)+b_{1,0}\lambda(\lambda-1)+b_{2,0}\lambda+b_{3,0}\right]\equiv A_0 I_3(\lambda)$$

Thus for this to vanish we must choose λ such that $I_3(\lambda)=0$ The particular case we are interested in is when $b_{1,0}=b_{2,0}=b_{3,0}=0$, as then the three roots are $0,1,2$ We order these highest first and label them $\lambda_1,\lambda_2,\lambda_3$ as $2,1,0$ respectively

Setting the coefficient of each power of x in turn to zero, we derive a recursive relationship for each coefficient $A_m(\lambda)$, such as

$$\begin{aligned}
A_1(\lambda) &= -\frac{A_0}{I_3(\lambda+1)}\left[b_{1,1}\lambda(\lambda-1)+b_{2,1}\lambda+b_{3,1}\right], \\
A_2(\lambda) &= -\frac{A_0}{I_3(\lambda+2)}\left[b_{1,2}\lambda(\lambda-1)+b_{2,2}\lambda+b_{3,2}\right] \\
&\qquad -\frac{A_1(\lambda)}{I_3(\lambda+2)}\left[b_{1,1}(\lambda+1)\lambda+b_{2,1}(\lambda+1)+b_{3,1}\right],
\end{aligned}$$

and in general,

$$A_m(\lambda)=$$
$$-\frac{1}{I_3(\lambda+m)}\sum_{n=1}^{m}A_{m-n}\Big[b_{1,n}(m-n+\lambda)(m-n+\lambda-1)+b_{2,n}(m-n+\lambda)+b_{3,n}\Big]$$

The first solution is easily described, simply

$$f_1(x)=\phi\,|_{\lambda=\lambda_1}=A_0x^2+A_1(\lambda_1)x^3+A_2(\lambda_1)x^4+$$

In fact this is the procedure for all solutions if the indices λ do not differ by integers However, if they do as in this situation, when we examine $\phi\,|_{\lambda=\lambda_2}$ we see $A_1(\lambda_2)$ contains in the denominator $I_3(\lambda_2+1)=I_3(\lambda_1)=0$, and thus is undefined

Instead we consider as a possible second solution

$$\phi=A_0(\lambda-1)x^\lambda+\sum_{m=1}^{\infty}B_m(\lambda)x^{m+\lambda}$$

From $L[\phi]$ we find the coefficient of x^λ is $A_0(\lambda-1)I_3(\lambda)$, and the next term vanishes if

$$B_1(\lambda) \quad = \quad -\frac{(\lambda-1)A_0}{I_3(\lambda+1)}\left[b_{1,1}\lambda(\lambda-1)+b_{2,1}\lambda+b_{3,1}\right],$$

but since $I_3(\lambda+1) = (\lambda+1)\lambda(\lambda-1)$, the diverging term cancels and so $B_1(\lambda_2)$ exists Thus the diverging term is removed from every coefficient thereafter, for example

$$B_2(\lambda_2) = \frac{A_1(\lambda_1)}{A_0} B_1(\lambda_2)$$

and so on But when we examine the full solution when $\lambda = \lambda_2$, we find

$$\begin{aligned}
\phi|_{\lambda=\lambda_2} &= (\lambda_2 - 1)A_0 x + B_1(\lambda_2)x^2 + B_2(\lambda_2)x^3 + \\
&= \frac{B_1(\lambda_2)}{A_0}\left[A_0 x^2 + \frac{B_2(\lambda_2)A_0}{B_1(\lambda_2)}x^3 + \quad\right] \\
&= \frac{B_1(\lambda_2)}{A_0}\left[A_0 x^2 + A_1(\lambda_1)x^3 + \quad\right] \\
&= \frac{B_1(\lambda_2)}{A_0} f_1(x),
\end{aligned}$$

that is to say, this proposed solution is not linearly independent of the first solution In general we write

$$\frac{B_1(\lambda_2)}{A_0} = \lim_{\lambda\to\lambda_2}\left[\frac{(\lambda-\lambda_2)A_1(\lambda_2)}{A_0}\right]$$

For the *actual* second solution we take a hint from the Euler equation, and find

$$\begin{aligned}
f_2(x) = \frac{\partial\phi}{\partial\lambda}\Big|_{\lambda=\lambda_2} &= \Big(A_0 x^\lambda + A_0(\lambda-1)x^\lambda \ln x \\
&\quad + \sum_{m=1}^{\infty}\frac{dB_m(\lambda)}{d\lambda}x^{m+\lambda} + \ln x\sum_{m=1}^{\infty}B_m(\lambda)x^{m+\lambda}\Big)\Big|_{\lambda=\lambda_2}
\end{aligned}$$

But from the previous work we know the last series is a multiple of the first solution Thus our second solution is

$$f_2(x) = \lim_{\lambda\to\lambda_2}\left[\frac{(\lambda-\lambda_2)A_1(\lambda_2)}{A_0}\right]\ln x\, f_1(x) + \sum_{m=0}^{\infty}\frac{dB_m}{d\lambda}\Big|_{\lambda=\lambda_2} x^{m+\lambda_2},$$

where $B_0 = A_0(\lambda - 1)$

The procedure for the third solution is similar we posit a solution as

$$\phi = \lambda^2 A_0 x^\lambda + \sum_{m=1}^{\infty} C_m(\lambda)x^{m+\lambda}$$

The coefficient of x^λ is $\lambda^2 A_0 I_3(\lambda)$, the coefficient of $x^{\lambda+1}$ is zero if

$$C_1 = -\frac{\lambda^2 A_0}{I_3(\lambda+1)}\left[b_{1,1}\lambda(\lambda-1) + b_{2,1}\lambda + b_{3,1}\right]$$

and so on Note the λ^2 means that $C_1(\lambda_3)$ vanishes, and thus $\phi|_{\lambda=\lambda_3}$ begins at x^2, just as $f_1(x)$ does In fact, we can show that $\phi|_{\lambda=\lambda_3}$ is a multiple of $f_1(x)$, to be precise

$$\phi|_{\lambda=\lambda_3} = \lim_{\lambda\to\lambda_3}\left[\frac{\lambda^2 A_2(\lambda)}{A_0}\right] f_1(x)$$

What's more, the derivative is a multiple of the second solution, so in fact the third solution is given by the second derivative,

$$f_3(x) = \left.\frac{\partial^2\phi}{\partial\lambda^2}\right|_{\lambda=\lambda_3}$$

Thus our three linearly independent solutions to (1 2 4) are (with a slight renaming of coefficients)

$$\begin{aligned}
f_1(x) &= \sum_{m=0}^{\infty} A_m x^{m+2} \\
f_2(x) &= \lim_{\lambda\to 1}\left[\frac{(\lambda-1)A_1(\lambda)}{A_0}\right]\ln x\sum_{m=0}^{\infty} A_m x^{m+2} + \sum_{m=0}^{\infty} B_m x^{m+1} \\
f_3(x) &= \lim_{\lambda\to 0}\left[\frac{\lambda^2 A_2(\lambda)}{A_0}\right]\ln^2 x\sum_{m=0}^{\infty} A_m x^{m+2} \\
&\quad +2\lim_{\lambda\to 0}\left[\frac{d}{d\lambda}\left(\frac{\lambda^2 A_1(\lambda)}{A_0}\right)\right]\ln x\sum_{m=0}^{\infty} B_m x^{m+1} + \sum_{m=0}^{\infty} C_m x^m
\end{aligned} \tag{1 2 5}$$

In terms of the coefficients of the differential equation, these limits are

$$\begin{aligned}
\lim_{\lambda\to 1}\left[\frac{(\lambda-1)A_1(\lambda)}{A_0}\right] &= -\tfrac{1}{2}(b_{2,1}+b_{3,1}), \\
\lim_{\lambda\to 0}\left[\frac{\lambda^2 A_2(\lambda)}{A_0}\right] &= -\tfrac{1}{2}b_{3,1}(b_{2,1}+b_{3,1}), \\
\lim_{\lambda\to 0}\left[\frac{d}{d\lambda}\left(\frac{\lambda^2 A_1(\lambda)}{A_0}\right)\right] &= b_{3,1}
\end{aligned}$$

Now we give a theorem by Littlefield and Desai [32] to generalise this analysis to nth order equations, however we only describe the case when the roots of the indicial equation differ by integers and do not repeat Suffice it to say that when the roots repeat there *must* be a logarithmic term in the solution

Theorem 1 2 3 (nth order Frobenius theorem)
Let $f(x)$ solve an ODE of the form (1 2 3) *Then the indicial equation is*

$$I_n(\lambda) \equiv \lambda(\lambda-1) \quad (\lambda-n+1) + b_{1,0}\lambda(\lambda-1) \quad (\lambda-n+2) + \quad + b_{n-1,0}\lambda + b_{n,0}$$

Arrange the roots of the indicial equation into groups differing by integers, and order them

$$\{\lambda_1, \lambda_2, \quad , \lambda_j, \quad \}$$

such that $\lambda_i > \lambda_{i+1}$ Then a linearly independent solution corresponding to λ_j is

$$f_j(x) = \sum_{i=1}^{j} \left(\beta_i K_i \log^{(j-i)} x \sum_{m=0}^{\infty} \frac{\partial^{(i-1)}}{\partial\lambda^{(i-1)}} \Big[(\lambda - \lambda_i) A_m \Big]_{\lambda_i} x^{m+\lambda_i} \right)$$

where

$$K_i = \lim_{\lambda\to\lambda_j} \left(\frac{\partial^{(i-1)}}{\partial\lambda^{(i-1)}} \left[(\lambda-\lambda_j)^{(j-1)} \frac{A_{\delta_i}}{A_0} \right] \right), \quad K_j = 1$$

and

$$\delta_i = \lambda_j - \lambda_i \in \mathbb{N}, \qquad \beta_i = \frac{(j-1)(j-2) \quad (j-i+1)}{i-1}, \quad \beta_1 = \beta_j = 1$$

1 2 3 Methods for systems of ordinary differential equations with singular points

For a first order system $Y' = M(x)Y$ we define p as the least number such that the system can be written near $x = 0$ as

$$Y' = \frac{1}{x^p}\left(J + \sum_{m=1}^{\infty} A_m x^m \right) Y, \tag{1 2 6}$$

and near $x = \infty$ as

$$Y' = -x^{p-2}\left(J + \sum_{m=1}^{\infty} A_m x^{-m} \right) Y \tag{1 2 7}$$

with $J \neq 0$ and constant

We classify singular points and describe the solution as

p = 0, Regular point It is sufficient for the sake of this thesis to know that the solutions near a regular point are themselves regular, as the following theorem (Coddington and Levinson [9]) shows

Theorem 1 2 4 *Consider the linear system $Y' = MY$ where $Y \in \mathbb{R}^n$ and M is an $n \times n$ matrix If the coefficients of M are continuous on some open interval I, which may be unbounded, there exists on I one and only one solution φ to the system satisfying*

$$\varphi(\tau) = \xi \qquad \forall \tau \in I,$$

with $\xi \in \mathbb{R}^n$ satisfying $|\xi| < \infty$ is arbitrary

p = 1, Regular singular point Also known as a simple singularity or a singularity of the first kind Here we distinguish solutions depending on whether the eigenvalues of J given above differ by an integer or not If they do not, we apply Theorem 1 2 5 immediately If they do, we reduce those eigenvalues individually until they are equal using Theorem 1 2 6, and then apply Theorem 1 2 5 (see [9])

Theorem 1 2 5 *In the system (1 2 6) with $p = 1$, if J has eigenvalues which do not differ by positive integers, then, in a disc around $x = 0$ not containing another singular point, (1 2 6) has a fundamental matrix Φ (whose columns are linearly independent solutions) of the form*

$$\Phi = Px^J, \qquad \text{where} \quad P(x) = \sum_{m=0}^{\infty} x^m P_m, \qquad P_0 = I$$

Theorem 1 2 6 *Let the distinct eigenvalues of J in (1 2 6) be (disregarding multiplicity) $\mu_1, \quad \mu_k$, ($k \leq N$, where N is the order of the system), and let the repeating eigenvalue be μ_i There is a matrix $V(x)$ such that $Y = VQ$ transforms (1 2 6) into*

$$Q' = \frac{1}{x}\left(\widehat{J} + \sum_{m=1}^{\infty} \widehat{A}_m x^m\right) Q,$$

where $\widehat{J}$ has eigenvalues $\mu_1, \quad , \mu_i - 1, \quad , \mu_k$ V is given by the $n \times n$ matrix $V = diag(1, \quad , 1, x, \quad , x, 1, \quad , 1)$, with x appearing in entry i to $i + j - 1$, where

μ_i *is of multiplicity* j

p $\geq$ 2, Irregular singular point Also known as a non-simple singularity or a singularity of the second kind Here we distinguish solutions depending on whether you can diagonalize J given in (1 2 7) If J has distinct eigenvalues, then J is diagonalizable and we apply Theorem 1 2 7 If J has multiple eigenvalues and J can only be reduced to Jordan normal form, then we apply Theorem 1 2 8 to remove off-diagonal terms (see Eastham [10]) When the eigenvalues are repeated zeroes, this has the effect of reducing the order of the singularity, as happens in this thesis in Chapter 6

There is a class of problems in between sometimes a matrix has multiple eigenvalues and yet can still be diagonalized In this case, there is a straightforward theorem given by [10] if $A_1 = 0$ (as is sometimes the case when a high order equation is written as a first order system) If not there is a cumbersome solution given by [9]

Theorem 1 2 7 *Let J have distinct eigenvalues $\mu_1, \quad , \mu_N$ Then (1 2 6) with $p \geq 2$ has a fundamental matrix*

$$\Phi = P x^R e^H, \qquad \text{where} \quad P(x) = \sum_{m=0}^{\infty} x^m P_m, \qquad P_0 = I,$$

R is a diagonal matrix of complex constants, and H is a matrix polynomial ($r = p - 2$)

$$H = \frac{x^{r+1}}{r+1} H_0 + \frac{x^r}{r} H_1 + \quad + x H_r, \quad H_i = diag\left(\mu_1^{(i)}, \quad , \mu_N^{(i)}\right), \quad \mu_j^{(0)} = \mu_j$$

For brevity's sake, we give the following theorem only for $p = 2$, as this is the case that arises in this thesis

Theorem 1 2 8 *We transform J to its Jordan normal form $\widehat{J}$, and write the blocks of $\widehat{J}$ as $\mu I + \rho E$, where E is the matrix with 1's along its super-diagonal and zeroes elsewhere For each block of $\widehat{J}$, define the matrices*

$$D = diag(1, \rho x, \quad , (\rho x)^{N-1}), \quad B = \begin{pmatrix} 1 & 1 & 1/2! & & 1/(N-1)! \\ 0 & 1 & 1 & & 1/(N-2)! \\ & & & & \\ & & & & 1 \end{pmatrix},$$

where N is the order of the system Then the transformation $Y = D^{-1}BW$ gives the system

$$W' = \left[\mu I + D'D^{-1} + B^{-1}D\left(\sum_{m=1}^{\infty} A_m x^{-m}\right) D^{-1}B\right] W,$$

and the leading order coefficient matrix has had its off-diagonal terms removed

Chapter 2

Self-similar spherically symmetric (4-S) spacetimes

2 1 Definitions and a line element for 4-S spacetimes

The Einstein field equations for a completely general spacetime are a system of quasi-linear, coupled, four-dimensional partial differential equations Any attempt to draw useful information about the solutions from the field equations is severely hampered without making some simplifying assumptions about the nature of the solution The two assumptions we will make use of in this thesis are spherical symmetry and self-similarity We will begin by defining rigorously what these notions mean, beginning with spherical symmetry

A spacetime is defined as a four dimensional Riemannian manifold $\mathcal{M}$ endowed with a metric tensor g_{ab} with signature $+2$, and is denoted $(\mathcal{M}, g_{ab})$ Spherical symmetry means that there is a preferred curve in $\mathcal{M}$, called the axis, at each point (moment in time) along this axis, the spatial part of the metric tensor is invariant under any rotation about this point Thus $\mathcal{M}$ can be considered to be a two dimensional Lorentzian manifold $\mathcal{M}^2$ crossed with two-spheres $\mathcal{S}^2$ Setting the coordinates on $\mathcal{M}^2$ constant, the metric tensor reduces to the metric of a two-sphere,

$$ds^2\big|_{\mathcal{S}^2} = r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \equiv r^2 d\Omega^2,$$

where (θ, ϕ) are angular coordinates on the two-sphere with the ranges $0 \leq \theta \leq \pi$ (colatitude) and $0 \leq \phi < 2\pi$ (azimuth) r is a function of the coordinates of $\mathcal{M}^2$, and is defined in terms of the surface area of the two-spheres A,

$$r = \left(\frac{A}{4\pi}\right)^{1/2},$$

and thus is called the areal radius There are no cross terms in $d\theta, d\phi$ as the spacetime is invariant under reflections in the angular coordinates (this is implicit in the definition of spherical symmetry)

$$\theta \to \pi - \theta, \qquad \phi \to 2\pi - \phi$$

The spacetime must further be invariant under translations in the angular coordinates (rotations),

$$\theta \to \theta + \theta_0, \qquad \phi \to \phi + \phi_0,$$

and thus θ, ϕ may only appear in the metric functions in the two-sphere portion

Thus we have derived the "2+2" split of Gerlach and Sengupta [13],[14] which we shall use later in Chapter 4 if the coordinates on $\mathcal{M}^2$ are denoted x^A, then the metric for a spherically symmetric spacetime can be written

$$ds^2 = g_{AB}dx^A dx^B + r^2(x^C)\big(d\theta^2 + \sin^2\theta d\phi^2\big),$$

where $g_{AB} = g_{AB}(x^C)$ is the metric on $\mathcal{M}^2$

It is always possible, though not always useful, to let the function r be one of the coordinates on the manifold $\mathcal{M}^2$ We will let t denote the other coordinate on $\mathcal{M}^2$, and find its nature below We wish $\mathcal{M}^2$, a 2-dimensional manifold, to be Lorentzian, thus g_{AB} must have one positive and one negative eigenvalue With A, B, C arbitrary functions of t, r we can write the metric therefore as

$$ds^2 = -A^2dt^2 + 2A\,B\,dt\,dr + C^2dr^2 + r^2d\Omega^2$$

There is freedom in the t coordinate to make the transformation

$$e^{\nu/2}dt' = A\,dt - B\,dr,$$

which knocks out the cross term $dt\,dr$ This diagonalizes the line element and is equivalent to choosing t' orthogonal to r, that is, if $\nabla_a t'$ is a covariant vector pointing in the t'-direction, and $\nabla_a r$ points in the r-direction, then $g^{ab}\nabla_a t'\nabla_b r = 0$ Renaming $B^2 + C^2 = e^\lambda$ and $t' = t$, we have the canonical form for the metric tensor of a general spherically symmetric spacetime,

$$ds^2 = -e^{\nu(t,r)}dt^2 + e^{\lambda(t,r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{2 1 1}$$

We see the length of a covariant vector pointing in the t-direction, $\nabla_a t$, is $-e^{-\nu} < 0$, and thus t is a timelike coordinate, whereas r, θ, ϕ are spacelike (The nature of coordinates can vary as we move into different regions of spacetime For example, when you cross the Schwarzschild radius, the t coordinate becomes spacelike and the r coordinate timelike More on this later)

In deriving this line element, we have chosen the coordinates on the sphere to be θ, ϕ, and the coordinates on $\mathcal{M}^2$ to be t, r These may not always be the most convenient coordinates, often it is useful to use a null coordinate, that is, a coordinate u or v such that $u, v =$constant describes a null geodesic Spherically symmetric spacetimes are Petrov type D (see §1 2) and thus we can always define two radial null directions One of these is ingoing and the other is outgoing

Suppose we wanted to write the metric for a spherically symmetric spacetime using the ingoing null coordinate We consider the geodesic equations given in §1 2 From (2 1 1), we see a radial ($\theta = \phi = 0$) null ($ds^2 = 0$) geodesic solves

$$\frac{dt}{dr} = \pm e^{\frac{\lambda-\nu}{2}},$$

with the plus and minus denoting outgoing and ingoing null geodesics respectively Assuming that λ and ν are C^1 functions of t and r, the theory of ordinary differential equations guarantees that there is a solution of the form $t = \pm T(r) + c$, where c is a constant If we transform to a new time coordinate, $t \to \bar{t} = t - r + T(r)$, then the ingoing null geodesics are simply $\bar{t} + r = c$ The constant of integration labels each ingoing null geodesic, and we denote with v therefore the ingoing null coordinate $v = \bar{t} + r$ To remove t from the line element (2 1 1) we let $t = v - T(r)$, and $dt = dv - (dT/dr)dr$ The line element then becomes

$$ds^2 = -e^\nu\left[dv^2 - 2\frac{dT}{dr}dv\,dr + \left(\frac{dT}{dr}\right)^2 dr^2\right] + e^\lambda dr^2 + r^2 d\Omega^2$$

But since $t = T + c$, then $dt/dr = dT/dr = e^{\frac{\lambda-\nu}{2}}$, and thus the dr^2 term cancels Renaming some terms, the line element for a general spherically symmetric spacetime in (v, r) coordinates will be

$$ds^2 = -2Fe^{2\psi}dv^2 + 2e^{\psi}dv\,dr + r^2d\Omega^2 \qquad (2\ 1\ 2)$$

We will call these advanced Bondi coordinates (Note there is some coordinate freedom left in v to make a transformation $v \to V(v)$) Similarly, we could choose to use both ingoing and outgoing null coordinates, in which case the line element becomes

$$ds^2 = -2e^{-2f}du\,dv + r^2d\Omega^2, \qquad (2\ 1\ 3)$$

where $f = f(u, v)$ and $r = r(u, v)$, the so-called double-null form

The method used above essentially involved choosing appropriate coordinates to place restrictions on the metric functions However we can describe symmetry in a covariant way, by using Killing vectors

A vector field ξ^a will describe a symmetry, or an isometry (distance preserving mapping), if the metric is unchanged by an infinitesimal motion in the direction ξ^a This motion is described in a covariant manner by the Lie derivative $\mathcal{L}$ with respect to ξ^a, which we will briefly outline as it is of importance here and in Chapter 4 the vector field ξ^a 'carries along' the metric tensor g_{ab} from the point with coordinates x^a to the point $x'^a = x^a + \varepsilon\xi^a$ (this is the active view of a coordinate transformation) The Lie derivative is defined as the difference between the metric tensor at x'^a, $g_{ab}(x')$, and the metric tensor from x^a carried along to x'^a at x'^a, $g'_{ab}(x')$, in the limit $\varepsilon \to 0$, that is

$$\mathcal{L}_\xi\, g_{ab} = \lim_{\varepsilon \to 0} \frac{g_{ab}(x') - g'_{ab}(x')}{\varepsilon}$$

Thus a spacetime will have a symmetry if there is some vector ξ^a such that

$$\mathcal{L}_\xi\, g_{ab} = \nabla_a\xi_b + \nabla_b\xi_a = 0,$$

where the formula $\mathcal{L}_\xi\, g_{ab} = \nabla_a\xi_b + \nabla_b\xi_a$ follows from the definition of the Lie derivative This is known as Killing's equation and any vector satisfying this equation is called a Killing vector

For example, consider the two-sphere with line element $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ If the components of ξ^a are $\alpha(\theta,\phi)$ and $\beta(\theta,\phi)$, then Killing's equations are

$$\alpha_{,\theta} = 0, \quad \alpha_{,\phi} + \sin^2\theta\beta_{,\theta} = 0, \quad \alpha\cos\theta + \sin\theta\beta_{,\phi} = 0$$

Differentiating the second equation w r t θ, we can solve to give $\beta = -\cot\theta f(\phi) + C$ and $\alpha' = -f$, and then the third equation gives $f'' + f = 0$ and $\alpha = f'$ The general solution is therefore

$$\xi^a = \begin{pmatrix} -A\sin\phi + B\cos\phi \\ -\cot\theta\big(A\cos\phi + B\sin\phi\big) + C \end{pmatrix},$$

and thus there is a three parameter family of solutions to Killing's equation As the Killing vector fields of $\mathcal{S}^2$ are also Killing vector fields of $\mathcal{M} = \mathcal{M}^2 \times \mathcal{S}^2$, in general a spherically symmetric spacetime will have (at least) three Killing vectors

The Lie derivative can be used to define another type of symmetry called self-similarity, or homothety A symmetry is when the metric does not change as we move in the direction of symmetry, $\mathcal{L}_\xi g_{ab} = 0$ On the other hand, a (continuous) self-similarity is when the metric changes in a manner proportional to itself as we move in the direction of self-similarity, that is ξ^a is a homothetic or self-similar vector field if

$$\mathcal{L}_\xi g_{ab} \propto g_{ab}$$

More precisely, a (proper) homothetic Killing vector field ξ^a is one such that $\mathcal{L}_\xi g_{ab} = k g_{ab}$, and a Killing vector field is a subset of this class with $k = 0$ By a constant rescaling of ξ^a we can assume without loss of generality that $k = 2$ for a proper homothetic Killing vector field (i e one for which $k \neq 0$)

For a spacetime to be both spherically symmetric and self-similar therefore, there must be three Killing vectors and a homothetic vector Working in (v,r) coordinates, with the metric as given in (2 1 2), a homothetic vector must solve the equations $\xi_{a,b} + \xi_{b,a} - 2g_{ab} = 0$ If $\xi^a = (\alpha(v,r), \beta(v,r))$, then the rr equation gives $\alpha = \alpha(v)$ and the $\theta\theta$ equation gives $\beta = r$ We make a further coordinate transformation $v \to v'$ to set $\alpha(v) = v$ Then the remaining equations are

$$v\psi_{,v} + r\psi_{,r} = 0, \qquad vF_{,v} + rF_{,r} = 0,$$

in other words, $\xi^a\psi_{,a} = \xi^a F_{,a} = 0$

We may solve these PDE's using the method of characteristics We define the characteristics of the PDE by $(v, r) = (v(s), r(s))$ satisfying

$$\frac{dv}{ds} = v, \qquad \frac{dr}{ds} = r \tag{2 1 4}$$

Then defining $F(s) = F(v(s), r(s))$ we find

$$\frac{dF}{ds} = \frac{\partial F}{\partial v}\frac{dv}{ds} + \frac{\partial F}{\partial r}\frac{dr}{ds} = vF_{,v} + rF_{,r} = 0$$

and so F is constant along the characteristics Integrating (2 1 4), we find that the characteristics are given by $\frac{v}{r}$ = constant, and thus $x \equiv v/r$ labels the different characteristics Since F is constant on characteristics, we can write $F = F(x)$, and similarly $\psi = \psi(x)$

We will call this x the similarity variable, or coordinate As a coordinate its nature is changeable however, going from timelike to null to spacelike and so on Surfaces over which a similarity variable changes from timelike to spacelike and vice versa are called similarity horizons, and will be discussed in more detail in the next section

To conclude, the canonical form for the line element of a self-similar, spherically symmetric spacetime is

$$ds^2 = -2Fe^{2\psi}dv^2 + 2e^{\psi}dv\,dr + r^2 d\Omega^2, \tag{2 1 5}$$

with $F = F(x)$ and $\psi = \psi(x)$, where $x = v/r$

2 2 Energy conditions

In Chapters 2 and 7 we will perturb a generic self-similar, spherically symmetric spacetime endowed with the metric described above The term 'generic' is key, as we wish to make the analysis very general and thus do not wish to specify the matter content of the spacetime However the background matter content cannot be completely arbitrary, there are a number of important restraints which any realistic physical system must obey These restraints will give us a handle on the metric functions defined in (2 1 5) and make the perturbation analysis feasible

There are three energy conditions, the weak, the strong and the dominant Consider Einstein's field equations,

$$R_{ab} - \tfrac{1}{2}g_{ab}R = 8\pi T_{ab}$$

Taking the trace gives $R = -8\pi T$, and we can write

$$R_{ab} = 8\pi\left(T_{ab} - \tfrac{1}{2}g_{ab}T\right)$$

This equation holds for all observers, for example an observer with unit timelike tangent vector ξ^a,

$$R_{ab}\xi^a\xi^b = 8\pi\left(T_{ab}\xi^a\xi^b + \tfrac{1}{2}T\right) \tag{2 2 1}$$

As mentioned in Section 1 2, the quantity $T_{ab}\xi^a\xi^b$ represents the energy density measured by this timelike observer (see below), and is considered to be non-negative for a physically realistic matter model This is the weak energy condition Requiring the entire term on the right hand side of (2 2 1) to be non-negative is the strong energy condition, and therefore is equivalent to $R_{ab}\xi^a\xi^b \geq 0$ Note some authors consider the weak energy condition to be $T_{ab}\ell^a\ell^b \geq 0$ for all null ℓ^a, in which case the strong energy condition implies the weak energy condition We will combine the two in the following definitions [28]

Weak energy condition $T_{ab}v^av^b \geq 0$ for all future-pointing causal (non-spacelike) vectors v^a

Strong energy condition $R_{ab}v^av^b \geq 0$ for all future-pointing causal (non-spacelike) vectors v^a

To interpret these conditions we note that all matter tensors representing what is believed to be physically reasonable matter models are diagonalizable (can be reduced to principle axes) [48] Thus T_{ab} takes the form

$$T_{ab} = \rho t_a t_b + p_1 x_a x_b + p_2 y_a y_b + p_3 z_a z_b,$$

where $\{t_a, x_a, y_a, z_a\}$ is an orthonormal basis with t_a timelike and x_a, y_a, z_a spacelike ρ represents the rest energy density of the matter field, and $p_{1,2,3}$ are called the principle pressures/stresses Then the weak energy condition is satisfied when

$$\rho \geq 0, \qquad \rho + p_i \geq 0, \quad i = 1, 2, 3,$$

and the strong energy condition is satisfied when

$$\rho + p_1 + p_2 + p_3 \geq 0, \qquad \rho + p_i \geq 0, \quad i = 1, 2, 3$$

There is a third energy condition, which requires

$$\rho \geq |p_i|, \quad i = 1, 2, 3,$$

that is, the energy density is non-negative and dominates the stresses present. This is known as the dominant energy condition, and derives from the fact that $-T^a{}_b\xi^b$ represents the density of momentum measured by a timelike observer with future-pointing tangent vector ξ^a. This leads to the definition

Dominant energy condition For all future-pointing timelike ξ^a, the vector $-T^a{}_b\xi^b$ is non-spacelike and future-pointing.

This can be shown to be equivalent to requiring that the local speed of sound is not greater than the local speed of light. The reason this condition is relaxed into the strong and weak energy conditions is that some effects violate the dominant energy condition, for example Hawking radiation (see Wald [48]). The dominant energy condition implies the weak energy condition, but there are no other implications between these conditions.

Now we will derive the appropriate energy conditions for a self-similar spherically symmetric spacetime. We wish our spacetime to satisfy the dominant energy condition, which as mentioned implies the weak energy condition. Thus we require

$$T_{ab}v^a v^b \geq 0 \qquad \Rightarrow \qquad R_{ab}v^a v^b \geq \tfrac{1}{2}R\, v^a v_a,$$

for all causal v^a. Taking $v^a = \ell^a$ to be null means we require $R_{ab}\ell^a\ell^b \geq 0$. Let us denote with ℓ^a_+ and ℓ^a_- the outgoing and ingoing null directions respectively. Solving $g_{ab}\ell^a_+\ell^b_+ = g_{ab}\ell^a_-\ell^b_- = 0$ and $g_{ab}\ell^a_+\ell^b_- = -1$ in the (v, r) coordinates gives

$$\ell^a_+ = (1, Fe^{\psi}), \qquad \ell^a_- = (0, -e^{-\psi}),$$

and, calculating the Ricci tensor in (v, r) coordinates, we find the strong and weak energy conditions imply $R_{ab}\ell^a_+\ell^b_+ \geq 0$ and $R_{ab}\ell^a_-\ell^b_- \geq 0$, which is equivalent to

$$x\psi' \leq 0, \tag{2.2.2a}$$

$$e^{\psi}(F' + xF^2e^{\psi}\psi') \leq 0 \tag{2.2.2b}$$

The dominant energy condition includes these inequalities as well as $R_{\theta\theta} \geq 0$. For a self-similar spherically symmetric spacetime this is equivalent to

$$1 - 2F + 2x\big(F' + F\psi'\big) \geq 0 \qquad (2\,2\,2c)$$

Note the dominant energy condition returns a number of other inequalities [40], however the three given here are sufficient for the analysis we wish to carry out Note also the energy conditions are ordinary differential equations in x

2 3 Conditions for a naked singularity

To consider a solution to the field equations as a possible counter-example to the Cosmic Censorship hypothesis, we must be certain any naked singularities do not appear because they are inserted 'by hand' in the initial data Thus we impose a number of regularity conditions on the region of spacetime prior to the formation of the singularity Then we will specify the conditions required for a naked singularity to emerge

We take our regularity conditions to be the following there is a regular axis where all curvature invariants are finite for a non-zero finite time before the formation of a singularity In addition there are no trapped or marginally trapped surfaces in the initial configuration (A trapped surface is a two-dimensional spacelike manifold where the expansion of both ingoing and outgoing null geodesics is negative The boundary of a region of trapped surfaces is called the apparent horizon Trapped surfaces signal the presence of singularities, which is why we wish to rule them out in the initial configuration) These conditions place further constraints on the metric functions

We will work in advanced Bondi coordinates using the line element given in (2 1 5),

$$ds^2 = -2Fe^{2\psi}dv^2 + 2e^{\psi}dv\,dr + r^2 d\Omega^2,$$

with $F = F(x)$ and $\psi = \psi(x)$, where $x = v/r$ We will call the point where the singularity forms the scaling origin $\mathcal{O}$, and denote the past null cone of $\mathcal{O}$ with $\mathcal{N}$ The past null cone of $\mathcal{O}$ may be called the threshold v labels the past null cones of $r = 0$ and is taken to increase into the future, and we can identify $v = 0$ with $\mathcal{N}$, so that $(v, r) = (0, 0)$ at $\mathcal{O}$ Taking v to measure proper time along the regular center $r = 0$ exhausts all remaining coordinate freedom

We require regularity of the axis to the past of $\mathcal{O}$ and of $\mathcal{N}$ The axis is given by $r = 0$

for $v < 0$ and thus $x = -\infty$ That v measures proper time along the axis we require

$$\lim_{x \to -\infty} 2Fe^{2\psi} = 1$$

The Misner-Sharp mass is defined by

$$m = \frac{r}{2}\left(1 - g^{ab}\nabla_a r \nabla_b r\right) = \frac{r}{2}\left(1 - 2F\right),$$

and thus we measure the invariant m/r^3 as it has the same units as other curvature invariants, for example the Ricci scalar For m/r^3 to be finite as $r \to 0$ we require $1 - 2F \to 0$, thus for a regular axis we require

$$F(-\infty) = \tfrac{1}{2}, \qquad \psi(-\infty) = 0$$

The interior to $\mathcal{N}$, and $\mathcal{N}$ itself, must be regular, that is $F, \psi \in C^2(-\infty, 0]$, since $\mathcal{N}$ is given by $v = 0 = x$ That there are no trapped surfaces interior to, or intersecting $\mathcal{N}$, demands

$$g^{ab}\nabla_a r \nabla_b r = 2F > 0,$$

that is, r remains spacelike Thus we require $F > 0$ for $x \in (-\infty, 0]$, and we note an apparent horizon will form for $F = 0$

We demonstrate that there must be a curvature singularity at $\mathcal{O}$ in the following way consider again the curvature invariant m/r^3 Since $F > 0$ for $x \in (-\infty, 0]$, we see $x = x_0 < 0$ is timelike and thus we can approach $\mathcal{O}$ along $x = x_0$ Then m/r^3 will diverge at $\mathcal{O}$ unless $F \equiv \frac{1}{2}$ for $x \in (-\infty, 0)$ Similarly the invariant

$$R_{\theta\theta} = \frac{1}{2r^2}\left(1 - 2F + 2x(F' + F\psi')\right)$$

will diverge at $\mathcal{O}$ unless ψ = constant for $x < 0$, however to match at the axis this constant must be zero Therefore to avoid a curvature singularity at $\mathcal{O}$ the region interior to $\mathcal{N}$ must be flat Avoiding this trivial case means there is a curvature singularity at $\mathcal{O}$ (Note the Vaidya spacetime *is* empty in the region interior to $\mathcal{O}$, thus we will prove the existence of the singularity for Vaidya spacetime in the appropriate section)

Now we will describe the conditions under which this singularity is naked A globally naked singularity is one from which null geodesics can escape to future null infinity $\mathcal{J}^+$ (the surface at which all future pointing null geodesics end) Locally naked singularities, on the other hand, admit null geodesics which leave the singularity but may not necessarily make it all the way to $\mathcal{J}^+$ All the backgrounds we consider in this thesis admit globally naked singularities, in opposition to the strong and weak CCH

The outgoing radial null geodesic (o r n g) equation of our self-similar spherically symmetric spacetime is

$$\frac{dr}{dv} = Fe^{\psi} \equiv G(x)$$

Since $x = v/r$, we may rephrase the geodesic equation as

$$\frac{dx}{dv} = \frac{r - v\frac{dr}{dv}}{r^2} = \frac{1}{r}(1 - xG) \tag{2 3 1}$$

We will prove the following proposition

Proposition 2 1 *There is an outgoing radial null geodesic emerging from $\mathcal{O}$ if and only if there exists a positive real root to $xG = 1$*

Proof Consider a point in spacetime with x-coordinate x_p such that $x_p > 0$ Since $1 - xG > 0$ at $x = 0$, therefore either $1 - xG > 0$ for $x \in [0, x_p]$, or $1 - xG = 0$ for some $x_0 \in (0, x_p]$ In the first case, through this point there is a unique o r n g , γ_p, by standard theorems of ordinary differential equations Since $1 - xG > 0$, by (2 3 1) we see $dx/dv > 0$ and thus x decreases as v decreases Tracking back along γ_p to $v = 0$ there are two possibilities either $r \to r* > 0$ in which case we miss the singularity, or $r \to 0$ If $r \to 0$ we calculate the limit $x_l = \lim_{v\downarrow 0} x(v)|_{\gamma_p}$ as

$$x_l = \lim_{v\downarrow 0} \frac{v}{r} = \lim_{v\downarrow 0} \frac{1}{dr/dv} = \lim_{v\downarrow 0} \frac{1}{G(x)} = \frac{1}{G(x_l)},$$

using l'Hôpital's rule, with $x_l < x_p$ Therefore $1 - x_l G(x_l) = 0$ which contradicts the criterion that $1 - xG > 0$ for $x \in [0, x_p]$ Thus there can be no outgoing radial null geodesics which reach the singularity in the past in the region $x \in [0, x_p]$ if $1 - xG > 0$ in this region

In the second case of $1 - xG = 0$ for some $x_0 \in (0, x_p]$, we see $x(v) = x_0$ solves (2 3 1) and therefore $x = x_0$ is an outgoing radial null geodesic Since $G = Fe^{\psi}$ and $F > 0$ prior to the formation of an apparent horizon, we must have $x_0 > 0$ Moreover, since all curves $x =$ constant focus at the origin, $x = x_0$ represents an outgoing radial null geodesic which emanates from the singularity at $\mathcal{O}$, concluding the proof $\square$

We will call the first real root to $xG = 1$ the Cauchy horizon, and denote it $x = x_c$

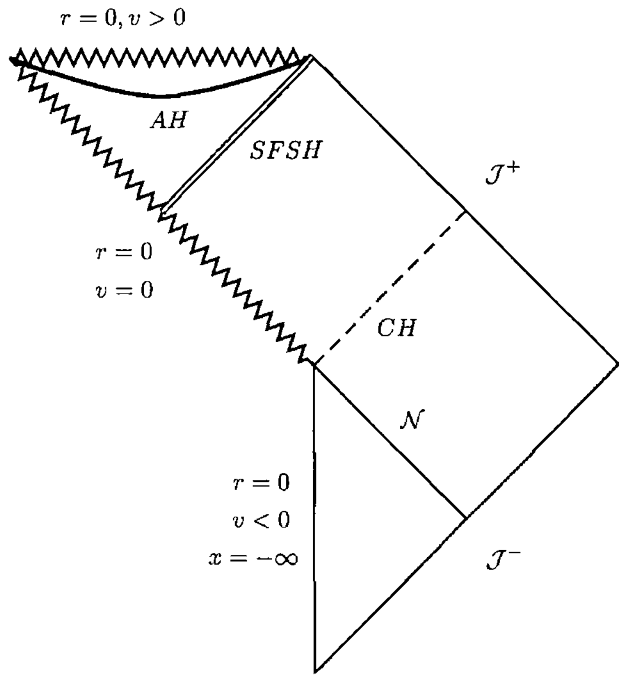


Figure 2 1 Possible conformal diagram for a 4-S spacetime admitting a globally naked singularity There are three similarity horizons at which the similarity coordinate x is null $x = 0$ denoted $\mathcal{N}$, $x = x_c$ shown dashed, and $x = x_e$ shown as a double line We identify $x = x_c$ as the Cauchy horizon, and will call $x = x_e$ the second future similarity horizon (SFSH) The apparent horizon is shown as a bold curve

We note the length of the vector pointing in the x-direction is

$$g^{ab}\nabla_a x \nabla_b x = \frac{2e^{-\psi}x(xG-1)}{r}, \tag{2 3 2}$$

and thus $x = 0$ and the roots of $xG = 1$ are null hypersurfaces These hypersurfaces mark the transition of the similarity coordinate x from timelike ($xG < 1$) to spacelike ($xG > 1$) and back again, and thus we call them 'similarity horizons' The Cauchy horizon is the first (future) similarity horizon, and subsequent roots of $xG = 1$ represent additional (future) similarity horizons If there are no real roots then the singularity is censored

A general picture of collapse in 4-S spacetimes is beginning to emerge there is a regular axis on which all curvature invariants are finite for a non-zero finite time, and in addition there are no trapped surfaces in the initial configuration Then there is an

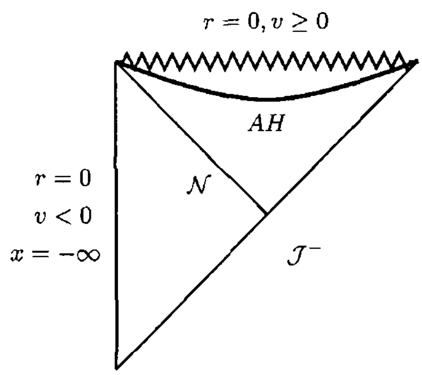


Figure 2 2 Conformal diagram for 4-S spacetime with censored singularity, corresponding to the case of no real roots to $xG = 1$ A spacelike singularity forms at $r = 0$ for $v \geq 0$

inevitable curvature singularity at the scaling origin, which is naked if and only if there are real roots to an equation involving the metric functions

There are still a number of possible configurations of this collapse, for example the nature of the singularity and apparent horizons and so on Let us consider the case where $xG = 1$ has two positive real roots, as this matches the self-similar Vaidya and Lemaître-Tolman-Bondi dust models considered in this thesis (see next sections) We will denote the first root $x = x_c$ and the second $x = x_e$ Consider an o r n g passing through a point which has x-coordinate $x = x_d$ where $x_c < x_d < x_e$ As we follow this curve into the past, the uniqueness of solutions to (2 3 1) prevents this curve from crossing $x = x_c$ or $x = x_e$, and thus each o r n g in this region must meet the singularity in the past Therefore there is a family of outgoing radial null geodesics through the point $v = 0, r = 0$, and thus the singularity is null The spacetime diagram for this 4-S collapse scenario is given in Figure 2 1 We see from (2 3 2) that the apparent horizon, given by $F = 0 = G$, is spacelike

The possible end-states of collapse have been classified by Nolan [36] and Carr and Gundlach [5] Figure 2 1 corresponds to there being two distinct real roots to the equation $xG = 1$ As the roots approach one another, the first and second similarity horizons draw closer A double root means the first and second horizons coincide, resulting in an instantaneously or marginally naked singularity Aside from briefly in Chapter 3, we will not consider marginally naked singularities in this thesis

The other most likely outcome of collapse is for a singularity to form which is spacelike and allows no null geodesic to escape This black hole case arises when there are no real solutions to $xG = 1$, and is shown in Figure 2 2

2 4 Special cases

In Chapters 3 and 7 we will test the stability of the naked singularities arising from 4-S collapse by coupling with a scalar field and by perturbing using an *odd* parity perturbation In the second case it is possible to perturb only the metric tensor and not the matter tensor, thus in both these cases it is not necessary to specify the matter content explicitly, merely requiring the matter field to satisfy the dominant energy condition is sufficient However, in order to perturb using an *even* parity perturbation, we must perturb both the metric and matter tensor and thus we must specify what the background matter content is

In choosing the background solution we have a number of requirements we seek a self-similar spherically symmetric spacetime, admitting naked singularities for a non-zero measure set of initial data, whose metric and matter functions may be solved for in closed form A good introductory model which contains all the essential features is the self-similar collapsing null dust (Vaidya) solution This is not the most physical solution, so we further consider the self-similar timelike dust (Lemaître-Tolman-Bondi) solution In the following sections we give the essential structure of these two solutions, and in Chapters 5 and 6 we perturb them each in turn

2 4 1 Self-similar Vaidya spacetime

This models a dust fluid following ingoing null geodesics Thus the matter tensor has no pressure/stress terms, only energy density ρ Since we are considering a null fluid the matter tensor is built from null vectors, and thus the Vaidya matter tensor is

$$T_{ab} = \rho \, \ell_a \ell_b,$$

where ℓ_a is an ingoing null vector Using the advanced Bondi coordinates described above we set $\ell_a = -\partial_a v = -\delta_a^v$ Thus we can use the self-similar metric given in (2 1 5) to calculate the Einstein tensor, and the field equations tell us the only non-zero entry in the Einstein tensor is in the vv slot The rr component of the field equations gives $\psi =$ constant which w l o g we can set equal to zero The $\theta\theta$ component returns $F'' = 0$, and thus we let $F = ax + b$ The vr component fixes $b = 1/2$, and the remaining equation is

$$8\pi\rho = -\frac{2}{r^2} F'$$

Setting $\lambda = -\frac{1}{2}a$ gives the self-similar Vaidya solution as

$$F = \tfrac{1}{2}(1 - \lambda x), \qquad \psi(x) = 0, \qquad \rho = \frac{\lambda}{8\pi r^2} \tag{2 4 1}$$

The Vaidya solution can alternatively be found by taking the Schwarzschild solution in advanced Bondi coordinates and replacing the mass term with a function of v To make the solution self-similar this must be a linear function of v, $m(v) = \lambda v$ Thus we recover the Schwarzschild solution by setting λv = constant, and we recover flat spacetime by setting $\lambda = 0$ Note the energy conditions (2 2 2) restrict $\lambda \geq 0$

For $v < 0$ we set $\lambda = 0$ and thus the interior of $v = 0$ is flat As we cross $v = 0$ we enter the matter filled region and hence the title 'threshold' for the past null cone of the origin $\mathcal{O}$ When the matter collapses to the origin $\mathcal{O}$ the density, $\rho = \lambda/8\pi r^2$, diverges Alternatively, if we consider the Kretschmann scalar given by $K = R_{abcd}R^{abcd} = \left(2x\lambda/r^2\right)^2$, we see this scalar diverges as we approach the singularity along $x =$ constant (unless $\lambda = 0$ of course) Thus there is a singularity at $\mathcal{O}$

If we consider the equation describing outgoing null geodesics, (2 3 1), we see there is a naked singularity if and only if there is a real solution to

$$\lambda x^2 - x + 2 = 0$$

The lowest root to this equation is

$$x = \frac{1}{2\lambda}(1 - \sqrt{1 - 8\lambda}) \equiv x_c, \tag{2 4 2}$$

and exists, i e is real, for $0 < \lambda < 1/8$ This similarity horizon represents the first null geodesic to leave the singularity and escape to future null infinity, and thus $x = x_c$ is the Cauchy horizon

For the self-similar Vaidya spacetime, there is one more similarity horizon,

$$x = \frac{1}{2\lambda}(1 + \sqrt{1 - 8\lambda}) \equiv x_e \tag{2 4 3}$$

For $0 < \lambda < 1/8$ these similarity horizons are distinct and the singularity is globally naked, the causal structure is as shown in Figure 2 1 For $\lambda = 1/8$ these horizons coincide and the singularity is instantaneously (marginally) naked, we will not consider this case in this thesis For $\lambda > 1/8$ a black hole forms, see Figure 2 2

The second similarity horizon is, in the purely self-similar Vaidya case, the last null geodesic to leave the singularity and escape to future null infinity, and thus can be called the event horizon However, to have an asymptotically flat model, we can match across $v = v_+ > 0$ with the exterior Schwarzschild spacetime, by setting $m(v > v_+) =$ constant, in this case $x = x_e$ would not be the event horizon Thus we will call $x = x_e$ the second future similarity horizon (SFSH)

2 4 2 Self-similar Lemaître-Tolman-Bondi (LTB) spacetime

This solution describes dust particles which move along timelike geodesics, and thus has a matter tensor of the form

$$T_{ab} = \rho\, u_a u_b,$$

where $u_a u^a = -1$ The advanced Bondi coordinates which we found so useful in the preceding sections are not suitable for this spacetime, as the fluid no longer moves along null geodesics Instead we use co-moving coordinates that is, we let coordinate t point in the direction of fluid flow (and increase into the future) such that $u^a \propto \delta^a_t$, and choose r orthogonal to t, that is $u^a \nabla_a r = 0$ This r coordinate is no longer necessarily the areal radius, which we instead denote with $R = R(t,r)$ Thus a line element for a spherically symmetric timelike dust in co-moving coordinates can be written

$$ds^2 = -e^{\nu(t,r)}dt^2 + e^{\lambda(t,r)}dr^2 + R^2(t,r)d\Omega^2$$

To normalise u^a such that $u^a u_a = -1$ we find $u^a = e^{-\nu/2}\delta^a_t$ Conservation of energy momentum requires u^a to be geodesic, that is $u^a \nabla_a u^b = 0$ The $b = r$ component of this equation gives $\partial\nu/\partial r = 0$, and thus $\nu = \nu(t)$ There is freedom in the t coordinate to set $\nu = 0$ without loss of generality, and our line element becomes

$$ds^2 = -dt^2 + e^{\lambda(t,r)}dr^2 + R^2(t,r)d\Omega^2$$

We generate the Einstein tensor from this line element and solve the field equations

The tr component of the field equations is

$$\lambda = \frac{2R'}{R'} = \frac{\partial}{\partial t}\left[2\ln R'\right],$$

where dot and prime denote differentiation w r t t and r respectively Integrating gives $e^\lambda = H(r)R'^2$, where $H(r)$ is a constant function of integration The rr component gives $H\left(1 + R^2 + 2RR\right) - 1 = 0$ Since $\left(RR^2\right) = R\left(R^2 + 2RR\right)$, we can integrate this equation to give

$$R = -\sqrt{f(r) + \frac{2\alpha(r)}{R}}, \tag{2 4 4}$$

where $\alpha(r)$ is another constant function of integration (we take the negative root for collapse), and $f(r) = 1/H - 1$ Finally the field equation $G_{tt} = 8\pi T_{tt}$ gives

$$\rho = \frac{1}{8\pi}\frac{2\alpha'}{R^2 R'} \tag{2 4 5}$$

Thus the field equations are solved up to two functions, $f(r)$ and $\alpha(r)$

We interpret these functions following [23] We recall the definition for the Misner-Sharp mass as $m = \frac{R}{2}(1 - g^{ab}\nabla_a R \nabla_b R) = \frac{R}{2}(1 + R^2 - e^{-\lambda}R'^2) = \alpha$ Alternatively, if we consider the dust cloud to be made up of infinitesimally thin spherical shells, and integrate the density times the surface area of each of these shells from 0 to R, we find the mass inside a sphere of dust particles of radius R Thus

$$m(r) = \int_0^{R(t,r)} 4\pi R^2 \rho dR = \int_0^r 4\pi\rho R^2 R' dr = \alpha(r)$$

The other function $f(r)$ is called the specific energy of the dust fluid, and for the sake of simplicity we will consider the case of marginally bound collapse, by setting $f(r) = 0$

With $f(r) = 0$ we can solve (2 4 4) as

$$R^3 = \frac{9}{2}m(r)\left[t_c(r) - t\right]^2,$$

for some function $t_c(r)$ Finally our remaining coordinate freedom allows us to choose r such that $R(t = 0, r) = r$ Using this, we find $t_c(r) = \frac{2}{3}\sqrt{r^3/2m}$, and thus once we have specified $m(r)$ (or alternatively $\rho(r)$) we have completely determined all the unknowns

From (2 4 5) we see the density diverges when $R = 0$, that is when $t = t_c(r)$ This is the curvature singularity known as the shell-focusing singularity, and we can interpret the function $t_c(r)$ then as the time of arrival of each shell of fluid to the singularity

Note there is an additional singularity known as the shell-crossing singularity when $R' = 0$ We will not consider this singularity as one may extend spacetime non-uniquely through the shell crossing singularity, see Nolan [37] To rule out the occurrence of the shell-crossing singularity we take $R' > 0$ for all $r > 0$, see Nolan and Mena [39]

Thus the line element for marginally bound timelike dust collapse is

$$ds^2 = -dt^2 + R'^2 dr^2 + R^2 d\Omega^2$$

We, however, are interested in self-similar collapse, and thus we look for a homothetic Killing vector field ξ^a which solves the equation $\nabla_a \xi_b + \nabla_b \xi_a - 2g_{ab} = 0$ If $\xi^a = \big(\alpha(t,r), \beta(t,r)\big)$, this returns four equations,

$$\alpha = 1, \qquad R'^2 \beta = \alpha', \qquad \beta R'' + \beta' R' - R' + \alpha R' = 0, \qquad \beta R' + \alpha R - R = 0$$

From the first equation we can write $\alpha = t + F_1(r)$, for arbitrary F_1 Since $\beta R'' + \beta' R' = (\beta R')'$, we may combine the third and fourth equations to give $\alpha' = 0$, and thus we may change the origin of t to set $\alpha = t$ The second equation therefore gives $\beta = F_2(r)$, and we can make a coordinate transformation to set $\beta = r$ The remaining equations are

$$t(R') + r(R')' = 0, \qquad tR + rR' - R = 0$$

The first of these equations is $\xi^a \partial_a R' = 0$ which, as we saw before, is solved if and only if R' is a function of a similarity variable, in this case $y = t/r$ Thus if we set $R = rG(y)$, where G is a function of the similarity variable, we have $\partial R/\partial r = G - y(dG/dy)$, which is solely a function of y

Thus the line element for a self-similar spherically symmetric timelike dust will be

$$ds^2 = -dt^2 + (G - yG')^2 dr^2 + r^2 G^2 d\Omega^2, \tag{2 4 6}$$

where here a prime denotes differentiation w r t argument We may use this metric now to generate the Einstein tensor and examine the field equations, still using the co-moving coordinates The rr component of the field equations is $G'^2 + 2GG'' = 0$ Integrating yields $GG'^2 = p^2$, where p is some constant The tt component then gives

$$\rho = \frac{1}{8\pi} \frac{GG'^2}{r^2 G^2 (G - yG')},$$

which is why we chose $GG'^2 = p^2 \geq 0$ Finally integrating this equation and using $R|_{t=0} = r$ we can solve for G as

$$G(y) = (1 - \mu y)^{2/3}, \tag{2 4 7}$$

where $\mu = -\frac{3}{2}p$ We note that flat spacetime is recovered by setting $\mu = 0$

There is a shell-focussing singularity therefore at $y = \mu^{-1}$ Since

$$\partial R/\partial r = (1-\mu y)^{2/3}\left(1+\tfrac{2}{3}\mu y(1-\mu y)^{-1}\right),$$

we see that prior to the formation of the shell focussing-singularity, $y < \mu^{-1} \Rightarrow 1-\mu y > 0$, thus $R' > 0$ This rules out the formation of shell-crossing singularities prior to the formation of shell-focussing singularities

The last issue is to examine the causal structure of the spacetime Radial null geodesics satisfy

$$\frac{dt}{dr} = \pm R',$$

with the plus and the minus describing ingoing and outgoing null geodesics respectively Since $t = yr$ this equation may be rewritten as

$$\frac{dy}{dr} = \frac{1}{r}\left(\pm R' - y\right)$$

If there is some $y =$ constant which is a root of the right hand side of this equation, it represents a null geodesic which reaches the singularity in the future/past Thus the Cauchy horizon, $y = y_c$, is given by the first real zero of

$$G - yG' - y = 0, \tag{2 4 8}$$

if one exists, and the past null cone of the origin, $y = y_p$, is given by the root of

$$G - yG' + y = 0 \tag{2 4 9}$$

Since $G = (1-\mu y)^{2/3}$, we find there is a Cauchy horizon, and therefore a naked singularity, if μ is in the range

$$0 < \mu \leq \mu_*, \qquad \mu_* \approx 0\ 638014$$

Moreover, when μ is in this range, there is one past null cone of the origin y_p, there is an additional future similarity horizon at $y = y_e > y_c$, as $\mu \to \mu_*$, $y_e \to y_c$, and as $\mu \to 0$, $y_p \to -1, y_c \to 1$ and $y_e \to \infty$

Thus when $0 < \mu < \mu_*$, we again have a spacetime with the structure given in Figure

2.1 The scaling origin at which the singularity initially forms is the point $(t,r) = (0,0)$. The apparent horizon forms when $g^{ab}\nabla_a R \nabla_b R = 0$ which is equivalent to $dG/dy = 1$. This occurs at $y = \frac{1}{\mu}\left(1 + \left(\frac{2\mu}{3}\right)^3\right)$, that is, *after* the formation of the shell-focusing singularity at $y = \frac{1}{\mu}$.

Chapter 3

Scalar field propagating on 4-S spacetime

As discussed in the previous section, there is a class of self-similar spherically symmetric spacetimes which collapse to form a naked singularity when the metric functions satisfy certain conditions These solutions represent possible counter examples to the Cosmic Censorship hypothesis as discussed in Chapter 1 To better understand how much of a threat to the CCH these solutions are, we must examine their stability In this chapter we minimally couple a massless scalar field to a self-similar spherically symmetric spacetime whose matter tensor is undefined, save for satisfying the dominant energy condition Further we consider spacetimes which develop a naked singularity at some time, and which contain no trapped surfaces prior to $\mathcal{N}$, as set out in the previous chapter

We will use the (v, r) coordinates, and thus the line element is

$$ds^2 = -2Fe^{2\psi}dv^2 + 2e^{\psi}dv\,dr + r^2 d\Omega^2,$$

with $F = F(x)$ and $\psi = \psi(x)$ where $x = v/r$, and the metric functions must satisfy

$$\begin{aligned} x\psi' &\leq 0, \\ e^{\psi}\left(F' + xF^2e^{\psi}\psi'\right) &\leq 0, \\ 1 - 2F + 2x\left(F' + F\psi'\right) &\geq 0 \end{aligned}$$

A massless scalar field Φ must satisfy the massless Klein-Gordon (wave) equation,

$$\Box\Phi = \Phi_{,a}{}^{;a} = (-g)^{-\frac{1}{2}}\partial_a\left[(-g)^{\frac{1}{2}}g^{ab}\partial_b\Phi\right] = 0$$

We will measure the flux of this scalar field, $\mathcal{F} = u^a \nabla_a \Phi$, where u^a is tangent to a radial ($u^\theta = u^\phi = 0$) timelike observer Our initial regularity condition is that $\mathcal{F}$ must be finite on $\mathcal{N}$, and we allow this finite field to evolve then up to the Cauchy horizon and beyond

The flux of a scalar field in (v, r) coordinates is

$$\mathcal{F} = u^v \Phi_{,v} + u^r \Phi_{,r},$$

where $u_a u^a = -1$ Thus to measure the flux on relevant surfaces we must understand the nature of the components of the tangent vector u^a of arbitrary radial timelike geodesics Let the coordinates of a future pointing radial timelike observer be x^a, and let the geodesic describing the observer's motion be parameterised by the proper time τ such that $x^a = x^a(\tau)$ Then the components of the tangent to this timelike geodesic will be $u^a = dx^a/d\tau$, that is $u^v = v$ and $u^r = r$, where a dot denotes differentiation w r t proper time

A detailed analysis of these components was carried out by Nolan in [40], and we summarise the main findings here in Propositions 3 1 and 3 2 First some definitions of asymptotic relations (see e g Chapter 3 of [3])

- $f(x) = O(g(x))$ as $x \to x_0$ iff $\exists$ constant c s t $|f(x)| \leq c|g(x)|$ as $x \to x_0$
- $f(x) = o(g(x))$ as $x \to x_0$ iff for any $\epsilon > 0$, $|f(x)| \leq \epsilon |g(x)|$ as $x \to x_0$ Note this is often denoted $f(x) \ll g(x)$, $x \to x_0$
- We define $\sim$ by the condition $f(x) \sim g(x)$ as $x \to x_0$ iff $f(x) - g(x) = o(g(x))$ as $x \to x_0$

Proposition 3 1 *For any future pointing radial timelike geodesic crossing $\mathcal{N}$, we have*

$$\begin{aligned} v(\tau) &= u_0 \tau + v_2 \tau^2 + O(\tau^3), \\ r(\tau) &= r_0 + r_1 \tau + O(\tau^2), \end{aligned}$$

and thus $v \sim u_0$, $r \sim r_1$ as $\tau \to 0$, where $\tau = 0$ describes the point where the geodesic crosses $\mathcal{N}$

Proposition 3 2 *Suppose that G and ψ are differentiable at $x = x_c$, the Cauchy horizon Then all radial timelike geodesics whose initial points are sufficiently close to the Cauchy horizon will cross the horizon in finite time Using coordinates (x, v), for any radial timelike geodesic crossing the horizon, the components of the tangent x and v have finite non-zero values*

3.1 Solving the scalar wave equation

Now we examine the scalar wave equation for Φ We exploit the spherical symmetry of the background spacetime and split the scalar field,

$$\Phi(v, x, \theta, \phi) = T(v, x)A(\theta, \phi),$$

where we use the advanced null coordinate v, the homothetic coordinate x, and the standard angular coordinates θ, ϕ Then the line element in these coordinates reads

$$ds^2 = 2e^{\psi}\left(\frac{1}{x} - G\right)dv^2 - \frac{2e^{\psi}v}{x^2}dvdx + \frac{v^2}{x^2}d\Omega^2,$$

where $G = Fe^{\psi}$ By using separation of variables the scalar wave equation splits into two PDE's, the first in v, x,

$$2x^2\left(\frac{1}{x} - G\right)T_{,xx} + 2vT_{,xv} - 2x^2G'T_{,x} - 2\frac{v}{x}T_{,v} - ke^{\psi}T = 0 \qquad (3\ 1\ 1)$$

and the second m θ, ϕ,

$$A_{,\theta\theta} + \cot\theta A_{,\theta} + \csc^2\theta A_{,\phi\phi} + kA = 0, \qquad (3\ 1\ 2)$$

where k is the separation constant

We will see in the next chapter that the θ, ϕ equation is solved by the spherical harmonics, $A(\theta, \phi) = Y_l^m(\theta, \phi)$, where $l = 0, 1, 2$ is the multipole mode number, and $k = l(l+1)$ with $l \in \mathbb{N}$ for periodicity reasons Thus this separation is essentially a multipole decomposition (see Chapter 4)

Since the spherical harmonics are well understood, we need only solve the v, x equation We reduce this from a PDE to an ODE by taking the Mellin transform of the equation over v (see Section 1 3 1), defined by

$$H(x, s) = \mathbb{M}[T(v, x)] = \int_0^{\infty} T(v, x)v^{-s-1}dv,$$

which amounts to replacing $T(v, x)$ with $v^s H(x, s)$, where s is an as yet unconstrained complex parameter Equation (3 1 1) thus reduces to an ODE in $H(x, s)$,

$$2x^2\left(\frac{1}{x} - G\right)H'' + (2s - 2x^2G')H' - \left(\frac{2s}{x} + l(l+1)e^{\psi}\right)H = 0 \qquad (3\ 1\ 3)$$

Performing the inverse Mellin transform on the solution of this ODE over a contour in the viable range of s will return the solution to (3 1 1), and then summing over the spherical harmonics will return the scalar field Φ

This ODE has a number of singular points, namely $x = 0$ and the roots of $xG = 1$, the lowest of which we have defined to be x_c, and the second to be x_e The canonical form of a second order linear ODE in a neighborhood of $x = x_0$ is

$$(x - x_0)^2 H'' + (x - x_0) b_1(x) H' + b_2(x) H = 0,$$

and when we write equation (3 1 3) in its canonical form in the neighborhood of $x = x_0$, we find

$$b_1(x) = \frac{s - x^2 G'}{1 - xG} \left(\frac{x - x_0}{x} \right), \qquad b_2(x) = -\frac{2s + l(l+1)e^{\psi} x}{2(1 - xG)} \left(\frac{x - x_0}{x} \right)^2$$

We will examine $x_0 = 0$, or $\mathcal{N}$, first

Past null cone

Since $b_1(x)$ and $b_2(x)$ are both C^1 in a neighborhood of $x = 0$, we can use the method of Frobenius to solve (3 1 3) on $\mathcal{N}$ * (see Section 1 3 2) The indicial exponents are $1, -s$

As it stands we cannot make any assumptions about s, however later analysis shows if $-Re(s) \geq 1$ the flux of the scalar field will be always infinite on $\mathcal{N}$, thus we only consider $-Re(s) < 1$

It is possible for 1 and $-s$ to differ by an integer and so the method of Frobenius yields the following expression for the general solution to (3 1 3) in a neighborhood of $x = 0$,

$$H(x, s) = c_1 \sum_{m=0}^{\infty} a_m x^{m+1} + c_2 \left\{ k \ln x \sum_{m=0}^{\infty} a_m x^{m+1} + \sum_{m=0}^{\infty} b_m x^{m-s} \right\} \tag{3 1 4}$$

In this expression, c_1 and c_2 are arbitrary constants, $a_0 = b_0 = 1$ with $k = 0$ if 1 and $-s$ do not differ by an integer, $a_0 = 1, b_0 = 0$ with $k = 1$ if 1 and $-s$ are equal, and $a_0 = b_0 = 1$ and k may or may not vanish if $1 + s = p$ for some positive integer p

The flux of the scalar field, $\mathcal{F} = u^a \nabla_a \Phi$, given that $\Phi = v^s H(x, s)$ (we may omit the

*To use the method of Frobenius the coefficients $b_1(x), b_2(x)$ should be analytic at $x = 0$ However, to obtain the required information about H it is sufficient to use a finite expansion with appropriate remainder terms, i e with $b_1, b_2 \in C^1$ at $x = 0$ Thus we only require the metric coefficients to be C^2 at $x = 0$, and similarly at $x = x_c$ and $x = x_e$ We assume this henceforth angular part) is

$$\begin{aligned}\mathcal{F}(v,r) &= v\partial_v\Phi + r\partial_r\Phi,\\ &= v\left[\frac{\partial\Phi}{\partial v}+\frac{1}{r}\frac{\partial\Phi}{\partial x}\right]+r\left[\frac{-v}{r^2}\frac{\partial\Phi}{\partial x}\right],\\ &= v\left[sv^{s-1}H+\frac{1}{r}v^sH'\right]+r\left[-\frac{v^{s+1}}{r^2}H'\right]\end{aligned}$$

Using our general solution near the past null cone of the origin, the expression for the flux on $\mathcal{N}$ is

$$\mathcal{F}_1(v,r) = v\sum_{m=0}^{\infty}a_m(m+s+1)\frac{v^{m+s}}{r^{m+1}} - r\sum_{m=0}^{\infty}a_m(m+1)\frac{v^{m+s+1}}{r^{m+2}}, \quad (3\,1\,5)$$

$$\begin{aligned}\mathcal{F}_2(v,r) = v&\left\{\sum_{m=0}^{\infty}b_{m+1}(m+1)\frac{v^m}{r^{m-s+1}}+k\sum_{m=0}^{\infty}\left[1+(m+s+1)\ln\left(\frac{v}{r}\right)\right]a_m\frac{v^{m+s}}{r^{m+1}}\right\}\\ -r&\left\{\sum_{m=0}^{\infty}b_m(m-s)\frac{v^m}{r^{m-s+1}}-k\sum_{m=0}^{\infty}\left[1+(m+1)\ln\left(\frac{v}{r}\right)\right]a_m\frac{v^{m+s+1}}{r^{m+2}}\right\} \quad (3\,1\,6)\end{aligned}$$

where the 1 subscript denotes the c_1 part of the general solution for H given in (3 1 4), and likewise the 2 subscript The finiteness of v, r on $\mathcal{N}$ has been given in Proposition 1

We see that for the flux to have a finite measure on $\mathcal{N}$ (away from the singularity), that is when $v = 0$ and $r =$ constant, we require

$$Re(s) > 0$$

Under this condition we let the scalar field evolve towards the Cauchy horizon, and examine its flux there (Technically, we have shown that each mode of the Mellin transform of the flux is finite on the past null cone, rather than the flux itself See §3 2)

Cauchy horizon

We seek the indicial exponents near $x = x_c$, however near this singular point b_1 and b_2 are in the form $\frac{0}{0}$ Thus we use l'Hôpital's rule to give

$$b_1(x_c) = \lim_{x\to x_c}\frac{x^2G'-s}{xG+x^2G'}, \qquad b_2(x_c) = \lim_{x\to x_c} -\frac{(x-x_c)}{xG+x_c^2G'(x_c)}\left[2s+\rho e^{\psi}x\right]$$

With $x_c G(x_c) = 1$ by definition, the denominators are $1 + x_c^2 G'(x_c)$, however it is unclear whether this term vanishes or not

Consider the function $W(x) = xG - 1$ Then $W(x_c) = 0$, and from our previous assumptions on F, ψ we have $W \in C^2(-\infty, x_c)$ If W has a *double* root at $x = x_c$, then $x = x_c$ is a local max/min, and thus $W'(x_c) = 0$ Then

$$W'(x_c) = \frac{1}{x_c}\left(1 + x_c^2 G'(x_c)\right) = 0,$$

and thus the denominators in b_1 and b_2 given above are zero On the other hand, we note that since we have assumed a regular axis,

$$\lim_{x \to -\infty} W(x) = \lim_{x \to -\infty} xFe^{\psi} - 1 < 0$$

Thus if $x = x_c$ is the *distinct* lowest root of W, then W crosses the x-axis from below and thus $W' > 0$, that is

$$1 + x_c^2 G'(x_c) > 0,$$

since $x_c > 0$ Physically we interpret the difference so a distinct lowest root describes the collapse pictured in Figure 2 1, with two distinct future similarity horizons and a globally naked singularity in between Multiple lowest roots means the first and second similarity horizons coincide, giving a marginally, or instantaneously, naked singularity

The two cases will lead to very different analyses, thus we treat them separately

(i) Unique lowest root

In this case $b_1(x)$, $b_2(x)$ are C^1 on $x = x_c$, thus x_c is a regular singular point and we can use the method of Frobenius Since $b_2(x_c) = 0$, the indicial exponents are $0, 1 - b_c$ where

$$b_c = b_1(x_c) = \frac{x_c^2 G'(x_c) - s}{x_c^2 G'(x_c) + 1}$$

We find the sign of $G'(x_c)$ in the following way

Since $\lim_{x \to -\infty} W = -\infty$, and the first zero of W is at x_c, therefore $W < 0$ for $x \in (-\infty, x_c)$ Also, since $G = Fe^{\psi}$ and $F > 0$ for $x \in (-\infty, x_c)$ to rule out trapped surfaces before the formation of the singularity, we have $0 < xG < 1$ for $x \in (0, x_c)$

Thus, since $\psi' < 0$ for $x > 0$ from energy condition (2 2 2a), we find

$$\begin{aligned} & xGF\psi' > F\psi', \\ \Rightarrow\quad & xF^2e^{\psi}\psi' > F\psi', \\ \Rightarrow\quad & F' + xF^2e^{\psi}\psi' > F' + F\psi', \\ \Rightarrow\quad & 0 \geq F' + xF^2e^{\psi}\psi' > F' + F\psi', \end{aligned}$$

where in the last line we have used the second energy condition (2 2 2b) Thus $G' = e^{\psi}(F' + F\psi') < 0$ for $x \in (0, x_c)$, and we conclude that

$$G'(x_c) \leq 0$$

Our initial regularity condition required $Re(s) > 0$ Under this condition, and using the work above, we can say that $b_c < 0$, hence $1 - b_c > 0$, which allows us to order the indicial exponents Thus a general solution near $x = x_c$ will be

$$H(x,s) = C_1 \sum_{m=0}^{\infty} A_m \zeta^{m+1-b_c} + C_2 \left\{ k \ln\zeta \sum_{m=0}^{\infty} A_m \zeta^{m+1-b_c} + \sum_{m=0}^{\infty} B_m \zeta^m \right\} \qquad (3\,1\,7)$$

where $\zeta = x - x_c$, and the coefficients have the same structure as (3 1 4) From this we calculate each component of the flux,

$$\mathcal{F}_1 = xv^s \sum_{m=0}^{\infty} (m+1-b_c) A_m \zeta^{m-b_c} + vsv^{s-1} \sum_{m=0}^{\infty} A_m \zeta^{m+1-b_c}, \qquad (3\,1\,8)$$

$$\begin{aligned} \mathcal{F}_2 = xv^s & \left[k \sum_{m=0}^{\infty} A_m \left[\ln\zeta(m+1-b_c) + 1\right] \zeta^{m-b_c} + \sum_{m=0}^{\infty} B_m m \zeta^{m-1} \right] \\ & + vsv^{s-1} \left[\sum_{m=0}^{\infty} B_m \zeta^m + k \ln\zeta \sum_{m=0}^{\infty} A_m \zeta^{m+1-b_c} \right] \qquad (3\,1\,9) \end{aligned}$$

Using the finiteness of v, x given in Proposition 2, we see that if $b_c < 0$, that is if $s > 0$, this expression is finite on the Cauchy horizon, i e when $x - x_c = \zeta = 0$

Thus in the case of $xG = 1$ having a unique lowest root, a scalar field with a finite flux on the past null cone of the origin $\mathcal{N}$, will have a finite flux on the Cauchy horizon

(ii) Multiple lowest root

If $x_c^2 G'(x_c) + 1 = 0$, x_c is an irregular singular point of (3 1 3) and the method of

Frobenius no longer applies Note that this is a special case which one would expect to correspond to a set of measure zero in the class of spacetimes under consideration, and thus is of lesser interest from the context of the Cosmic Censorship hypothesis, we give the analysis merely for completeness, and to present some of the methods used for irregular singular points

We label $\eta = x_c - x$ and examine solutions to the ODE in the asymptotic limit $\eta \downarrow 0$ We assume the solution to (3 1 3) can be written in the form

$$H(\eta) = e^{h(\eta)},$$

transforming (3 1 3) to an ODE in $h(\eta)$ Now we assume the common property near irregular singular points [3],

$$h = o(h^2), \qquad \eta \downarrow 0$$

where the overdot denotes differentiation with respect to η (3 1 3) becomes a quadratic in h,

$$h^2 \left\{(x_c - \eta) - (x_c - \eta)^2 G\right\} - \left(s + (x_c - \eta)^2 G\right) h \sim \frac{s}{x_c - \eta} + \frac{l(l+1)e^{\psi}}{2}, \quad \eta \downarrow 0 \qquad (3\ 1\ 10)$$

If we consider $xG = 1$ to have a lowest root of multiplicity k, then we can write its Taylor series around $\eta = 0$ as

$$1 - (x_c - \eta)G(\eta) = P(\eta) = \eta^k \frac{P^{(k)}(0)}{k!} + O(\eta^{k+1})$$

This means if the lowest root is of multiplicity k, we need the metric functions to be C^k This is not too much of a restriction however, since the class of functions with roots of multiplicity k becomes very small as k increases, meaning we are dealing with a very special case in this analysis

We can make the approximation

$$s + (x_c - \eta)^2 G \sim s + 1, \qquad \eta \downarrow 0,$$

and since we assume the metric coefficients are at least C^2, we can approximate e^{ψ} by the first term in its expansion, c_0, in the limit $\eta \downarrow 0$ Thus we arrive at a quadratic in h,

$$\eta^k (h)^2 - \alpha h \sim \beta, \qquad \eta \downarrow 0, \tag{3 1 11}$$

$$\alpha = \frac{k!(s+1)}{x_c P^{(k)}(0)}, \quad \beta = \frac{k!}{x_c P^{(k)}(0)} \left(\frac{s}{x_c} + \frac{l(l+1)c_0}{2} \right),$$

where $\alpha, \beta > 0$ (if $Re(s) > 0$) and constant in the limit $\eta \downarrow 0$, and $k > 1$ This quadratic has two solutions corresponding to two linearly independent solutions of (3 1 3), which are

$$\begin{aligned} h_1 &= -\frac{\alpha}{(k-1)}\eta^{1-k} + O(\eta), \\ h_2 &= -\frac{\beta}{\alpha}\eta + \frac{\beta^2}{\alpha^3}\frac{\eta^{k+1}}{(k+1)} + O(\eta^{2k+1}), \qquad \eta \downarrow 0 \end{aligned}$$

At this point we verify our earlier assumption, namely

$$h = o(h^2), \qquad \eta \downarrow 0$$

Thus we have constructed two solutions to (3 1 3),

$$H_1(\eta) = \eta^k \exp\left\{ -\frac{\alpha}{(k-1)}\eta^{1-k} + O(\eta) \right\} \tag{3 1 12}$$

$$H_2(\eta) = \exp\{O(\eta)\} \tag{3 1 13}$$

Both of these functions and their derivatives are finite in the limit $\eta \downarrow 0$, $x \to x_c$ if $Re(s) > 0$, and thus the resulting expressions for the flux of the scalar field on the Cauchy horizon are finite

We summarize thus

Proposition 3 3 *Let spacetime ($\mathcal{M}$,g) be self-similar and spherically symmetric, satisfy the dominant energy condition, and admit a Cauchy horizon $x = x_c$ Assume also that $g_{ab} \in C^2$ at $x = x_c$ Then a scalar field which has a finite flux on $\mathcal{N}$, the past null cone of $\mathcal{O}$, will also have a finite flux on the Cauchy horizon*

Second future similarity horizon

We have found that the Cauchy horizon formed in the collapse of self-similar spherically symmetric spacetimes is stable with respect to an infalling scalar field But what comes after the Cauchy horizon? When there is a unique lowest root to $xG = 1$, the Cauchy horizon is followed by a distinct second future similarity horizon (SFSH), denoted $x = x_e$, which is the next root to $xG = 1$

Since the scalar field evolved through the Cauchy horizon without divergence, we will assume $W \in C^2(-\infty, x_e)$, where $W = xG - 1$ We will consider only the case where $x = x_e$ is a distinct root of W, and thus we have $W(x_e) = 0$, $W > 0$ for $x \in (x_c, x_e)$, and $W'(x_e) < 0$ In the same manner as at the Cauchy horizon, the method of Frobenius gives indicial exponents near $x = x_e$ as 0 and $1 - b_e$, where

$$b_e = b_1(x_e) = \frac{x_e^2 G'(x_e) - s}{x_e^2 G'(x_e) + 1}$$

Since $W'(x_e) < 0$ and $x_e > 0$, we have $x_e^2 G'(x_e) + 1 < 0$ Also, writing $G = \frac{1}{x}(W + 1)$, we have

$$G'(x_e) = \frac{1}{x_e} W'(x_e) - \frac{1}{x_e^2}\Big[W(x_e) + 1\Big],$$

and since $W(x_e) = 0$ and $W'(x_e) < 0$, this means $G'(x_e) < 0$ Finally since we are only considering $Re(s) > 0$, we have $b_e > 0$

Our expression for the flux of a scalar field near the SFSH will be exactly as in (3 1 8), (3 1 9), only with x_c replaced everywhere with x_e Consider the first term in $\mathcal{F}_1$,

$$x v^s \sum_{m=0}^{\infty} (m + 1 - b_e) \mathcal{A}_m (x - x_e)^{m - b_e}$$

This term will diverge on $x = x_e$ (unless $b_e = 1$, but this would require $s = -1$)

The final picture is this a scalar field with a finite flux on the past null cone $\mathcal{N}$ will evolve onto and through the Cauchy horizon without divergence However, when the scalar field reaches the second future similarity horizon, its flux will diverge There will be a distinct SFSH in the most general scenario of self-similar spherically symmetric collapse, which is of most interest from the point of view of Cosmic Censorship

In brief the naked singularity persists after perturbation by a scalar field, but only for a finite time We will see that this is a general feature of perturbations of naked singularities formed in 4-S spacetimes

3 2 The stabilıty of modes

In thıs sectıon we dıscuss some of the finer poınts of the ıssue of performıng the ınverse Mellın transform The comments of thıs sectıon are relevant for subsequent Chapters

We have derıved an ınvarıant scalar whıch we wısh to examıne on certaın surfaces ın spacetıme In thıs case, the scalar ıs the flux, $\mathcal{F}$ We have found the condıtıons under whıch the basıs of solutıons for the Mellın transform of $\mathcal{F}$, whıch we wıll denote $\mathbb{F}_1$ and $\mathbb{F}_2$, are finıte on the past null cone of the orıgın, however thıs ıs not quıte enough to guarantee that we may use the Mellın ınversıon theorem to recover the flux $\mathcal{F}$ The full solutıon for the Mellın transform of $\mathcal{F}$ on $v = 0$ ıs a lınear combınatıon of the form

$$\mathbb{F}|_{pnc} = c_1(s)\,\mathbb{F}_1^p(v,r,s) + c_2(s)\,\mathbb{F}_2^p(v,r,s),$$

where p denotes the solutıons on the past null cone of the orıgın Importantly, the 'constants' c_1 and c_2 may depend on s To perform the ınverse Mellm transform of $\mathbb{F}$ we would need to know somethıng about these coefficıents The ıssue becomes more complıcated on the Cauchy horızon, as each solutıon on the past null cone ıs 'scattered' to the solutıons on the Cauchy horızon,

$$\mathbb{F}_1^p = d_1(s)\,\mathbb{F}_1^c + d_2(s)\,\mathbb{F}_2^c, \qquad \mathbb{F}_2^p = d_3(s)\,\mathbb{F}_1^c + d_4(s)\,\mathbb{F}_2^c,$$

where c denotes a solutıon near the Cauchy horızon Now to perform the ınverse Mellın transform of $\mathbb{F}$ we would need to know ınformatıon about the scatterıng coefficıents $d_\imath(s)$

Thıs scatterıng problem ıs technıcally extremely dıfficult to solve, and ıs beyond the scope of thıs thesıs However, whıle a finıte $\mathbb{F}_1, \mathbb{F}_2$ may not be a *sufficıent* condıtıon to guarantee the ınverse Mellın transform exısts, ıt ıs clear that ıt ıs an absolutely *necessary* condıtıon, as ıf $\mathbb{F}_1$ or $\mathbb{F}_2$ were to dıverge at some poınt then there would be no hope of the ınverse Mellın transform exıstıng at that poınt

We wıll adopt the followıng as our **mınımum stabılıty requırement** that for the ınverse Mellın transform to exıst we must at least have each component of the basıs of solutıons for the Mellm transformed quantıtıes finıte on the surface ın questıon Thıs ıs equıvalent to assertıng that each ındıvıdual mode remaıns finıte Indeed every solutıon of the ODE corresponds to a general solutıon of the PDE obeyıng the ansatz $T(v,x) = v^s H(x,s)$ What we have shown ıs that every mode whıch ıs finıte on the past null cone ıs also finıte on the Cauchy horızon

From (3 1 5),(3 1 6), we see that thıs mınımum stabılıty requırement ıs satısfied on the past null cone for $Re(s) > 0$, and that under this condition the minimum stability requirement is *automatically* satisfied on the Cauchy horizon While this is not conclusive proof that the flux of the scalar field, as recovered from the inverse Mellin transform, is finite on the Cauchy horizon, it is a strong indication that the flux of the scalar field does not diverge there On the second future similarity horizon however, we see that even this minimum stability requirement is not satisfied, and thus the flux certainly diverges there

For the rest of this thesis we will use this minimum stability requirement, and examine the finiteness of individual modes rather than attempt to perform the inverse Mellin transform While Chapters 5 and 6 use the minimum stability requirement, it is only when we reach Chapter 7 that this becomes an issue and needs more discussion

Chapter 4

Formalism describing gauge invariant perturbations of spherically symmetric spacetimes

4.1 Spherical harmonics - Multipole decompositions

While spherical symmetry is a good approximation for stars, planets, moons and so on, the fact is that stars are *not* spheres. The huge energies inside the star, and the fact that stars are not isolated in an otherwise empty universe, will cause distortions of the star's surface. These distortions are in fact crucial for gravitational radiation. Birkhoff's theorem says that the vacuum outside a spherically symmetric mass will be static, that is, no radiation can be present. But on an even more basic level, as soon as a star begins to rotate it will stop being spherical and will bulge at the equator, a deformation called the quadrupole.

So while stars are not spherical, they are close to spherical and thus we may consider them as a sphere plus 'bulges'. These 'bulges' may be quantified by the spherical harmonics, an infinite series of functions defined over the sphere which contain all possible deformations, to describe the shape of a particular star one simply chooses the relevant spherical harmonics to include.

The quickest way to get at the spherical harmonics, indeed their definition, is as the angular part to the solution of Laplace's equation in spherical coordinates,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0,$$

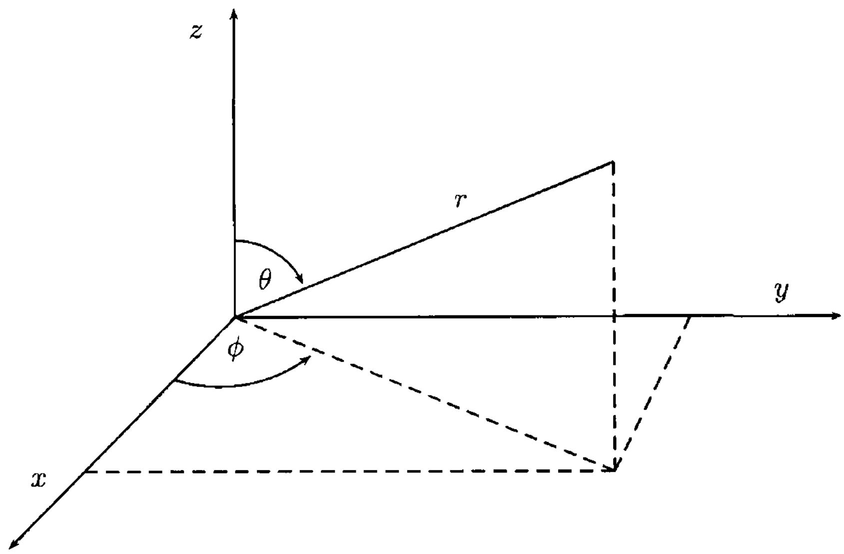


Figure 4 1 The convention used for the spherical coordinates

where $r \in [0, \infty)$ is the radius, $\theta \in [0, \pi]$ is the colatitude measured from the z-axis, and $\phi \in [0, 2\pi)$ is the azimuth, see Figure 4 1

Solutions are found from separation of variables, $V(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$, giving three equations,

$$\begin{aligned} r^2R'' + 2rR' - l(l+1)R &= 0, \\ \Phi'' + m^2\Phi &= 0, \\ \frac{1}{\sin\theta}\left(\sin\theta\Theta'\right)' + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta &= 0, \end{aligned}$$

where a prime denotes differentiation w r t argument, and l, m are separation parameters The first equation is simply solved*, as is the second, and after a change of dependent variable $\cos\theta = x \in (1, -1)$ we recognise the third equation as the associated Legendre equation of degree l and order m The solutions to these two second equations will be bounded w r t θ (i e at the poles) and periodic w r t ϕ (i e $\Phi(0) = \Phi(2\pi)$) if l, m are integers, $l \geq 0$ and $-l \leq m \leq l$, and are called the (surface) spherical harmonics We will

*Not so in cylindrical coordinates the radial equation is Bessel's equation, which is why Bessel's functions are sometimes called cylinder (harmonic) functions define the normalised spherical harmonic as [49]

$$Y_l^m = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l-m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}, \tag{4.11}$$

where $P(x)$ is the solution to Legendre's equation. The spherical harmonics are orthonormal,

$$\int_0^\pi \int_0^{2\pi} Y_l^{m*} Y_{l'}^{m'} \sin\theta d\theta d\phi = \delta_{l\,l'} \delta^{m\,m'},$$

where $*$ represents complex conjugation, and form a complete set, that is every function, whether it is a solution of Laplace's equation or not, continuous over a sphere may be decomposed into an infinite series of spherical harmonics over the mode numbers l, m,

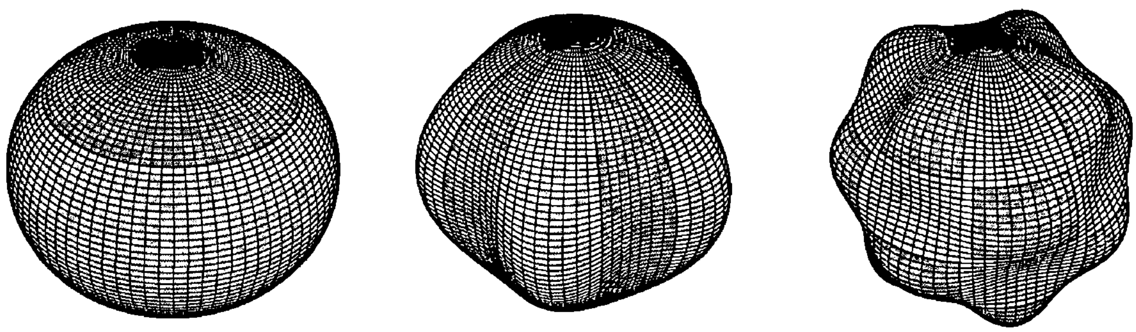
$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l^m Y_l^m(\theta, \phi)$$

Picking out individual modes will specify which sort of deformation from the sphere you want. When $m = 0$, the ϕ dependence in the spherical harmonic drops out, and thus these modes represent axisymmetric deformations. The regions over the sphere where the spherical harmonic has the same sign divide the sphere into bands of latitude, or zones, and thus the $m = 0$ modes are called zonal harmonics. See, for example, Fig 4.2 (a). This is the $l = 2, m = 0$ mode. In this case $P_2^0 = \frac{1}{2}(3\cos^2\theta - 1)$, and thus the spherical harmonic has one sign near the poles and another near the equator. We can use this mode to describe how oblate/prolate the deformed sphere is.

For $|m| = l$ modes, we find P_m^m goes like $\sin^m\theta$, which does not change sign as θ varies. Thus these modes only change sign as ϕ varies, and the regions over the sphere with the same sign are divided into wedges defined by lines of longitude, and are called sectoral harmonics. The other modes divide the sphere into tiles or tesserae and thus are called tesseral harmonics (see Fig. 4.2).

These spherical harmonics have a very practical application in the multipole expansion. Imagine a haphazard collection of matter particles. For an observer outside of this cloud of particles, the gravitational potential in Newtonian gravity will solve Laplace's equation, and thus we can write the potential, V, as

$$V(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_l r^l + B_l r^{-l-1}\right) C_{lm} Y_l^m(\theta, \phi),$$



(a) (b) (c)

Figure 4 2 Spherical harmonics plotted on the sphere as $r = r_0 + Re(Y_l^m)$ (a) The $l = 2$, $m = 0$ zonal harmonic This describes a bulging at the equator and is typical of rotating bodies (b) The $l = m = 5$ sectoral harmonic (c) The $l = 7$, $m = 4$ tesseral harmonic

with A, B, C constants Let us consider the gravitational potential of a star Requiring the potential to vanish as $r \to \infty$ sets $A_l = 0$ Renaming some constants gives

$$V(r, \theta, \phi) = -\frac{GM}{r} \left\{ B_0 - \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left(\frac{R}{r}\right)^l C_{lm} Y_l^m \right\},$$

where R is the equatorial radius of the star For large $r > R$ the first term dominates, therefore to leading order in $\frac{1}{r}$ we could treat the system of masses as a point source, and thus this lowest mode is called the monopole Since minus the gradient of the potential gives the force, minus the gradient of the first term returns the familiar inverse square law of force for point sources As the potential of a point source extends in all directions equally we see this is the spherically symmetric mode

Including further terms will describe more accurately the distribution of matter in the source Consider a source with azimuthal symmetry (axisymmetry), then the gravitational potential is given by

$$V(r, \theta) = -\frac{GM}{r} \left\{ 1 - \sum_{l=1}^{\infty} J_l \left(\frac{R}{r}\right)^l P_l(\cos\theta) \right\},$$

where the numbers J_l are the gravitational moments The first moment, $l = 1$, is called the dipole, and can be set to zero by choosing the center of mass as the origin of coordinates The $l = 2$ mode is called the quadrupole and describes the amount of bulge at the equator

All rotating objects will bulge at their equator, for example [44] the sun has $J_2 \approx 2 \times 10^{-7}$ whereas the earth has a larger quadrupole†, with $J_2 \approx 1 \times 10^{-3}$. The next term quantifies how much north/south asymmetry there is. For the earth, $J_3 \approx -2 \times 10^{-6}$, since there are more land masses in the earth's northern hemisphere, and so on.

Multipole expansions can also be used to describe the electrostatic potential, and this is where the names for the modes come from. The $l = 1$ mode describes two oppositely charged particles and thus is called the dipole. The $l = 2$ mode describes two pairs of oppositely charged particles, and thus is called the quadrupole. The next is the octupole, and so on. In fact, all physical potentials satisfying a $1/d$ law can be expressed using a multipole expansion,

$$V(x) = \int \frac{\rho(x')}{|x - x'|} d^3x',$$

where ρ is the charge or mass density, for example [29].

Finally some important considerations from the point of view of radiation in general relativity. Fields are classified from the form of the potential generating the field, thus Newtonian theory is a scalar field as its potential is the scalar described above, electromagnetism is a vector field as its potential is a vector, and general relativity treats gravity as a tensor field as the metric tensor can be thought of as the potential.

Every field has an integer spin number associated with it, a scalar field has spin 0, a vector field spin 1 and a tensor field spin 2. An important theorem [34] says that a radiation field of spin S will manifest at modes $l = S$ and above. A scalar field therefore may be spherically symmetric, such as the monopole of the Newtonian potential discussed above. A vector field, such as electromagnetism, can not be spherically symmetric, there must be two poles, north and south. Thus electromagnetic radiation begins at the dipole. Gravitational radiation in general relativity begins at the quadrupole, that is in vacuum the monopole and dipole terms do not evolve with time. Birkhoff's theorem guarantees spherically symmetric spacetimes, which contain only monopole terms, are static. Conservation of momentum can be used to rule out the dipole term in vacuum [34], however we will explicitly show in §5.1.2 and §5.2.2 that a choice of coordinates can set the dipole to zero.

† Interestingly, the earth's quadrupole is in fact decreasing, for the following reason: during the ice-age the weight of ice around the poles caused the earth to bulge more at the equator, with the receding of the ice-sheets the earth is returning to a more spherical shape. This is called glacial rebound.

To visualise this consider the first three zonal harmonics projected onto a plane containing the r-axis The $l = 0$ mode is a circle, and thus any rotation about the origin will preserve symmetry spin 0 The $l = 1$ mode bulges in the top half, and thus a 360° rotation is needed to preserve symmetry spin 1 Lastly the $l = 2$ mode bulges at the equator but is still north/south symmetric, and thus a 180° rotation will preserve symmetry spin 2

The reason we mention this is that perturbations in the metric and matter tensor can be used to model gravitational radiation, the physical scenario we examine is a spacetime which is acted on by gravitational waves

For completeness we describe how to calculate directly the mass quadrupole for a nearly-Newtonian system of masses Analogous to electromagnetism, we define the reduced (traceless) quadrupole moment tensor as [34]

$$I_{ij} = \int \rho \left(x_i x_j - \tfrac{1}{3}\delta_{ij} r^2\right) d^3x,$$

where ρ is the density, indices i, j run over the three spatial coordinates, and the x_i are the components of the position vector r for each mass in the system The energy radiated from this system is proportional to the third time derivative of the reduced quadrupole moment Note that when this tensor is reduced to principle axes (i e diagonalized), the tensor given above has one independent component, and this is sometimes called the quadrupole moment [31]

4.2 Gauge invariant perturbations

The basic unit of GR from which everything else is built is the metric tensor, which can be considered the potential for the Riemann tensor, representing the gravitational field We seek to perturb this potential Since the unperturbed quantity (the background) is spherically symmetric, the perturbed metric will be close to spherical and thus the spherical harmonics are the obvious basis to describe the perturbation

We perform a multipole decomposition of the perturbation, splitting it into an infinite series of modes the monopole, dipole, quadrupole and so on, just as in the previous section Importantly however, the perturbation is a tensor and thus we must construct scalar, vector and tensor bases over which to decompose the perturbation A formalism for such a process has been given by Gerlach and Sengupta [13, 14], and we will review that formalism in the next sections First we will give a useful description of the background quantities

We perform a 2+2 split of spacetime into a manifold spanned by $x^A = (t, r)$ coordinates denoted $(\mathcal{M}^2, g_{AB})$, crossed with unit two spheres spanned by $x^a = (\theta, \phi)$ coordinates and denoted $(\mathcal{S}^2, \gamma_{ab})$ A spherically symmetric spacetime will therefore have a metric and matter tensor given by

$$
\begin{aligned}
g_{\mu\nu} dx^\mu dx^\nu &= g_{AB}(x^C) dx^A dx^B + r^2(x^C) \gamma_{ab} dx^a dx^b, \\
t_{\mu\nu} dx^\mu dx^\nu &= t_{AB}(x^C) dx^A dx^B + \tfrac{1}{2} t^c{}_c r^2(x^C) \gamma_{ab} dx^a dx^b
\end{aligned}
$$

For the remainder of this thesis, capital Latin indices will denote coordinates on $\mathcal{M}^2$, lowercase Latin indices will denote coordinates on $\mathcal{S}^2$, and Greek indices the 4-dimensional spacetime (i e $x^\mu = (x^A, x^a)$) r is a function on $\mathcal{M}^2$ and gives the areal radius (see §2 1) Covariant derivatives on $\mathcal{M}, \mathcal{M}^2$ and $\mathcal{S}^2$ are respectively denoted

$$
g_{\mu\nu;\lambda} = 0, \quad g_{AB|C} = 0, \quad g_{ab:c} = 0,
$$

and a comma defines a partial derivative

The field equations (1 1 4) can be separated then into an equation on $\mathcal{M}^2$,

$$
G_{AB} = -2(v_{A|B} + v_A v_B) + (2 v_C{}^{|C} + 3 v_C v^C - r^{-2}) g_{AB} = 8\pi t_{AB},
$$

and an equation on $\mathcal{S}^2$

$$
G^b{}_b = 2(v_C{}^{|C} + v_C v^C - \mathcal{R}) = 8\pi t^b{}_b,
$$

where $v_A = r_{,A}/r$ $\mathcal{R}$ is the Gaussian curvature of $\mathcal{M}^2$, the manifold spanned by the time and radial coordinates, and thus equals half the Ricci scalar of $\mathcal{M}^2$

4 2 1 Angular decomposition

We write a non-spherical metric perturbation

$$
g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}(t, r, \theta, \phi),
$$

where from now on an over-tilde denotes background quantities, similarly for the matter perturbation

$$
t_{\mu\nu} = \tilde{t}_{\mu\nu} + \Delta t_{\mu\nu}(t, r, \theta, \phi)
$$

If we consider the perturbation to be small, we can extract the linear part of the field equations for the perturbed spacetime

$$(h_{\mu\nu,\alpha}{}^{,\alpha} - h_{\mu\alpha,\nu}{}^{,\alpha} - h_{\alpha\nu,\mu}{}^{,\alpha} + h_{\alpha}{}^{\alpha}{}_{,\mu,\nu}) + h_{\mu\nu}\tilde{R} + \tilde{g}_{\mu\nu}(h_{\alpha\beta}{}^{,\alpha,\beta} - h_{\alpha}{}^{\beta}{}_{,\beta}{}^{,\beta} - h_{\alpha\beta}\tilde{R}^{\alpha\beta}) = -16\pi\Delta t_{\mu\nu} \quad (4\,2\,1)$$

The spherical harmonics form a basis for functions, and from the spherical harmonics we can construct bases for vectors,

$$\left\{ Y_{,a} \quad , \quad S_a \equiv \epsilon_a{}^b Y_{,b} \right\} \quad (4\,2\,2)$$

and tensors,

$$\left\{ Y\gamma_{ab} \quad , \quad Z_{ab} \equiv Y_{,a\,b} + \frac{1}{2}l(l+1)Y\gamma_{ab} \quad , \quad S_{(a\,b)} \right\} \quad (4\,2\,3)$$

where we have suppressed the mode numbers l, m, $X_{(ab)} = \frac{1}{2}(X_{ab}+X_{ba})$ is the symmetric part of a tensor, and ϵ_{ab} is the anti-symmetric pseudo-tensor with respect to $\mathcal{S}^2$ such that $\epsilon_{ab\,c} = 0$ Using these, we decompose the perturbation in terms of scalar, vector and tensor objects defined on $\mathcal{M}^2$, times scalar, vector and tensor bases defined on $\mathcal{S}^2$

When we compute the linearized Einstein equations for a perturbed spacetime decomposed in this way, we find they naturally decouple into two sectors, even and odd Gundlach and Martin-Garcia's [20] definition is the most straightforward ‡ that sector whose bases are in even powers of $\epsilon_a{}^b$ is called even (or polar or spheroidal), that sector whose bases are in odd powers of $\epsilon_a{}^b$ is called odd (or axial or toroidal) We will denote with e, o even and odd parity objects respectively where there may be confusion As these two sectors naturally decouple we may consider them separately *a priori* In subsequent sections there is no need for the e, o markers as the parity will be clear

We write the even metric and matter perturbation as

$$h^{\mathsf{e}}_{\mu\nu} = \begin{pmatrix} h_{AB}Y & h^{\mathsf{e}}_A Y_{,a} \\ Symm & r^2(KY\gamma_{ab} + GZ_{ab}) \end{pmatrix}, \quad \Delta t^{\mathsf{e}}_{\mu\nu} = \begin{pmatrix} \Delta t_{AB}Y & \Delta t^{\mathsf{e}}_A Y_{,a} \\ Symm & r^2\Delta t^1 Y\gamma_{ab} + \Delta t^2 Z_{ab} \end{pmatrix},$$

‡The standard definition is found from spatial inversion $\vec{x} \to -\vec{x}$ objects which transform with parity $(-1)^{l+1}$ are odd, those with parity $(-1)^l$ are even

and the odd metric and matter perturbation as

$$h^{\circ}_{\mu\nu} = \begin{pmatrix} 0 & h^{\circ}_{A} S_a \\ Symm & h\, S_{(a\,b)} \end{pmatrix}, \qquad \Delta t^{\circ}_{\mu\nu} = \begin{pmatrix} 0 & \Delta t^{\circ}_{A} S_a \\ Symm & \Delta t\, S_{(a\,b)} \end{pmatrix}$$

Note that we will use r and v^A, rather than the more rigorously appropriate $\tilde{r}, \tilde{v}^A$, to denote those defined on the background, for ease of notation

Thus the even parity perturbation is defined by two symmetric two-tensors, two two-vectors and four scalars,

$$\left\{ h_{AB}, \quad \Delta t_{AB}, \quad h^{e}_{A}, \quad \Delta t^{e}_{A}, \quad G, \quad K, \quad \Delta t^1, \quad \Delta t^2 \right\} \tag{4 2 4}$$

whereas the odd parity perturbation is defined by two two-vectors and two scalars,

$$\left\{ h^{\circ}_{A}, \quad \Delta t^{\circ}_{A}, \quad h, \quad \Delta t \right\} \tag{4 2 5}$$

We will call these the "bare" perturbation objects

When we write out the field equations (4 2 1) for this metric and matter tensor, we identify the left and right hand side coefficients of the scalar, vector and tensor spherical harmonic bases given in (4 2 2),(4 2 3), and these are our evolution equations for the perturbation

The great simplification is that these equations are in terms of two-dimensional objects (4 2 4),(4 2 5), and their derivatives, defined on the (background) manifold $\mathcal{M}^2$, and in particular since we are working to linear order, all the connections used to calculate derivatives, e g $h_{A|B}$, are those defined on the background This makes actually calculating the perturbation equations much easier

It is important to note that the bases given above in (4 2 2),(4 2 3) are not always linearly independent For $l = 0$ all of the basis functions vanish except for $Y\gamma_{ab}$, and for $l = 1$, Z_{ab} and $S_{(a\,b)}$ vanish This is clear for $l = 0$, the monopole, since then the spherical harmonic is a constant,

$$Y_0^0 = \frac{1}{2\sqrt{\pi}}$$

For the dipole mode $l = 1$, this involves a little more calculation so we will just give an example the three spherical harmonics for $l = 1$ are

$$Y_0^{-1} = e^{-i\phi}\sqrt{\frac{3}{8\pi}}\sin\theta, \quad Y_0^0 = \sqrt{\frac{3}{8\pi}}\sin\theta, \quad Y_0^1 = -e^{i\phi}\sqrt{\frac{3}{8\pi}}\sin\theta$$

The tensor basis Z_{ab} is

$$Z_{ab} = Y_{,a\,b} + Y\gamma_{ab} = Y_{,ab} - \Gamma^c{}_{ab}Y_{,c} + Y\gamma_{ab},$$

the components of which in (θ, ϕ) coordinates are

$$Y_{,\theta\theta} + Y, \qquad Y_{,\phi\phi} + \cos\theta\,\sin\theta Y_{,\theta} + \sin^2\theta\, Y, \qquad Y_{,\theta\phi} - \cot\theta\, Y_{,\phi},$$

each of which vanish for $l = 1$

Thus some equations do not exist in these cases, and we must consider $l = 0$ and $l = 1$ separately from the more general $l \geq 2$ case This is not much of a problem when the background is vacuum since, as already noted at the end of Section 4 1, gravitational radiation (which is what these perturbations model) manifests at the quadrupole mode and above (we will explicitly show this in Sections 5 1 2, 5 1 3 and 5 2 2) In matter filled backgrounds however this is not necessarily the case, and thus requires further analysis

We delay giving the perturbation evolution equations since the perturbation variables, as we have defined them thus far, are not gauge invariant

4 2 2 Gauge invariance

Two spacetimes are identical if they only differ by a diffeomorphism [48] (we take the passive view of a diffeomorphism as a coordinate transformation) There is a danger that if you add a "perturbation"

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}, \tag{4 2 6}$$

you are in fact still looking at the same spacetime after undergoing a coordinate transformation, rather than after being perturbed in a physically meaningful way To escape this problem we must only interest ourselves in those objects which do not change under an infinitesimal coordinate (gauge) transformation These are called gauge invariants, and are the true measure of a physically meaningful perturbation

More precisely they are identification gauge invariant (see Stewart and Walker [46]) The reason we use the term *identification* gauge invariant is that general relativity has a fundamental ambiguity in how you identify points in different spacetimes Consider the five-dimensional manifold $\mathcal{M}_\varepsilon$ containing the unperturbed spacetime, $(\mathcal{M}_0, \tilde{g}_{\mu\nu})$, and the perturbed spacetime, $(\mathcal{M}_1, \tilde{g}_{\mu\nu} + h_{\mu\nu})$ Thus ε parameterizes a family of perturbed spacetimes, $(\mathcal{M}_\varepsilon, \tilde{g}_{\mu\nu} + \varepsilon h_{\mu\nu})$, with $\varepsilon = 0$ denoting the background spacetime and $\varepsilon = 1$

denoting the perturbed spacetime we are considering

We define the identification mapping as a vector field $\vec{u}$, defined on $\mathcal{M}_\varepsilon$, such that two points $p_0 \in \mathcal{M}_0$ and $p_1 \in \mathcal{M}_1$ are the same if they lie on the same integral curve of $\vec{u}$ To linear order, the metric tensor on $\mathcal{M}_1$ is then the Lie derivative (see §2 1) of the metric tensor on $\mathcal{M}_0$, in the direction of $\vec{u}$, evaluated on $\mathcal{M}_0$ That is,

$$\tilde{g}_{\mu\nu} + h_{\mu\nu} = \mathcal{L}_u \, \tilde{g}_{\mu\nu}|_{\varepsilon=0}$$

Another vector field, $\vec{v}$, would define another identification mapping, this time of points in $(\mathcal{M}_0, \tilde{g}_{\mu\nu})$ to points in $(\mathcal{M}_1, \tilde{g}_{\mu\nu} + \overline{h}_{\mu\nu})$, again by the Lie derivative, but now in the direction of $\vec{v}$ The difference between these two identifications is

$$\left(\tilde{g}_{\mu\nu} + h_{\mu\nu}\right) - \left(\tilde{g}_{\mu\nu} + \overline{h}_{\mu\nu}\right) = \left(\mathcal{L}_u \tilde{g}_{\mu\nu} - \mathcal{L}_v \tilde{g}_{\mu\nu}\right)\Big|_{\varepsilon=0}$$

If we define the vector field $\xi^\mu = (\vec{u} - \vec{v})|_{\varepsilon=0}$, then we can say that the gauge change induced on the metric perturbation by any vector field ξ^μ is

$$h_{\mu\nu} \rightarrow \overline{h}_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_\xi \tilde{g}_{\mu\nu}, \tag{4 2 7}$$

where an overbar will represent gauge transformed objects Importantly, this is the Lie derivative of the background metric tensor $\tilde{g}_{\mu\nu}$, and thus we have

$$\mathcal{L}_\xi \tilde{g}_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu,$$

with ∇_μ the covariant derivative associated with the *background* metric

We perform a multipole decomposition of the vector field ξ^μ using the spherical harmonics and the vector bases given in (4 2 2) Again there is an even and an odd part,

$$\xi^{\mathrm{e}}_\mu = \begin{pmatrix} \xi_A Y \\ \xi^{\mathrm{e}} Y_{,a} \end{pmatrix}, \qquad \xi^{\mathrm{o}}_\mu = \begin{pmatrix} 0 \\ \xi^{\mathrm{o}} S_a \end{pmatrix}$$

Now we can write down how all of the perturbation objects m $h_{\mu\nu}$ and $\Delta t_{\mu\nu}$ transform For example, we give the gauge transformation of the even parity off-diagonal metric perturbations

$$h_{Ab} - \overline{h}_{Ab} = \nabla_A \xi_b + \nabla_b \xi_A = \left\{\xi_{b,A} - \Gamma^\mu{}_{Ab}\xi_\mu\right\} + \left\{\xi_{A,b} - \Gamma^\mu{}_{bA}\xi_\mu\right\}$$

Since we are working to linear order, the connection coefficients are those defined on the background,

$$\tilde{\Gamma}^{\mu}{}_{Ab} = \tfrac{1}{2}\tilde{g}^{\mu\nu}\{\tilde{g}_{A\nu,b} + \tilde{g}_{b\nu,A} - \tilde{g}_{Ab,\nu}\} = \tfrac{1}{2}\tilde{g}^{\mu c}\left(r^2\gamma_{bc}\right)_{,A}$$

Thus $\tilde{\Gamma}^{B}{}_{Ab} = 0$, and since $\tilde{g}^{dc} = r^{-2}\gamma^{dc}$ we have $\tilde{\Gamma}^{d}{}_{Ab} = v_A\delta^d_b$ Thus

$$h_{Ab} - \overline{h}_{Ab} = h^{e}_{A}Y_{,b} - \overline{h}^{e}_{A}Y_{,b} = \left(\xi^{e}Y_{,b}\right)_{,A} + \left(\xi_A Y\right)_{,b} - 2v_A\xi^{e}Y_{,b}$$

and so

$$h^{e}_{A} - \overline{h}^{e}_{A} = \xi_A + r^2\left(\xi^{e}/r^2\right)_{,A}$$

We give the gauge transformations for all the bare perturbation objects Firstly the even parity,

$$\left.\begin{array}{rcl} h_{AB} - \overline{h}_{AB} &=& \xi_{A|B} + \xi_{B|A} \\ h^{e}_{A} - \overline{h}^{e}_{A} &=& \xi_A + r^2\left(\xi^{e}/r^2\right)_{,A} \\ K - \overline{K} &=& 2v^A\xi_A \\ G - \overline{G} &=& 2\xi^{e}/r^2 \end{array}\right\} (metric)$$

$$\left.\begin{array}{rcl} \Delta t_{AB} - \Delta\bar{t}_{AB} &=& \tilde{T}_{AB|C}\xi^C + \tilde{T}_{CB}\xi^C{}_{|A}\tilde{T}_{CA}\xi^C{}_{|B} \\ \Delta t^{e}_{A} - \Delta\bar{t}^{e}_{A} &=& \tilde{t}_{AB}\xi^B + \tfrac{1}{2}r^2\tilde{t}^a{}_a\left(\xi^{e}/r^2\right)_{,A} \\ \Delta t^1 - \Delta\bar{t}^1 &=& \tfrac{1}{2}r^{-2}\left(r^2\tilde{t}^a{}_a\right)_{,A}\xi^A \\ \Delta t^2 - \Delta\bar{t}^2 &=& \tilde{t}^a{}_a\xi^{e} \end{array}\right\} (matter)$$

and then odd parity,

$$\left.\begin{array}{rcl} h^{o}_{A} - \overline{h}^{o}_{A} &=& r^2(\xi^{o}/r^2)_{,A} \\ h - \overline{h} &=& \xi^{o} \end{array}\right\} (metric)$$

$$\left.\begin{array}{rcl} \Delta t^{o}_{A} - \Delta\bar{t}^{o}_{A} &=& \tfrac{1}{2}\tilde{t}^a{}_a r^2(\xi^{o}/r^2)_{,A} \\ \Delta t - \Delta\bar{t} &=& \tfrac{1}{2}\tilde{t}^a{}_a\xi^{o} \end{array}\right\} (matter)$$

The next step is to construct the gauge invariants, that is we take linear combinations of (4 2 4), (4 2 5) to form objects which do not change under a gauge transformation For example, consider the combination

$$K - 2v^A\left(h^{e}_{A} - \tfrac{1}{2}r^2G_{,A}\right)$$

How does this change under a gauge transformation? Using the above we find

$$
\begin{aligned}
K - 2v^A\big(h^{\bullet}_A - \tfrac{1}{2}r^2 G_{,A}\big) - \Big\{\overline{K} - 2v^A\big(\overline{h}^{\bullet}_A - \tfrac{1}{2}r^2\overline{G}_{,A}\big)\Big\}& \\
&= 2v^A\xi_A - 2v^A\Big[\xi_A + r^2\big(\xi^{\bullet}/r^2\big)_{,A} - \tfrac{1}{2}r^2\big(2\xi^{\bullet}/r^2\big)_{,A}\Big] \\
&= 0
\end{aligned}
$$

Thus this combination does *not* change under a gauge transformation, it is gauge invariant We give the complete set of gauge invariants, firstly even parity

$$
\left.\begin{aligned}
k_{AB} &= h_{AB} - (p_{A|B} + p_{B|A}) \\
k &= K - 2v^A p_A
\end{aligned}\right\} (metric) \qquad (4\ 2\ 10a)
$$

$$
\left.\begin{aligned}
T_{AB} &= \Delta t_{AB} - \tilde{t}_{AB|C}p^C - \tilde{t}_A{}^C p_{C|B} - \tilde{t}_B{}^C p_{C|A} \\
T_A &= \Delta t^{\bullet}_A - \tilde{t}_A{}^C p_C - r^2(\tilde{t}^a{}_a/4)G_{,A} \\
T^1 &= \Delta t^1 - (p^C/r^2)(r^2\tilde{t}^a{}_a/2)_{,C} + l(l+1)(\tilde{t}^a{}_a/4)G \\
T^2 &= \Delta t^2 - (r^2\tilde{t}^a{}_a/2)G
\end{aligned}\right\} (matter) \qquad (4\ 2\ 10b)
$$

where $p_A = h^{\bullet}_A - \frac{1}{2}r^2 G_{,A}$, and secondly odd parity

$$
\left. k_A = h^{\circ}_A - r^2\big(h/r^2\big)_{,A}\right\} (metric) \qquad (4\ 2\ 11a)
$$

$$
\left.\begin{aligned}
L_A &= \Delta t^{\circ}_A - \big(\tilde{t}^a{}_a/2\big)h^{\circ}_A \\
L &= \Delta t - \big(\tilde{t}^a{}_a/2\big)h
\end{aligned}\right\} (matter) \qquad (4\ 2\ 11b)
$$

Before proceeding a caveat when $l = 0, 1$, we cannot construct a set of gauge invariant objects like those given above for the same reason that there are less equations in these sectors the vanishing of some or all of the bases given in (4 2 2),(4 2 3) Thus for $l = 1$ modes we can at best construct only partially gauge invariant objects, and for $l = 0$ all remnants of gauge invariance are lost This will be discussed in more detail in the relevant sections ahead

The perturbation evolution equations are then recast entirely in terms of the gauge invariants [13, 14] As previously mentioned, not all equations apply in each sector and so we denote the mode numbers for which each equation is valid Again, firstly the even parity equations,

$$2v^C(k_{AB|C} - k_{CA|B} - k_{CB|A} + 2\tilde{g}_{AB}k_{CD}{}^{|D}) - 2\tilde{g}_{AB}v^C k_D{}^D{}_{|C}$$
$$+\tilde{g}_{AB}\left(\frac{l(l+1)}{r^2} + \frac{1}{2}(\tilde{G}_C{}^C + \tilde{G}_a{}^a) + \tilde{\mathcal{R}}\right)k_D{}^D + 2(v_A k_{,B} + v_B k_{,A} + k_{,A|B})$$
$$+\tilde{g}_{AB}(2v^{C|D} + 4v^C v^D - \tilde{G}^{CD})k_{CD}$$
$$-\tilde{g}_{AB}\left(2k_{,C}{}^{|C} + 6v^C k_{,C} - \frac{(l-1)(l+2)}{r^2}k\right)$$
$$-\left(\frac{l(l+1)}{r^2} + \tilde{G}_C{}^C + \tilde{G}_a{}^a + 2\tilde{\mathcal{R}}\right)k_{AB} = -16\pi T_{AB}, \quad (l \geq 0) \quad (4\ 2\ 12a)$$

$$-\left(k_C{}^C{}_{|D}{}^{|D} + \tilde{\mathcal{R}}k_C{}^C - \frac{l(l+1)}{2r^2}k_C{}^C\right) - (k_{,C}{}^{|C} + 2v^C k_{,C} + \tilde{G}_a{}^a k)$$
$$+\left(k_{CD}{}^{|C|D} + 2v^C k_{CD}{}^{|D} + 2(v^{C|D} + v^C v^D)k_{CD}\right) = -16\pi T^1, \quad (l \geq 0) \quad (4\ 2\ 12b)$$

$$k_{,A} - k_{AC}{}^{|C} + k_C{}^C{}_{|A} - v_A k_C{}^C = -16\pi T_A, \quad (l \geq 1) \quad (4\ 2\ 12c)$$

$$k_A{}^A = -16\pi T^2 \quad (l \geq 2) \quad (4\ 2\ 12d)$$

and then odd parity,

$$-\left[r^4(k^A/r^2)^{|C} - r^4(k^C/r^2)^{|A}\right]_{|C} + (l-1)(l+2)k^A = 16\pi r^2 L^A, \quad (l \geq 1) \quad (4\ 2\ 13a)$$

$$k^A{}_{|A} = 16\pi L \quad (l \geq 2) \quad (4\ 2\ 13b)$$

Note there are no odd parity equations for $l = 0$, this is because there is no $l = 0$ odd perturbation A spherically symmetric perturbation would have a scalar angular part and thus be of even parity (see (4 2 4) with $l = 0$) Note also that the divergence of the term in square brackets in (4 2 13a) is identically zero, thus the two odd parity equations imply a conservation equation,

$$\left(r^2 L^A\right)_{|A} = (l-1)(l+2)L \qquad (l \geq 1) \qquad (4\ 2\ 14)$$

Now we have our procedure we may choose any spherically symmetric background whatsoever, calculate the background quantities such as $\tilde{\mathcal{R}}$, v^A etc, and then simply(!) generate, for $l \geq 2$ modes, the complete set of equations given above for the compete set of unknown gauge invariants There are eleven even parity unknowns (4 metric and 7 matter), and five odd parity unknowns (2 metric and 3 matter) However, there are only seven equations in the even sector, and three in the odd sector Clearly we need more information, and we find this information from the background by choosing a specific gauge

The objects we are measuring are gauge invariant, that means we can perform any gauge transformation on them and they remain the same There is an especially useful gauge choice we can make, called the Regge-Wheeler or longitudinal gauge This consists of transforming to a specific gauge via the gauge transformation generated by

$$\xi_\mu dx^\mu = \left(h^{\mathrm{e}}_A - \frac{r^2}{2}G,_A\right) Y dx^A + \left(\frac{r^2}{2}GY,_a + hS_a\right) dx^a, \tag{4 2 15}$$

in which $\overline{h^{\mathrm{e}}_A} = \overline{G} = \overline{h} = 0$ For example, consider the function G A gauge transformation takes G to $G - 2\xi^{\mathrm{e}}/r^2$ But we are setting $\xi^{\mathrm{e}} = r^2 G/2$, thus we have transformed to a coordinate system in which $\overline{G} = 0$ Now, not only are there less unknowns, but also the bare perturbations (4 2 4), (4 2 5) and the gauge invariants match, that is, we can make the identifications

$$\begin{array}{r|l} h_{AB} = k_{AB} & \Delta t_{AB} = T_{AB} \\ h^{\mathrm{e}}_A = 0 & \Delta t^{\mathrm{e}}_A = T_A \\ K = k & \Delta t^1 = T^1 \\ G = 0 & \Delta t^2 = T^2 \end{array} , \quad \text{and} \quad \begin{array}{r|l} h^{\mathrm{o}}_A = k_A & \Delta t^{\mathrm{o}}_A = L_A \\ h = 0 & \Delta t_A = L \end{array}$$

Therefore using the Regge-Wheeler gauge means our bare matter perturbations are automatically gauge invariant, and thus we can use information about the background to simplify some perturbation terms These two bonuses make solving the system of perturbation equations feasible

Code used to calculate the field equations

The perturbation field equations given in (4 2 12), (4 2 13) would be extremely laborious to calculate by hand To save time and to reduce the risk of error we have calculated these equations using the Mathematica package

We define the coordinates, and the metric and matter tensors, on $\mathcal{M}^2$ using a vector and 2×2 matrices respectively Using these we calculate the (background) metric connection, covariant derivative, Riemann tensor and so on Summation over indices is performed using the `Sum` command in Mathematica, for example

$$v^A v_A = \texttt{Sum[vup[a]*vdown[a],\{a,1,2\}]}$$

The perturbation variables are unknown functions defined on $\mathcal{M}^2$, e g $k = k(x^A)$, and we calculate the field equations in terms of these unknowns Each component of the field equations is then taken as an individual equation

We perform straightforward algebraic manipulations of these equations to cast them in a more tractable form, for example removing the second derivatives The use of Mathematica is purely to save time and does not represent an interesting coding problem Therefore, we will not dwell on the details of reducing the field equations, instead we will sketch the procedure where it is informative

4 2 3 Gauge invariant scalars

Finally we must consider what to measure on the relevant surfaces in order to test for stability Following Chandrasekhar [6], we use the Newman-Penrose Weyl scalars (see §1 2) to measure the flux of energy of the perturbations For a detailed discussion on how the Weyl scalars relate to the scalars of Zerilli, Regge-Wheeler, Moncrief etc see Lousto [33] Stewart and Walker [46] showed that the only Weyl scalars which are both identification gauge invariant, which is the sense described above, and tetrad gauge invariant (independent of the choice of null tetrad with which the Weyl scalars are defined), are the Petrov type N terms Furthermore, they are only tetrad and identification gauge invariant if the background is type D or conformally flat Since all of the background spacetimes in this thesis are spherically symmetric, and therefore type D (or conformally flat), this means there are two fully (tetrad and identification) gauge invariant Weyl scalars (for $l \geq 2$), $\delta\Psi_0$ and $\delta\Psi_4$ These scalars represent pure transverse gravitational waves propagating in the radial inward (respectively outward) null directions of a spherically symmetric background However, Nolan [38] has shown that in fact all of the Weyl scalars are gauge invariant with respect to odd parity perturbations, and thus in the odd sector we will consider all five Weyl scalars

With the invariantly defined choice of null tetrad described below, these scalars can be given as

$$\delta\Psi_0 = \frac{Q_0}{2r^2}\tilde{\ell}^A\tilde{\ell}^B k_{AB}, \quad \delta\Psi_4 = \frac{Q_0^*}{2r^2}\tilde{n}^A\tilde{n}^B k_{AB}, \tag{4 2 16a}$$

in the even sector and, defining $\Pi = \epsilon^{AB}(r^{-2}k_A)_{|B}$,

$$\begin{aligned}
\delta\Psi_0 &= \frac{Q_1}{r^2}\tilde{\ell}^A\tilde{\ell}^B k_{A|B}, &(4\,2\,16b)\\
\delta\Psi_1 &= \frac{Q_2}{r}\left[(r^2\Pi)_{|A}\tilde{l}^A - \frac{4}{r^2}k_A\tilde{l}^A\right], &(4\,2\,16c)\\
\delta\Psi_2 &= Q_3\Pi, &(4\,2\,16d)\\
\delta\Psi_3 &= \frac{Q_2^*}{r}\left[(r^2\Pi)_{|A}\tilde{n}^A - \frac{4}{r^2}k_A\tilde{n}^A\right], &(4\,2\,16e)\\
\delta\Psi_4 &= \frac{Q_1^*}{r^2}\tilde{n}^A\tilde{n}^B k_{A|B}, &(4\,2\,16f)
\end{aligned}$$

in the odd sector, where $\tilde{\ell}^\mu, \tilde{n}^\mu, \tilde{m}^\mu = r^{-1}\tilde{w}^\mu(\theta,\phi)$ and $\tilde{m}^{*\mu}$ are a null tetrad of the background and the $*$ represents complex conjugation The angular parts are

$$Q_0 = \tilde{w}^a\tilde{w}^b Y_{ab}, \quad Q_1 = -2\tilde{w}^a\tilde{w}^b S_{a\,b}, \quad Q_2 = -\tfrac{1}{4}\tilde{w}^a S_a, \quad Q_3 = -\tfrac{i}{4}l(l+1)Y$$

These scalars are fully gauge invariant for $l \geq 2$

Tetrad gauge invariance of these objects must be carefully interpreted In the type D background, there is an obvious choice of null tetrad we take $\tilde{\ell}^\mu, \tilde{n}^\mu$ to be the principal null directions of the Weyl tensor, and take $\tilde{m}^\mu$ and its conjugate to be unit space-like vectors orthogonal to $\tilde{\ell}^\mu, \tilde{n}^\mu$ to complete the tetrad Then $\delta\Psi_0$ and $\delta\Psi_4$ in the even sector, and all five scalars in the odd sector, are identification gauge invariant and also tetrad gauge invariant with respect to any *infinitesimal* Lorentz transformation of the tetrad, and also with respect to any finite null rotation that leaves the directions of $\tilde{\ell}^\mu, \tilde{n}^\mu$ fixed However these scalars are *not* preserved under the *finite* boost-rotations

$$\tilde{\ell}^\mu \to A\tilde{\ell}^\mu, \quad \tilde{n}^\mu \to A^{-1}\tilde{n}^\mu, \quad \tilde{m} \to e^{i\omega}\tilde{m}, \quad A > 0, \omega \in \mathbb{R}$$

under which they rescale as

$$\delta\Psi_n \to A^{2-n}\delta\Psi_n, \qquad n = 0,1,2,3,4$$

For the sake of boundary conditions however we must take this scale covariance into account, (see Beetle and Burko [2] for a generalized discussion), and thus to first order our "master" functions will be

$$\delta P_{-1} = |\delta\Psi_0\delta\Psi_4|^{1/2}, \qquad (4\,2\,17a)$$

in the even sector and

$$\delta P_{-1} = |\delta\Psi_0 \delta\Psi_4|^{1/2}, \tag{4 2 17b}$$

$$\delta P_0 = \delta\Psi_2, \tag{4 2 17c}$$

$$\delta P_1 = |\delta\Psi_1 \delta\Psi_3|^{1/2}, \tag{4 2 17d}$$

in the odd sector

For $l = 0$ modes, the angular part of all of these scalars vanishes, for $l = 1$, the angular part of $\delta\Psi_0, \delta\Psi_4$ is zero Thus there is no pure transverse radiation in the $l = 0, 1$ sectors, entirely as expected (see Section 1 2) We will show in §6 2 2 that the only gauge invariant in the $l = 1$ sector is $\delta\Psi_2$

We note that gravitational radiation is often measured using the transverse traceless or radiation gauge For a discussion on how these gauges and their particular scalars $h_+, h_\times$, are related to our choice see Harada et al [25]

Chapter 5

Perturbations of Minkowski spacetime

In this chapter we analyse gauge invariant perturbations, having the structure described in the previous chapter, acting on Minkowski background There are three main reasons for doing so firstly, this simple background will provide a case study, containing all the essential characteristics of a perturbation problem, such as solving the field equations, calculating the scalars, gauge freedom in the low modes, and so on Secondly, since the Vaidya spacetime contains a portion of Minkowski spacetime in the interior to $\mathcal{N}$, our initial data requires a continuous matching across $\mathcal{N}$ and thus we must understand perturbations in this interior And finally, it is important to ensure that singularities do not develop when the background is flat, if they did, we could not be certain an unstable Cauchy horizon, for example, was due to a feature of the naked singularity or merely due to singularities being inserted 'by hand'

This chapter will be divided in a manner which is followed in the next three chapters even and odd parity perturbations decouple and thus may be studied independently, and the vanishing of some vector/tensor bases requires the individual study of the modes $l = 0$, $l = 1$ and $l \geq 2$

In all cases we consider the matter tensor of the perturbed spacetime to be the same as the background spacetime, in this case vacuum Thus there are no matter perturbation terms, and the right hand side of all perturbation equations is zero Note this also means there is no need to use the Regge-Wheeler gauge

We will use two coordinate systems in the following chapter In the standard 'orthogonal' system the (background) metric tensor for flat spacetime is given by

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5 0 1}$$

and using the similarity coordinate $x = v/r$, where $v = t+r$ is the ingoing null coordinate, the metric is given by

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -r^2dx^2 + 2r(1-x)dx\,dr + x(2-x)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5 0 2}$$

In terms of the definitions given in Chapter 2, the past and future null cones of the origin are given by $x = 0$ and $x = 2$ respectively

We derive the (background) radial ingoing and outgoing null geodesic tangents by setting $ds^2 = 0 = d\theta^2 = d\phi^2$, and solving as a quadratic to give

$$\frac{dx}{du} = \frac{-x}{r}\frac{dr}{du}, \qquad \frac{dx}{du'} = \frac{2-x}{r}\frac{dr}{du'},$$

where u, u' parameterize the ingoing and outgoing null geodesics respectively Since a vector is defined by $v^{\mu} = dx^{\mu}/du$, this means

$$\ell^A = \left(-\frac{x}{r}, 1\right)\frac{dr}{du}, \qquad n^A = \left(\frac{2-x}{r}, 1\right)\frac{dr}{du'},$$

where ℓ^A and n^A are the ingoing and outgoing null geodesics respectively These definitions give $\ell^A\ell_A = n^A n_A = 0$, but $\ell^A n_A = 2(dr/du)(dr/du')$, so we simply choose u, u' to give $\ell^A n_A = -1$ (that is, both point into the future (or past), and one is ingoing, the other outgoing) *

5 1 Even parity perturbations

5 1 1 $l \geq 2$ modes, even parity, Minkowski

In this sector, all the vector and tensor bases given in (4 2 2),(4 2 3) are linearly independent, and thus we may use all of the equations given in (4 2 12) Thus the perturbation

*Alternatively, we note the ingoing null geodesic in v, r coordinates is simply $\ell^A = -\delta^A_v$, transform to x, r coordinates, and solve for n^A such that $n^A n_A = 0$ and $\ell^A n_A = -1$ This is equivalent to setting $dr/du = -1$ and $dr/du' = 1/2$, but it is important to note that it doesn't matter what dr/du and dr/du' are, as long as $(dr/du)(dr/du') = -1/2$

equations for Minkowski background are

$$2v^C(k_{AB|C} - k_{CA|B} - k_{CB|A} - \tilde{g}_{AB}k_{CD}{}^{|D} + \tilde{g}_{AB}k_{CD}v^D) - \frac{l(l+1)}{r^2}k_{AB}$$
$$-\tilde{g}_{AB}\left(2k_{,C}{}^{|C} - \frac{(l-1)(l+2)}{r^2}k\right) + 2(v_A k_{,B} + v_B k_{,A} + k_{,A|B}) = 0,$$
$$(v^C v^D + v^{C|D})k_{CD} = 0,$$
$$k_{,A} - k_{AC}{}^{|C} = 0,$$
$$k_A{}^A = 0$$

We write these equations in (x, r) coordinates and perform a Mellin transform (see §1 2 1) over r There are four unknowns, which, after the Mellin transform, we denote

$$k_{AB} = \begin{pmatrix} r^{s+1}A(x,s) & r^s B(x,s) \\ r^s B(x,s) & r^{s-1}C(x,s) \end{pmatrix}, \qquad k = r^{s-1}K(x,s) \tag{5 1 1}$$

We must solve the field equations for these four variables, and then use the solutions to calculate the appropriate Weyl scalars on the axis, the past null cone, and so on These will be the individual modes of the Mellin transform of the Weyl scalars, however as discussed in §3 2 we will not perform the inverse Mellin transform, and instead use our minimum stability requirement that each mode must be finite on the relevant surfaces

Using the background null vectors given above, we can calculate the $\mathcal{M}^2$ portion of the scalars we wish to measure (see §4 2 2) Defining a new variable $D = B - xA$ and removing B, they become

$$\delta\Psi_0 = -2r^{s-3}D, \qquad \delta\Psi_4 = \frac{1}{2}r^{s-3}(2A + D), \qquad \delta P_{-1} = |\delta\Psi_0\delta\Psi_4|^{1/2} \tag{5 1 2}$$

The perturbation field equations are a system of coupled ODE's in four unknowns, A, D, C and K We can decouple the system to get an ODE in D,

$$x(x-2)D'' + 2(1 + s + x - sx)D' - (l + l^2 + s - s^2)D = 0, \tag{5 1 3}$$

which is hypergeometric,

$$z(1-z)D''(z) + (\gamma - [\alpha + \beta + 1]z)D'(z) - \alpha\beta D(z) = 0,$$

with

$$\alpha = 1 + l - s, \qquad \beta = -l - s, \qquad \gamma = -1 - s, \qquad \text{and} \qquad z = x/2 \tag{5 1 4}$$

The hypergeometric equation has been extensively studied, and has three regular singular points, $z = -\infty, 0$ and 1 We can describe concisely the leading behavior of the solutions near these singular points using the P-symbol [49] For all but the last column in the P-symbol, the first row's entry denotes the location of the regular singular point, and the second and third rows' entries denote the leading order exponents of two infinite power series solutions at that point If these exponents do not differ by an integer, then these series are two linearly independent solutions of the differential equation near that point, if the exponents do differ by an integer, then a logarithmic term may be required for the construction of the general solution The final column denotes the dependent variable Thus the solutions to the ODE in D are

$$P\left\{\begin{array}{cccc} -\infty & 0 & 1 & \\ \alpha & 0 & 0 & , z \\ \beta & 1-\gamma & \gamma-\alpha-\beta & \end{array}\right\}$$

Axis

First we consider the axis given by $x = -\infty$ Here the indicial exponents do differ by an integer, since $\alpha - \beta = 2l + 1 \in \mathbb{Z}$ Thus our two solutions near the axis, denoted by the subscript A, are [49]

$$\begin{aligned} {}_{\mathrm{A}}D_1 &= (-z)^{s-l-1} {}_2F_1(1 + l - s, 3 + l, 2 + 2l, z^{-1}), \\ {}_{\mathrm{A}}D_2 &= -\ln(-z)\, {}_{\mathrm{A}}D_1 + (-z)^{s+l}(1 + O(z^{-1})), \end{aligned}$$

where again we use the 'big O' notation described in Chapter 3 Here, ${}_2F_1$ (or, for brevity, F) is the hypergeometric function, or series, given near $z = 0$ by

$$F(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n!\,(\gamma)_n} z^n,$$

where $(\lambda)_n$ is the Pochhammer symbol, or rising factorial, with

$$(\lambda)_0 = 1, \qquad (\lambda)_n = \lambda(\lambda + 1) \quad (\lambda + n - 1) = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}$$

If we form the general solution for D,

$$D|_{axis} = d_1\,{}_{\mathrm{A}}D_1 + d_2\,{}_{\mathrm{A}}D_2,$$

we can use this to find the solutions for the other variables near the axis, firstly for A in terms of D and D', then for K and C in terms of A, A' and D, D' Using these solutions we find the leading behavior of the Weyl scalars To leading order near $x = -\infty$ $(r = 0)$,

$$\delta\Psi_{0,4} = d_1 r^{l-2}(1 + O(r)) + d_2 r^{-l-3}(1 + O(r))$$

Thus in order to make $\delta\Psi_{0,4}$, and hence δP_{-1}, regular at the axis, we need to set $d_2 = 0$ in the general solution for D

Past null cone

Now we allow only the first solution for D to evolve up to the past null cone of the origin $\mathcal{N}$, the threshold When we do so we use the nature of the hypergeometric equation to write the acceptable solution at the regular axis as a linear combination of the two solutions on $\mathcal{N}$ That is to say, near $x = 0$ this solution has the form

$${}_{\mathrm{A}}D_1 = d_3\,{}_{\mathrm{T}}D_1 + d_4\,{}_{\mathrm{T}}D_2, \tag{5 1 5}$$

where ${}_{\mathrm{T}}D_1, {}_{\mathrm{T}}D_2$ are two naturally arising linearly independent solutions of the hypergeometric equation near $x = 0$ In finding these solutions, the relation between α, β and γ is key so we must consider two cases $s \in \mathbb{Z}$ and $s \notin \mathbb{Z}$

The more straightforward case is when $s \notin \mathbb{Z}$ Then $1 - \gamma \notin \mathbb{Z}$ and we can take

$${}_{\mathrm{T}}D_1 = F(\alpha, \beta, \gamma, z), \qquad {}_{\mathrm{T}}D_2 = z^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z), \tag{5 1 6}$$

and (5 1 5) holds with [1]

$$d_3 = \frac{\Gamma(1-\gamma)\Gamma(\alpha - \beta + 1)}{\Gamma(1-\beta)\Gamma(\alpha - \gamma + 1)}, \qquad d_4 = -\frac{\Gamma(\gamma)\Gamma(1-\gamma)\Gamma(\alpha - \beta + 1)}{\Gamma(2-\gamma)\Gamma(\gamma - \beta)\Gamma(\alpha)} e^{i\pi(\gamma - 1)} \tag{5 1 7}$$

Again the solutions for A, C and K can be recovered from these expressions for D When we calculate the scalars due to ${}_{\mathrm{T}}D_1, {}_{\mathrm{T}}D_2$ near $x = 0$ (and away from the singularity at

$r = 0$) we find

$$\delta\Psi_0 = d_3(1 + O(x)) + d_4 x^{s+2}(1 + O(x)), \quad \delta\Psi_4 = d_3(1 + O(x)) + d_4 x^{s-2}(1 + O(x))$$

Since $d_3, d_4 \neq 0$, these two scalars and δP_{-1} will be finite on $x = 0$ iff

$$Re(s) > 2, \quad s \notin \mathbb{Z} \tag{5 1 8}$$

The case $s \in \mathbb{Z}$ is cumbersome and we will only summarize the results here This case is further split according to the sign of $1 - \gamma \in \mathbb{Z}$, however there are no solutions for $1 - \gamma \leq 0$ which are finite on $\mathcal{N}$ and thus we only present here the solutions for $1 - \gamma > 0$ $(s + 2 > 0)$

If $\gamma = 1 - m$ where m is a natural number, and α, β are different from the numbers $0, -1, -2, \quad , 1 - m$, then there are two linearly independent solutions near $x = 0$ given by [15]

$$\begin{aligned} {}_{\mathrm{T}}D_1 &= z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z), \\ {}_{\mathrm{T}}D_2 &= \ln(z)\, {}_{\mathrm{T}}D_1 - \sum_{n=1}^{m} \frac{(n-1)!(-m)_n}{(\gamma - \alpha)_n (\gamma - \beta)_n} z^{m-n} \\ &\quad + \sum_{n=0}^{\infty} \frac{(\alpha + m)_n (\beta + m)_n}{(1 + m)_n n!} [h^*(n) - h^*(0)] z^{m-n}, \end{aligned} \tag{5 1 9}$$

where

$$h^*(n) = \psi(\alpha + m + n) + \psi(\beta + m + n) - \psi(1 + m + n) - \psi(1 + n),$$

and $\psi = \psi^{(0)} = \frac{\Gamma'}{\Gamma}$ is the Digamma function The scalars $\delta\Psi_{0,4}$ and P_{-1} will be finite on $x = 0$ due to these solutions if $Re(s) \geq 2$

If, however, α or β is equal to one of the numbers $0, -1, -2, \quad , 1 - m$, then the solution given above loses meaning, this will occur for $s = l + m$, where m is a natural number (≥ 1) In this case two linearly independent solutions are [15]

$$\begin{aligned} {}_{\mathrm{T}}D_1 &= F(\alpha, \beta, \gamma, z), \\ {}_{\mathrm{T}}D_2 &= F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z) \end{aligned}$$

The scalars due to these solutions *will* diverge on $x = 0$ Therefore we must *not* consider the modes $s = l+m$ when integrating to find the general solution This does not challenge the generality of the result however, as when performing the inverse Mellin transform we may simply choose our contour of integration to avoid the points in the complex plane of s where $Re(s) = l + m$ (see below)

Future null cone

Finally we consider the future null cone of the origin, given by $x = 2$ or $z = 1$ Since $\gamma - \alpha - \beta = s - 2$, we again must consider separately $s \notin \mathbb{Z}$ and $s \in \mathbb{Z}$

Firstly if $s \notin \mathbb{Z}$, then two linearly independent solutions near $z = 1$, denoted with F, are given by [49]

$$
\begin{aligned}
{}_{\mathrm{F}}D_1 &= F(\alpha, \beta, 1 + \alpha + \beta - \gamma, 1 - z), \\
{}_{\mathrm{F}}D_2 &= (1 - z)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta, 1 - \alpha - \beta + \gamma, 1 - z)
\end{aligned}
$$

If $D|_{fnc} = d_5\, {}_{\mathrm{F}}D_1 + d_6\, {}_{\mathrm{F}}D_2$, then the scalars go like

$$
\begin{aligned}
\delta\Psi_0 &= d_5\big(1 + O(x - 2)\big) + d_6(x - 2)^{s-2}\big(1 + O(x - 2)\big), \\
\delta\Psi_4 &= d_5\big(1 + O(x - 2)\big) + d_6(x - 2)^{s+2}\big(1 + O(x - 2)\big)
\end{aligned}
$$

Since we have constrained $Re(s) \geq 2$ in order to guarantee finiteness on $\mathcal{N}$, we see these scalars and δP_{-1} are automatically finite on the future null cone

If $s \in \mathbb{Z}$ the general solution contains logarithmic terms Since $Re(s) \geq 2$, therefore $\gamma - \alpha - \beta \geq 0$ and thus the first solution near $z = 1$ is ${}_{\mathrm{F}}D_2$, and the second contains $\ln(1 - z)\, {}_{\mathrm{F}}D_2$, plus an analytic series, that is, the general solution is

$$
D|_{fnc} = d_7\, {}_{\mathrm{F}}D_2 + d_8 \big[{}_{\mathrm{F}}D_2 \ln(1 - z) + O(1)\big]
$$

We find the scalars due to this solution go like

$$
\begin{aligned}
\delta\Psi_0 &\sim d_7\, (x - 2)^{s-2} + d_8 \big[(x - 2)^{s-2} \ln(x - 2) + O(1)\big], \\
\delta\Psi_4 &\sim d_7\, (x - 2)^{s+2} + d_8 \big[(x - 2)^{s-2} + O(1)\big]
\end{aligned}
$$

near $x = 2$ These scalars will diverge logarithmically if $Re(s) = 2$, however all other integer values of s are fine

We arrive at the following proposition restricting the range of $Re(s)$, which we will use in the next chapter

Proposition 5 1 *For even parity perturbations ($l \geq 2$) of Minkowski spacetime, all modes of the form (5 1 1) are admitted with $s \in \mathbb{C}$ such that $Re(s) > 2$, except for $s = l + m$ where m is a natural number*

These solutions can be matched across $x = 0$ into the Vaidya spacetime, and then allowed evolve up to the Cauchy horizon which that spacetime admits, and beyond

A note on the inverse Mellin transform

The analysis of the previous section became complicated when considering the form of the general solution when the indicial exponents differ by integers, that is when $Re(s) \in \mathbb{Z}$ But technically these cumbersome calculations could have been avoided by looking more carefully at the inverse Mellin transform

To perform the inverse Mellin transform and recover the original function, we integrate over a vertical contour in the complex plane of s We are free to choose this contour, and as noted before we are not summing over *all* contours Thus we could simply have chosen our integration contour to be such that $Re(s) \notin \mathbb{Z}$ However, at this stage we are looking for the *largest* class of solutions which are finite on the relevant surfaces, and thus we go the extra distance to include $Re(s) \in \mathbb{Z}$, and similarly in the next Chapter In Chapter 7 on the other hand, we *must* avoid certain values of $Re(s)$ for the modes to be convergent, and thus in that chapter we will use arguments regarding the careful choosing of our integration contour

5 1 2 $l = 1$ mode, even parity, Minkowski

While the result in this sector is well known, for the sake of completeness we briefly give the analysis in terms of the formalism defined above

In this sector not all the perturbation equations apply, and we can only define partially gauge invariant objects For $l = 1$ we find $Z_{ab} = 0$, and thus

$$h_{\mu\nu}dx^{\mu}dx^{\nu} = \quad + r^2 KY\gamma_{ab}dx^a dx^b,$$

which is the same as setting $G = 0$ How does the scalar K transform now, under the gauge transformation generated by $\xi_{\mu}dx^{\mu} = \xi_A Y dx^A + \xi^e Y_{,a}\, dx^a$? (As this section only deals with even parity perturbations, we will drop the label e)

$$
\begin{aligned}
h_{ab} - \overline{h}_{ab} &= \nabla_a \xi_b + \nabla_b \xi_a \\
&= 2\xi Y_{,a\,b} + 2r^2 v^A \xi_A Y \gamma_{ab} \\
&= \left(2r^2 v^A \xi_A - 2\xi\right) Y \gamma_{ab} \\
&= r^2 (K - \overline{K}) Y \gamma_{ab},
\end{aligned}
$$

where in the third line we have used the fact that $Y_{,a\,b} = -Y\gamma_{ab}$ when $l = 1$ Since $G = 0$ we have $p_A = h_A$, where h_A transforms as before Defining the objects k_{AB} and k as in (4 2 10), we see they are no longer gauge invariant They are sensitive to the angular part of the gauge transformation in the following way

$$
k_{AB} \rightarrow k_{AB} + \left[r^2(\xi/r^2)_{,A}\right]_{|B} + \left[r^2(\xi/r^2)_{,B}\right]_{|A}, \tag{5 1 10}
$$

$$
k \rightarrow k + 2\xi/r^2 + 2v^A r^2 (\xi/r^2)_{,A} \tag{5 1 11}
$$

We can, however, use this to our advantage Following Sarbach and Tiglio [45], we look to transform into a coordinate system in which $k_A{}^A = 0$ To do this we choose ξ such that

$$
\left[r^2(\xi/r^2)_{,A}\right]^{|A} = -k_A{}^A
$$

Then we are free to make further gauge transformations, provided

$$
\left[r^2(\xi/r^2)_{,A}\right]^{|A} = 0 \tag{5 1 12}
$$

Using this, we can reinstate the second scalar perturbation equation, $k_A{}^A = 0$, which was not available in the $l = 1$ sector, as a gauge choice

Using this, we split the tensor perturbation equation (4 2 12a) into its trace and trace-free parts Then the set of equations for $l = 1$ is given by

$$
2v^C(k_{AB|C} - k_{CA|B} - k_{CB|A}) - \frac{2}{r^2}k_{AB} + 2(v^A k_{,B} + v^B k_{,A} + k_{,A|B}) - \tilde{g}_{AB} k_{;C}{}^{|C} = 0,
$$

$$
2v^C v^D k_{CD} = k_{;C}{}^{|C} + 2v^C k_{,C} \qquad k_{AC}{}^{|C} = k_{,A}
$$

where the first equation is the trace-free part of the tensor perturbation equation, and the second is its trace

Now we must consider what to solve the equations for The scalars $\delta\Psi_{0,4}$ given in

Section 4 2 2 have an angular dependence which is zero for $l = 1$, and thus they vanish in this sector The other options for true scalars to measure are $k_A{}^A$ and k, however we have chosen a gauge in which the trace of k_{AB} is zero, and thus the only scalar left to measure is k We will show this scalar is pure gauge, that is there is enough residual gauge freedom to transform into a gauge in which $k = 0$ In this gauge, all the components of k_{AB} are also zero, and thus this perturbation sector is empty

For convenience we look at the perturbation equations in orthogonal coordinates,

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2,$$

and in this coordinate system we label components

$$k_{AB} = \begin{pmatrix} A(t,r) & B(t,r) \\ B(t,r) & A(t,r) \end{pmatrix},$$

since k_{AB} is symmetric and trace-free Now when we look at the perturbation equations, the trace equation gives A, and one of the trace-free equations gives B, in terms of k and its derivatives,

$$\begin{aligned} A(t,r) &= \frac{r}{2}\Big(2k_{,r} + r(k_{,rr} - k_{,tt})\Big), \qquad (5\ 1\ 13a) \\ B(t,r) &= \frac{r}{2}\Big(2k_{,t} + r^2(k_{,tt} - k_{,rr})_{,t}\Big) \qquad (5\ 1\ 13b) \end{aligned}$$

The remaining equations give

$$r\,(\Box k)_{,r} = -4\,(\Box k)\,,$$

which we solve as

$$\Box k = f(t)\, r^{-4},$$

where $\Box$ is the two-dimensional d'Alambertian (wave) operator of flat spacetime, $\Box k = k_{|A}{}^{|A}$, and $f(t)$ is a constant of integration k should satisfy this equation with initial data $k(0,r) = \alpha(r)$, $k(0,r) = \beta(r)$ satisfying α, $\beta \in C^1$ and $k(t < 0, r = 0)$ finite This implies $f(t) = 0$, and thus k solves the homogeneous wave equation

$$\Box k = 0$$

k however is not gauge invariant. Looking at (5.1.11), we see $k \to \overline{k} = k + \eta$ where $\eta = 2\xi/r^2 + 2v^A r^2(\xi/r^2)_{,A}$. Thus

$$\Box k \quad \to \quad \Box\overline{k} = \Box k + \Box\eta,$$

but $\Box\eta = 0$ (provided (5.1.12) holds), and thus k and η satisfy the same equation. Therefore we can choose $\eta = -k$, and thus we can always transform to a gauge in which $k = 0$. In this gauge we also have $k_{AB} = 0$, from (5.1.13), and thus the entire $l = 1$ perturbation is pure gauge.

5.1.3 $l = 0$ mode, even parity, Minkowski

The $l = 0$ mode represents a spherically symmetric perturbation. As we are considering zero matter perturbation, we have the perturbed spacetime is spherically symmetric vacuum, and thus Birkhoff's Theorem applies, that is the perturbed spacetime is Schwarschild. We will recover the specifics using our perturbation formalism described above.

For $l = 0$, $Y_{,a} = 0$ and thus $h_A = G = 0$. We cannot form gauge invariants and thus use K and h_{AB}, which are fully dependent on gauge transformations $\xi_\mu dx^\mu = \xi_A dx^A$ as

$$\begin{aligned} h_{AB} &\to h_{AB} - (\xi_{A|B} + \xi_{B|A}), \\ K &\to K - 2v^A\xi_A \end{aligned}$$

We can transform to a gauge in which $K = h^A{}_A = 0$ by choosing ξ_A such that

$$\xi_A{}^{|A} = \tfrac{1}{2}h^A{}_A, \qquad v^A\xi_A = \tfrac{1}{2}K$$

In (t,r) coordinates, this means making a gauge transformation $\xi_r = \frac{1}{2}rK$ and $\xi_{t,t} = \xi_{r,r} - \frac{1}{2}h_A{}^A$. Further transformations preserving $K = h^A{}_A = 0$ must be of the form $\xi_A = f(r)\delta^t_A$ for any arbitrary function $f(r)$. Thus of the remaining perturbation terms, $A(t,r)$ is gauge invariant, whereas $B(t,r) \to B(t,r) - f'(r)$, where

$$h_{AB} = \begin{pmatrix} A(t,r) & B(t,r) \\ B(t,r) & A(t,r) \end{pmatrix}$$

The perturbation equations reduce to $A_{,t} = B_{,t} = 0$ and $rA_{,r} = -A$. Since B is a function of r alone, we can choose $f(r)$ to set $B = 0$. Thus the only perturbation term which cannot be gauged away is

$$A = \frac{c}{r}$$

Renaming the constant $c = 2m$, and noting $(1 - 2m/r)^{-1} \approx (1 + 2m/r)$ for m small (which is the case in this linear model), we have recovered the Schwarzschild line element There is an intrinsic singularity on the axis $r = 0$, and thus this solution contradicts our initial regularity condition, therefore there is no $l = 0$ perturbation

5 2 Odd Parity perturbations

5 2 1 $l \geq 2$ modes, odd parity, Minkowski

Here there is one metric gauge invariant, k^A, and no matter perturbation The second field equation, (4 2 13b), tells us k^A is divergence free, and thus we can write

$$k^A = \left(\frac{\gamma_{,r}}{r}, -\frac{\gamma_{,x}}{r}\right), \tag{5 2 1}$$

for some scalar $\gamma = \gamma(x, r)$ A Mellin transform of the perturbation equations is equivalent to decomposing $\gamma = r^s K(x, s)$ Using

$$\epsilon_{AB} = \begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix} \tag{5 2 2}$$

in these coordinates, we can write Π and thus the five Weyl scalars in terms of K

The vector field equation, (4 2 13a), returns two equations, one a derivative of the other, and thus there is a second order ode for K,

$$x(x-2)K'' - 2(s-2)(x-1)K' - (l^2 + l - 2 + 3s - s^2)K = 0$$

Π involves K, K' and K'', but we can use the K equation to simplify Π to

$$\Pi = (l^2 + l - 2)r^{s-4}K \tag{5 2 3}$$

The K equation is hypergeometric with

$$\alpha = 2 + l - s, \qquad \beta = 1 - l - s, \qquad \gamma = 2 - s, \qquad z = x/2$$

Axis

There are two solutions near the axis $z = -\infty$, which we denote with a subscript A, and since $\alpha - \beta = 2l + 1 \in \mathbb{Z}$, these are given by [49]

$$
\begin{aligned}
{}_{\mathrm{A}}K_1 &= (-z)^{-\alpha} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, z^{-1}), \\
{}_{\mathrm{A}}K_2 &= {}_{\mathrm{A}}K_1 \ln(-z) + (-z)^{-\beta}\big(1 + O(z)\big)
\end{aligned}
$$

The general solution for K therefore is $K|_{axis} = k_1\, {}_{\mathrm{A}}K_1 + k_2\, {}_{\mathrm{A}}K_2$, and the perturbed Weyl scalars go like

$$
\delta\Psi_i \sim k_1 r^{l-2} + k_2 r^{-l-3}, \qquad i = 1, \quad , 5 \tag{5 2 4}
$$

Thus to guarantee a regular axis we must set $k_2 = 0$ We are left with a one parameter family of solutions emerging from the regular axis, and this specifies the allowable combinations of solutions near the threshold

Past null cone

Near the threshold, our linearly independent solutions are distinguished by whether s is an integer or not We will examine the two possibilities separately

Firstly we consider $s \notin \mathbb{Z}$ Then $1 - \gamma \notin \mathbb{Z}$, and we have two linearly independent solutions

$$
\begin{aligned}
{}_{\mathrm{T}}K_1 &= z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z), \\
{}_{\mathrm{T}}K_2 &= F(\alpha, \beta, \gamma, z)
\end{aligned}
$$

A general solution is therefore $K|_{\mathcal{N}} = k_3\, {}_{\mathrm{T}}K_1 + k_4\, {}_{\mathrm{T}}K_2$ (where $k_3, k_4 \neq 0$), and the Weyl scalars go like

$$
\begin{gathered}
\delta\Psi_0 = k_3 O(x^{s+1}) + k_4 O(1), \quad \delta\Psi_1 = k_3 O(x^{s-1}) + k_4 O(1), \\
\delta\Psi_2 = k_3 O(x^{s-1}) + k_4 O(1), \\
\delta\Psi_3 = k_3 O(x^{s-2}) + k_4 O(1),\ \delta\Psi_4 = k_3 O(x^{s-3}) + k_4 O(1)
\end{gathered} \tag{5 2 5}
$$

Hence

$$
P_{-1} \sim x^{s-3}, \quad P_0 \sim x^{s-1}, \quad P_1 \sim x^{s-2},
$$

and therefore a finite threshold requires

$$Re(s) > 3, \quad s \notin \mathbb{Z}$$

Next we consider $s \in \mathbb{Z}$ This section is again split according to the sign of $1-\gamma$, however for $1-\gamma \leq 0$ (or $s \leq 1$) there are no solutions finite on the threshold When $1-\gamma > 0$ ($s > 1$), there are two classes of solutions

If $\gamma = 1-m$, where $m \in \mathbb{N}$, and neither $\alpha, \beta \in \{0, -1, -2, \quad , 1-m\}$, then

$$\begin{aligned} {}_{\mathrm{T}}K_1 &= z^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z), \\ {}_{\mathrm{T}}K_2 &= {}_{\mathrm{T}}K_1 \ln(z) + O(1) \end{aligned}$$

The $O(1)$ series is quite significant here, as now we find the scalars go like

$$\begin{aligned} \delta\Psi_0 &= k_3 O(x^{s+1}) + k_4 \left\{O(x^{s-1}) + O(1)\right\}, \\ \delta\Psi_1 &= k_3 O(x^{s-1}) + k_4 \left\{O(x^{s-1}\ln x) + O(1)\right\}, \\ \delta\Psi_2 &= k_3 O(x^{s-1}) + k_4 \left\{O(x^{s-1}\ln x) + O(1)\right\}, \\ \delta\Psi_3 &= k_3 O(x^{s-2}) + k_4 \left\{O(x^{s-2}\ln x) + O(1)\right\}, \\ \delta\Psi_4 &= k_3 O(x^{s-3}) + k_4 \left\{O(x^{s-3}\ln x) + O(1)\right\} \end{aligned} \tag{5 2 6}$$

Thus we must exclude $s = 2, 3$ for a finite threshold

If $\gamma = 1-m$, where $m \in \mathbb{N}$, and α or $\beta = -m'$ where $m' = 0, 1, \quad , m-1$, then two linearly independent solutions near $x = 0$ are given by

$$\begin{aligned} {}_{\mathrm{T}}K_1 &= F(\alpha, \beta, \gamma, z), \\ {}_{\mathrm{T}}K_2 &= F(\alpha-\gamma+1, \beta-\gamma+1, 1-\gamma, z) \end{aligned}$$

This will be the case for $l = 2, 3, \quad , s-2$ (i e only when $s \geq 4$) In this case all the scalars are $O(1)$

Thus the most general class of solutions which are finite on the threshold is

$$Re(s) > 3, \quad s \in \mathbb{C}$$

Future null cone

The indicial exponents near $x = 2$ or $z = 1$ are $\gamma - \alpha - \beta = s - 1$ and 0 A general solution is given by $D|_{fnc} = k_5\, {}_{\mathrm{F}}K_1 + k_6\, {}_{\mathrm{F}}K_2$, where

$$
\begin{aligned}
{}_{\mathrm{F}}K_1 &= (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, 1-\alpha-\beta+\gamma, 1-z), \\
{}_{\mathrm{F}}K_2 &= F(\alpha, \beta, 1+\alpha+\beta-\gamma, 1-z)
\end{aligned}
$$

if $Re(s) \notin \mathbb{Z}$, and

$$
{}_{\mathrm{F}}K_2 = {}_{\mathrm{F}}K_1 \ln(1-z) + O(1)
$$

if $Re(s) \in \mathbb{Z}$ The scalars due to these solution have the leading behaviour

$$
\begin{aligned}
\delta\Psi_0 &\sim k_5(x-2)^{s-3} + k_6[O(1) + (x-2)^{s-3}\ln(x-2)], \\
\delta\Psi_1 &\sim k_5(x-2)^{s-2} + k_6[O(1) + (x-2)^{s-2}\ln(x-2)], \\
\delta\Psi_2 &\sim k_5(x-1)^{s-3} + k_6[O(1) + (x-2)^{s-1}\ln(x-2)], \\
\delta\Psi_3 &\sim k_5(x-2)^{s-1} + k_6[O(1) + (x-2)^{s-1}\ln(x-2)], \\
\delta\Psi_4 &\sim k_5(x-2)^{s+1} + k_6[O(1) + (x-2)^{s-1}],
\end{aligned}
$$

where the logarithmic terms are omitted if $Re(s) \notin \mathbb{Z}$ If $Re(s) > 3$ then all of these scalars are finite on the future null cone

Proposition 5 2 *For odd parity perturbations ($l \geq 2$) of Minkowski spacetime, all modes of the form $\gamma = r^s K(x, s)$, $s \in \mathbb{C}$ are allowed with $Re(s) > 3$*

5 2 2 $l = 1$ mode, odd parity, Minkowski

Here we have one gauge dependent metric term k^A which transforms as

$$
k_A \to k_A - r^2(\xi^o/r^2)_{,A},
$$

(again we will drop the o label as we are only considering odd parity perturbations in the section), and one field equation

$$
\left[r^4(k^A/r^2)^{|C} - r^4(k^C/r^2)^{|A}\right]_{|C} = 0
$$

Let us examine this problem in orthogonal coordinates t, r, where $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$, and the connections vanish Defining $X_A = r^2(\xi/r^2)_{,A}$ gives

$$X^A = \left(-\xi, \xi' - 2\xi/r\right),$$

where dot and prime denote differentiation w r t t and r respectively Consider

$$W^{AC} \;=\; r^4 \left(X^A/r^2\right)^{|C} - r^4 \left(X^C/r^2\right)^{|A}$$

Obviously $W^{AA} = 0$, and

$$W^{tr} = -W^{rt} = r^2 \left(X^t{}_{,r} + X^r{}_{,t}\right) - 2rX^t = 0$$

Thus $W^{AC}{}_{|C} = 0$ and hence k_A and $X_A = r^2 \left(\xi/r^2\right)_{,A}$ solve the same equation Thus we can always transform to a gauge in which $k_A = 0$, and therefore the odd $l = 1$ perturbation is pure gauge in Minkowski spacetime

Chapter 6

Perturbations of Vaidya spacetime

In this chapter, we describe metric and matter perturbations of the self-similar Vaidya solution given in Section 2 4 1 Most convenient will be the coordinates (x, r, θ, ϕ), where $x = v/r$ is the similarity coordinate In this coordinate system the self-similar Vaidya metric and matter tensor are

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= r^2(-1+\lambda x)\,dx^2 + 2r\left(1-x+\lambda x^2\right)dxdr + x\left(2-x+\lambda x^2\right)dr^2 + r^2 d\Omega^2, \\ t_{\mu\nu}dx^\mu dx^\nu &= \rho \ell_\mu \ell_\nu dx^\mu dx^\nu = \frac{\lambda}{8\pi}dx^2 + \frac{\lambda x}{8\pi r}dxdr + \frac{\lambda x^2}{8\pi r^2}dr^2, \end{aligned}$$

where $\rho = \lambda/8\pi r^2$ is the energy density, $\ell_\mu = -\partial_\mu v$ is the ingoing null direction, and Minkowski spacetime is recovered in the limit $\lambda \to 0$

The ingoing and outgoing radial null geodesic tangents of the background spacetime are ℓ^μ and n^μ respectively Solving $\ell^\mu \ell_\mu = n^\mu n_\mu = 0$ and $\ell^\mu n_\mu = -1$, we find on $\mathcal{M}^2$ using (x, r) coordinates

$$\ell^A = \begin{pmatrix} x/r \\ -1 \end{pmatrix}, \qquad n^A = \begin{pmatrix} (\lambda x^2 - x + 2)/(2r) \\ (1-\lambda x)/2 \end{pmatrix}$$

Note these solutions are not unique due to the rescaling freedom $\ell^A \to A\ell^A, n^A \to A^{-1}n^A$

There is a globally naked singularity for $0 < \lambda < 1/8$, where the spacetime has the structure given in Fig 2 1, and the first and second future similarity horizons are given by

$$x_c \equiv \frac{1}{2\lambda}(1-\sqrt{1-8\lambda}), \qquad x_e \equiv \frac{1}{2\lambda}(1+\sqrt{1-8\lambda})$$

respectively.

Initial regularity conditions for perturbations of this spacetime are that any perturbations coming from the region of flat space preceding the threshold (the past null cone of the origin, $\mathcal{N}$) should be finite on, and match continuously across, this surface.

As before, we will need to split the analysis into the even and odd parity sector, and split the modes into the $l = 0, l = 1$, and $l \geq 2$ cases.

6.1 Even parity perturbations

6.1.1 $l \geq 2$ modes, even parity, Vaidya

We consider the case where the perturbed spacetime is that of a null dust, and thus has matter tensor

$$t_{\mu\nu} = (\bar{\rho} + \delta\rho)(\bar{\ell}_\mu + \delta\ell_\mu)(\bar{\ell}_\nu + \delta\ell_\nu), \tag{6.1.1}$$

where $\bar{\rho}+\delta\rho$ is the energy density and $\bar{\ell}_\mu+\delta\ell_\mu$ is the ingoing null direction of the perturbed spacetime. As before we decompose these perturbation objects in terms of the spherical harmonics. The energy conservation equation ${t_{\mu\nu}}^{;\nu} = 0$ requires $\bar{\ell}_\mu + \delta\ell_\mu$ to be geodesic; if $\delta\ell_\mu = \zeta_A Y + \zeta Y_{,a}$, then for $\bar{\ell}_\mu + \delta\ell_\mu$ to be a null geodesic of the perturbed spacetime we require $(\bar{\ell}_\mu + \delta\ell_\mu)(\bar{\ell}^\mu + \delta\ell^\mu) = 0$ and $(\bar{\ell}^\mu + \delta\ell^\mu)\nabla_\mu(\bar{\ell}^\nu + \delta\ell^\nu) = 0$. In the (x, r, θ, ϕ) coordinate system, this reduces to two equations,

$$\zeta_{x,r} = \zeta_{r,x}, \qquad x\zeta_x - x\zeta_{,x} + r\zeta_{,r} - r\zeta_r = 0.$$

This holds if $\zeta_x = \Gamma_{,x}, \zeta_r = \Gamma_{,r}$ and $\zeta = \Gamma$, for some function $\Gamma(x, r)$. We may consider Γ the perturbation in the null coordinate v; that is we can define an ingoing null coordinate of the perturbed spacetime as $V = v - \Gamma(x, r)Y(\theta, \phi)$. Then the ingoing null direction of the perturbed spacetime is $\bar{\ell}_\mu + \delta\ell_\mu = -\partial_\mu V$. Since $\delta\rho = \varrho(x, r)Y$, then we can write the 'bare' matter perturbation entirely in terms of Γ and ϱ.

Since we are only concerned with measuring gauge invariant quantities, we may examine the perturbations in any gauge we choose. For convenience, we will use the Regge-Wheeler (RW) gauge given in (4.2.15), the benefit being that the 'bare' perturbations and the gauge invariants match, as described in Section 4.2.2. In other words, when using the RW gauge, the 'bare' matter perturbations are automatically gauge invariant. Thus we can use (6.1.1) to write the matter gauge invariants in terms of Γ, ϱ, as

$$T_{AB} = \begin{pmatrix} r^2\varrho - \frac{\lambda\partial_x\Gamma}{4\pi r} & rx\varrho - \frac{\lambda\partial_r\Gamma}{8\pi r} - \frac{\lambda x\partial_x\Gamma}{8\pi r^2} \\ \text{Symm} & x^2\varrho - \frac{\lambda x\partial_r\Gamma}{4\pi r^2} \end{pmatrix}, \quad T_A = \begin{pmatrix} \frac{-\lambda\Gamma}{8\pi r} \\ \frac{-\lambda x\Gamma}{8\pi r^2} \end{pmatrix}, \quad T^1 = T^2 = 0. \tag{6.1.2}$$

Now we see the benefit in using the RW gauge: instead of 7 matter variables there are only 2.

We write out the perturbation equations as given in (4.2.12), and as before perform a Mellin transform over r. There are now six unknowns in total,

$$k_{AB} = \begin{pmatrix} r^{s+1}A(x;s) & r^sB(x;s) \\ r^sB(x;s) & r^{s-1}C(x;s) \end{pmatrix}, \quad k = r^{s-1}K(x;s), \quad \Gamma = r^sG(x;s), \quad \varrho = r^{s-3}E(x;s).$$

A constraint equation defining ϱ decouples, and so we are left with five unknowns in the evolution system, and six second order ordinary differential equations.

We use the scalar equation (4.2.12d) to remove the metric variable $C(x;s)$, and as in the previous chapter, we define the variable $D = B - xA$, and remove B. With some rearranging we can reduce the set of equations to a four dimensional first order linear system. With $Y = (A, D, K, G)^T$, we write the first order linear system as $dY/dx = M(x)Y$, with the coefficients of M given by

$$\begin{aligned}
M_{11} &= \frac{4-4x+2(1+\lambda)x^2-3\lambda x^3+\lambda^2x^4+2s^2\left(2-x+\lambda x^2\right)+s\left(-4+2x-\lambda x^3+\lambda^2x^4\right)}{x\left(2s+\lambda x^2\right)\left(2-x+\lambda x^2\right)} \\
M_{12} &= \frac{4(-1+x)-2s^2x(-1+x\lambda)^2+(l+l^2)(2+x(-1+x\lambda))(2+x(-1+x\lambda)(2+x(-1+x\lambda)))}{x^2(-1+x\lambda)(2s+x^2\lambda)(2+x(-1+x\lambda))} \\
&\quad \frac{-(s(-1+x\lambda)(4+x(-2+x(2+\lambda(-2+x(-5+3x\lambda))))))+x(8\lambda+x(-2+\lambda(-12+x(9+\lambda(8+7x(-2+x\lambda))))))}{x^2(-1+x\lambda)(2s+x^2\lambda)(2+x(-1+x\lambda))} \\
M_{13} &= \frac{\left(2+\left(-2+l+l^2\right)x\right)(2+(-2+x)x)-x^2(-6+(9+2l(1+l)(-1+x)-4x)x)\lambda+x^4\left(3+\left(-2+l+l^2\right)x\right)\lambda^2}{x^2(2s+x^2\lambda)(2+x(-1+x\lambda))} \\
&\quad \frac{-2s^2(2+x(-1+x\lambda))+sx(2-x(2+\lambda(4+x(-5+3x\lambda))))}{x^2(2s+x^2\lambda)(2+x(-1+x\lambda))} \\
M_{14} &= \frac{2\lambda(-4+2s(2+x(-1+x\lambda))-x(-2+x\lambda)(2+x(-1+x\lambda)))}{x(2s+x^2\lambda)(2+x(-1+x\lambda))} \\
M_{21} &= \frac{2x(-1+x\lambda)}{(2s+x^2\lambda)(2+x(-1+x\lambda))} \\
M_{22} &= \frac{2+l(1+l)(-2+x)-\left((-1+s)(-6+4s-x)+2l(-1+x)+2l^2(-1+x)\right)x\lambda}{(-1+x\lambda)(2s+x^2\lambda)(2+x(-1+x\lambda))} \\
&\quad + \frac{x^2\left(6+2s(-3+s-x)+\left(3+l+l^2\right)x\right)\lambda^2+(-2+s)x^4\lambda^3+2s(-2+s+2\lambda)}{(-1+x\lambda)(2s+x^2\lambda)(2+x(-1+x\lambda))} \\
M_{23} &= -\frac{\left(-2+2s-\left(-2+l+l^2\right)x\right)(-1+x\lambda)}{(2s+x^2\lambda)(2+x(-1+x\lambda))} \qquad M_{24} = \frac{-4x\lambda(-1+x\lambda)}{(2s+x^2\lambda)(2+x(-1+x\lambda))} \\
M_{31} &= \frac{-2}{2s+x^2\lambda} \qquad M_{32} = \frac{2-2s+l(1+l)(-2+x)-x(4-2s+l(1+l)x)\lambda}{x(-1+x\lambda)(2s+x^2\lambda)} \\
M_{33} &= \frac{-2+2s^2+sx^2\lambda-x\left(-2+l+l^2+x\lambda\right)}{2sx+x^3\lambda} \qquad M_{34} = \frac{4\lambda}{2s+x^2\lambda} \\
M_{41} &= 0 \qquad M_{42} = \frac{1}{-x+x^2\lambda} \\
M_{43} &= 0 \qquad M_{44} = \frac{s}{x}
\end{aligned} \tag{6.1.3}$$

Using the null vectors of the background given above, we calculate (each mode of) the $\mathcal{M}^2$ portion of the perturbed Weyl scalars $\delta\Psi_{0,4}$ to be

$$\delta\Psi_0 = \frac{2r^{s-3}D}{\lambda x - 1}, \quad \delta\Psi_4 = \frac{r^{s-3}(2A + (1-\lambda x)D)}{2}, \quad \delta P_{-1} = |\delta\Psi_0\delta\Psi_4|^{1/2}. \tag{6.1.4}$$

From our system of equations we can decouple a second order ordinary differential equation for D in the following way: the second row of the system gives the equation $D' = f_1(A, D, K, G)$. Differentiating w.r.t. x, we can use the system to remove A', D', K' and G' to give $D'' = f_2(A, D, K, G)$. Next we consider the second order equation (4.2.12b). Differentiating the system w.r.t. x we have $Y'' = MY' + M'Y = (M^2 + M')Y$. We use this to remove A'', K'' from (4.2.12b), and use $Y' = MY$ to remove A', D', K' and G', giving another equation $D'' = f_3(A, D, K, G)$. Combining with f_2 we are left with a second order linear ODE only in D,

$$D''(x) + p(x)D'(x) + q(x)D(x) = 0.$$

The coefficients are

$$\begin{aligned}
p(x) &= -\frac{s+1}{x} + \frac{2\lambda}{1-\lambda x} + \frac{s-3-(s-6)\lambda x}{\lambda x^2 - x + 2}, \\
q(x) &= \frac{(\lambda x - 1)^2(l + l^2 + s^2(\lambda x - 1)) + \lambda(x - 2 + 6\lambda x - \lambda x^2)}{x(\lambda x - 1)^2(\lambda x^2 - x + 2)} \\
&\qquad - \frac{s(1 + \lambda(-2 + x(2\lambda x - 3)))}{x(\lambda x - 1)(\lambda x^2 - x + 2)},
\end{aligned}$$

and we note that in the limit $\lambda \to 0$ we recover (5.1.3) exactly.

This equation has a number of singular points (see Section 1.3.2): the regular singular point at $x = 0$ is the past null cone of the origin of coordinates, and is the surface over which we move from flat spacetime to Vaidya spacetime (the threshold). The Cauchy horizon at $x = x_c$ and the second future similarity horizon (SFSH) at $x = x_e$, the first and second zeroes of $\lambda x^2 - x + 2$ respectively, are both regular singular points (for $\lambda < 1/8$). Finally the apparent horizon $x_a = 1/\lambda$ is also a regular singular point.

This equation for D is a second order equation coming from a fourth order system, therefore there must be two solutions with $D = 0$. We find these by setting $D = 0$ in the system, which simplifies the equations greatly, reducing them to

$$xK' - (s-1)K = 0, \qquad xG' - sG = 0, \qquad A = 2\lambda G + \left(\tfrac{s-1}{x} - \tfrac{1}{2}(l^2 + l - 2)\right)K.$$

Thus we can find the set of exact solutions corresponding to $D = 0$, which we call solutions III and IV. They are valid everywhere and irrespective of whether s is an integer or not:

$$\begin{array}{rc|c} & III & IV \\ A = & 2g_0\lambda x^s & (s-1)k_0x^{s-2} - \frac{1}{2}(l^2+l-2)k_0x^{s-1} \\ D = & 0 & 0 \\ K = & 0 & k_0x^{s-1} \\ G = & g_0x^s & 0, \end{array} \tag{6.1.5}$$

where k_0, g_0 are arbitrary constants.

We will begin the analysis by examining the threshold $x = 0$ and ensuring our initial regularity conditions are met. As the analysis will differ depending on whether s is an integer or not, we will consider these two cases separately. Then we will allow the acceptable solutions to evolve up to the Cauchy horizon, and then on to the SFSH.

Threshold, $s \notin \mathbb{Z}$

We can use the method of Frobenius to describe solutions to the second order equation in D near regular singular points, and we present these using the P-symbol notation (see Section 5.1.1). For $\lambda < 1/8$, there are four regular singular points, and the indicial exponents near each singular point are given by

$$P\left\{\begin{array}{ccccc} 0 & x_c & x_e & x_a & \\ 0 & 0 & 0 & 1 & ;x \\ s+2 & \frac{1}{2}\left(s-4+\frac{s}{\sqrt{1-8\lambda}}\right) & \frac{1}{2}\left(s-4-\frac{s}{\sqrt{1-8\lambda}}\right) & 2 & \end{array}\right\}. \tag{6.1.6}$$

The full set of solutions are found using system methods, as this approach has the benefit of solving for all four solutions at once, with the required accuracy found by simply taking further terms in the series. These system methods are described in Section 1.3.3, and we briefly outline their application here:

The system of perturbation equations can be written in the form

$$Y' = \frac{1}{x^2}\left(J + \sum_{m=1}^{\infty} A_m x^m\right) Y,$$

where $Y = (A, D, K, G)^T$, and $J \neq 0$. Thus $x = 0$ is an irregular singular point*. We

*And yet $x = 0$ was a *regular* singular point of the D equation. This is a common trade off: in reducing find that J has eigenvalue 0 multiplicity four. J cannot be diagonalised, and we use Theorem 1.2.8 given in Section 1.3.3 to remove off-diagonal terms; this effectively reduces the singularity to a regular singular point. Now the leading order coefficient matrix has eigenvalues $0, s, s, s-2$, and so we apply Theorem 1.2.6 twice to reduce the eigenvalues to 0 and $s-2$ multiplicity three. Finally we apply Theorem 1.2.5 to obtain the following:

To leading order, we find a full set of linearly independent solutions with asymptotic behaviour (as $x \to 0$)

$$
\begin{array}{r|c|c|c|c}
 & I & II & III & IV \\
A = & O(1) & O(x^{s-2}) & O(x^s) & O(x^{s-2}) \\
D = & O(1) & O(x^{s+2}) & 0 & 0 \\
K = & O(1) & O(x^{s+1}) & 0 & O(x^{s-1}) \\
G = & O(1) & O(x^{s+2}) & O(x^s) & 0.
\end{array}
\tag{6.1.7}
$$

Solutions I and II correspond with the Minkowski solutions (matching across $x = 0$ is dealt with in the next subsection), and III and IV are the $D = 0$ solutions given in (6.1.5) .

Taking A and D to be a linear combination of these solutions, we calculate the leading order of the scalars near $x = 0$ as

$$\delta\Psi_0 = O(1) + O(x^{s+2}), \quad \delta\Psi_4 = O(1) + O(x^{s-2}) + O(x^s),$$

(for brevity we have left out the constants of combination). Thus for our master function δP_{-1} to be finite on the threshold due to the Vaidya solutions, we find we must maintain the same constraint as that coming from the flat space solutions and given in Proposition 5.1, that is

$$Re(s) > 2, \quad \text{when} \quad s \notin \mathbb{Z}. \tag{6.1.8}$$

Threshold, $s \in \mathbb{Z}$

When s is an integer the system methods break down for the following reason: the eigenvalues of the leading order coefficient matrix of the regular singular point at $x = 0$ are 0 and $s - 2$ multiplicity three. These differ by an integer, and thus they must be repeatedly reduced until they are equal. However, each time we reduce an eigenvalue

the order of the system one often increases the singularity of the singular points.

we must diagonalise the leading order coefficient matrix, which prevents us from simply reducing the eigenvalue (the unspecified number) $s-2$ times.

Instead, we use the ordinary differential equation for D to write the solutions near $x=0$ as, using the fact that $Re(s)+2>0$ (since the flat space solutions constrained $Re(s)\geq 2$),

$$D = d_5 \sum_{m=0}^{\infty} A_m x^{m+s+2} + d_6 \left\{ \hat{k} \ln x \sum_{m=0}^{\infty} A_m x^{m+s+2} + \sum_{m=0}^{\infty} B_m x^m \right\}, \tag{6.1.9}$$

with d_5, d_6 constants and $\hat{k} = \lim_{\lambda\to 0}(\lambda A_{s-2}/A_0)$ possibly zero.

We use this solution as an inhomogeneous term to solve for the other variables by integration, and we find

$$G \sim g_0 x^s - d_5 \sum_{m=0}^{\infty} \frac{A_m x^{m+s+2}}{m+2} - d_6 \left\{ \sum_{m=0}^{\infty} \frac{B_m x^m}{m-s} + \hat{k} \sum_{m=0}^{\infty} \frac{A_m x^{m+s+2}}{m+2} \left(\ln x - \frac{1}{m+2} \right) \right\},$$

$$\begin{aligned} K \sim\ & k_0 x^{s-1} + d_5 \left\{ \sum_{m=0}^{\infty} \frac{A_m x^{m+s+1}}{m+2} \left(-2(m+s+2) + x(1-s-2\lambda) \right) \right\} \\ & + d_6 \left\{ \sum_{m=0}^{\infty} B_m x^m \left(\frac{1-s-2\lambda}{m-s+1} - \frac{2m}{(m-s)x} \right) + \hat{k} \sum_{m=0}^{\infty} A_m x^{m+s+1} \left(\frac{-2}{m+2} \right.\right. \\ & \left.\left. - \frac{2(m+s+2)}{m+2} \left(\ln x - \frac{1}{m+2} \right) + \frac{x(1-s-2\lambda)}{m+3} \left(\ln x - \frac{1}{m+3} \right) \right) \right\}, \end{aligned}$$

$$A \sim (s + l^2 + l - 1) x^{-1} D + 2\lambda G + (s^2 - 1) x^{-1} K - sK'.$$

Since both D and K have $O(1)$ terms, we see there is a solution for A which diverges on the threshold like x^{-1}. This divergent term cannot be switched off, for the following reason:

On the axis (see Chapter 5) there were two solutions for D, which we denoted

$$_{\rm A}D_1 \sim x^{s-l-1}, \quad _{\rm A}D_2 \sim x^{s+l}.$$

The scalars $\delta\Psi_{0,4}$ due to the first solution went like r^{l-2}, whereas the second solution gave $\delta\Psi_{0,4} \sim r^{-l-3}$. Thus we needed to switch off the divergent term in the general solution, $D|_{axis} = d_1\, {}_{\rm A}D_1 + d_2\, {}_{\rm A}D_2$, by setting $d_2 = 0$.

Now when this solution was allowed evolve to the threshold, $D|_{x=0} = d_3\, {}_{\rm T}D_1 + d_4\, {}_{\rm T}D_2$, the constants $d_3, d_4 \neq 0$ were fixed [see (5.1.7)]. To match across $x = 0$, we must have (since we are using a global coordinate system)

$$D^M|_{x=0} = D^V|_{x=0},$$

where M denotes solutions coming from Minkowski spacetime, and V denotes solutions coming from Vaidya spacetime. Thus we require

$$\lim_{x \uparrow 0}(d_3 \,{}_{\mathrm{T}}D_1^M + d_4 \,{}_{\mathrm{T}}D_2^M) = \lim_{x \downarrow 0}(d_5 \,{}_{\mathrm{T}}D_1^V + d_6 \,{}_{\mathrm{T}}D_2^V).$$

From (5.1.6),(5.1.9), we see the solutions for D from Minkowski spacetime are $O(1), O(x^{s+2})$. When $s \in \mathbb{Z} \geq 2$, we see from (6.1.9) the solutions from Vaidya spacetime are also $O(1), O(x^{s+2})$, and thus to match continuously across $x = 0$ we cannot switch off the $O(1)$ D-solution.

Thus when we calculate A, and hence $\delta\Psi_4$, there will be divergence as $x \downarrow 0$ due to this solution. This does not happen when $s \notin \mathbb{Z}$, since there is no divergent A-solution when $Re(s) > 2$.

There were four regularity conditions for initial data: each mode of the flux had to be (i) finite on the axis, (ii) finite on the threshold when approached from flat space, (iii) finite on the threshold when approached from Vaidya spacetime, and (iv) continuously matched across $x = 0$. The most general class of perturbations which satisfies all of these conditions are those modes with

$$Re(s) > 2, \quad s \notin \mathbb{Z}. \tag{6.1.10}$$

Cauchy Horizon

When $\lambda < 1/8$, the Cauchy horizon is a regular singular point of the system given in (6.1.3). Its leading order coefficient matrix has eigenvalues 0 multiplicity three and

$$\sigma \equiv \frac{1}{2}\left(s - 4 + \frac{s}{\sqrt{1-8\lambda}}\right).$$

When σ is not an integer, we can use the system methods outlined in Section 1.3.3. Applying Theorem 1.2.5, we find solutions with asymptotic behaviour

$$\begin{array}{rc|c|c|c} & I & II & III & IV \\ A = & O(1) & O(w^\sigma) & O(1) & O(1) \\ D = & O(1) & O(w^\sigma) & 0 & 0 \\ K = & O(w) & O(w^{\sigma+1}) & 0 & O(1) \\ G = & O(w) & O(w^{\sigma+1}) & O(1) & 0, \end{array} \tag{6.1.11}$$

as $w \to 0$ where $w = x - x_c$ (for consistency see (6.1.6)).

Now we make an important observation: since

$$0 < \sqrt{1 - 8\lambda} < 1$$

for $0 < \lambda < 1/8$, therefore

$$\sigma = \frac{1}{2}\left(s - 4 + \frac{s}{\sqrt{1 - 8\lambda}}\right) > \frac{1}{2}(2s - 4),$$

and thus $Re(\sigma) > 0$ for $Re(s) > 2$. Alternatively, we can say

$$\sigma = s - 2 + O(\lambda)$$

where each coefficient of λ^n is positive, and thus again $Re(\sigma) > 0$ for $Re(s) > 2$. Thus each solution for A and D as given in (6.1.11) is at most $O(1)$ near $x = x_c$; all the solutions for A and D which are series beginning with w, w^σ or $w^{\sigma+1}$ will decrease to zero as we approach the Cauchy horizon.

Since

$$\delta\Psi_0 = \frac{2r^{s-3}D}{\lambda x - 1}, \quad \delta\Psi_4 = \frac{r^{s-3}(2A + (1 - \lambda x)D)}{2},$$

and A and D near the Cauchy horizon are a linear combination of $O(1)$ solutions, the scalars $\delta\Psi_{0,4}$ representing the flux of the perturbation, and hence the scalar δP_{-1}, will be finite on the Cauchy horizon $x = x_c$. Thus when $\sigma \notin \mathbb{Z}$, the Cauchy horizon is stable under metric and matter perturbations.

However, for each value of the parameter $\lambda < 1/8$, there will be a mode number s such that $\sigma \in \mathbb{Z}$, and thus we must also consider this case. From (6.1.6), we see a general solution for D near $w = x - x_c = 0$ can be written as

$$D = d_7 \sum_{m=0}^{\infty} A_m w^{m+\sigma} + d_8 \left\{ \tilde{k} \ln w \sum_{m=0}^{\infty} A_m w^{m+\sigma} + \sum_{m=0}^{\infty} B_m w^m \right\},$$

where d_7, d_8 are constants and $\tilde{k}$ can be zero. Since we are considering $\lambda > 0$ ($\lambda = 0$ being vacuum spacetime) and $Re(s) > 2$, we have $\sigma \geq 1$ if $\sigma \in \mathbb{Z}$. Now we use this solution for D as an inhomogeneous term to integrate the perturbation equations. Near $w = 0$, we find a 4-parameter set of solutions:

$$\begin{aligned}
K &\sim k_0 + \frac{x_c^{-1}(x_c - x_e)}{1 - \lambda x_c} \int w D' dw - x_c^{-1}(x_c + 3 - n) \int D dw, \\
G &\sim g_0 + \frac{1}{x_c(\lambda x_c - 1)} \int D dw, \\
A &\sim \frac{1}{x_c(1 - \lambda x_c)} D + 2\lambda G + \tfrac{1}{2} x_c^{-1}[\lambda x_c - x_c(l^2 + l - 2) + 6]K - \tfrac{1}{2}(\lambda x_c^2 + 4)K',
\end{aligned}$$

where k_0, g_0 are constants, a prime denotes differentiation w.r.t. w, and

$$\begin{aligned}
\int D dw &= d_7 \sum_{m=0}^{\infty} \frac{A_m w^{m+\sigma+1}}{m + \sigma + 1} \\
&\quad + d_8 \left\{ \tilde{k} \sum_{m=0}^{\infty} \frac{A_m w^{m+\sigma+1}}{m + \sigma + 1} \left(\ln w - \frac{1}{m + \sigma + 1} \right) + \sum_{m=0}^{\infty} \frac{B_m w^{m+1}}{m + 1} \right\}, \\
\int w D' dw &= d_7 \sum_{m=0}^{\infty} \frac{A_m (m + \sigma) w^{m+\sigma+1}}{m + \sigma + 1} + d_8 \left\{ \sum_{m=0}^{\infty} \frac{B_m m w^{m+1}}{m + 1} \right. \\
&\quad \left. + \tilde{k} \sum_{m=0}^{\infty} \frac{A_m w^{m+\sigma+1}}{m + \sigma + 1} \left(1 + (m + \sigma) \ln w - \frac{m + \sigma}{m + \sigma + 1} \right) \right\}.
\end{aligned}$$

Since $\sigma \geq 1$ and $\lim_{w \to 0} w^\sigma \ln w = 0$, we see all of these variables A, D, K, G, and thus the scalars $\delta\Psi_{0,4}$ and δP_{-1}, are again finite in the limit $w \to 0$.

Thus every mode of the perturbation which is finite on the axis and on the threshold $\mathcal{N}$ will evolve up to the Cauchy horizon and beyond *without* diverging. Therefore in the case of self-similar null dust there is a naked singularity whose Cauchy horizon is stable under metric and matter perturbations.

Second Future Similarity Horizon

Now something interesting happens when we allow the solution to evolve past the Cauchy horizon and on to the next singular surface, the SFSH given by $x_e = \frac{1}{2\lambda}(1 + \sqrt{1 - 8\lambda})$. The first scalar depends only on D,

$$\delta\Psi_0 = \frac{2r^{s-3}}{\lambda x - 1} D,$$

and the solutions for D near $x = x_e$ can be found directly from (6.1.6) as

$$D = d_9 \sum_{m=0}^{\infty} A_m (x - x_e)^m + d_{10} \sum_{m=0}^{\infty} B_m (x - x_e)^{m+\varsigma},$$

where

$$\varsigma = \frac{1}{2}\left(s - 4 - \frac{s}{\sqrt{1 - 8\lambda}}\right).$$

Since $0 < \sqrt{1 - 8\lambda} < 1$, we see ς will always be negative for $Re(s) \geq 2$. Thus there is a class of solutions which are finite on the axis, finite on the threshold $\mathcal{N}$, finite on the Cauchy horizon, and then finally diverge on the SFSH. We emphasize that this instability is due to $x = x_e$ being a similarity horizon of the spacetime, and not an event horizon.

We arrive at the following result:

Proposition 6.1. *The Cauchy horizon formed in the self-similar collapse of the Vaidya spacetime (with $\lambda < 1/8$) is stable with respect to even parity perturbations in the metric and matter tensors of multipole mode $l \geq 2$. The second future similarity horizon following the Cauchy horizon is unstable.*

6.1.2 $l = 1$ mode, even parity, Vaidya

In this sector we can only define partially gauge invariant objects. As in Minkowski spacetime, Section 5.1.2, the metric perturbation objects are gauge sensitive to $\xi_\mu dx^\mu = \xi_A Y dx^A + \xi^{\mathsf{e}} Y_{,a}\, dx^a$ as (dropping the label e)

$$\begin{aligned} k_{AB} &\rightarrow k_{AB} + \left[r^2(\xi/r^2)_{,A}\right]_{|B} + \left[r^2(\xi/r^2)_{,B}\right]_{|A} \\ k &\rightarrow k + 2\xi/r^2 + 2v^A r^2(\xi/r^2)_{,A}\,. \end{aligned}$$

We examine the case where the perturbed spacetime is that of null dust, and thus the bare matter perturbations are as given in (6.1.2). Since $p_A = h_A$ in this sector, we can transform into the equivalent of the Regge-Wheeler gauge by choosing

$$\xi_A = h_A - r^2(\xi/r^2)_{|A},$$

the benefit being that in this gauge $p_A = 0$. Therefore $T_{AB} = \Delta t_{AB}$ etc. and we can, for the sake of consistency, use the same variables T_{AB}, T_A etc. as before. This set of variables however is no longer gauge invariant, as we see below. We can express the right hand side of the perturbation equations given in §4.2.2 in terms of ϱ, Γ, as in (6.1.2).

Further transformations maintain this condition provided $\xi_A = -r^2(\xi/r^2)_{|A}$. Importantly, this fixes ξ_A while keeping ξ completely free.

T_{AB} and T_A are not gauge invariant, they are sensitive to gauge transformations as

$$\begin{aligned}
\overline{T}_{AB} - T_{AB} &= \tilde{t}_{AB|C} r^2(\xi/r^2)^{|C} + \tilde{t}_{CB}(r^2(\xi/r^2)^{|C})_{|A} + \tilde{t}_{CA}(r^2(\xi/r^2)^{|C})_{|B}, \\
\overline{T}_A - T_A &= -\tilde{t}_{AB} r^2(\xi/r^2)^{|B}.
\end{aligned}$$

The vanishing of T^1 and T^2 is gauge invariant.

Now we look at the perturbation equations in (x, r) coordinates. As in the $l \geq 2$ sector, we perform a Mellin transformation of the equations, which is equivalent to parameterizing the perturbation components as before,

$$k_{AB} = \begin{pmatrix} r^{s+1}A(x;s) & r^s B(x;s) \\ r^s B(x;s) & r^{s-1}C(x;s) \end{pmatrix}, \quad k = r^{s-1}K(x;s), \quad \Gamma = r^s G(x;s), \quad \varrho = r^{s-3}E(x;s).$$

Again, we define the new variable $D = B - xA$.

We exploit the gauge freedom and transform into a gauge in which ${k_A}^A = 0$, by choosing ξ such that

$$\left[r^2(\xi/r^2)_{,A}\right]^{|A} = -{k_A}^A.$$

Then we are allowed make further transformations which preserve RW gauge and ${k_A}^A = 0$, provided

$$\begin{aligned}
\left[r^2(\xi/r^2)_{,A}\right]^{|A} &= 0, \qquad (\text{preserves}\, {k_A}^A = 0) \\
\xi_A + r^2(\xi/r^2)_{|A} &= 0. \qquad (\text{preserves}\, h_A = 0)
\end{aligned}$$

Thus we recover the perturbation equation (4.2.12d), which was not valid in the $l = 1$ sector, as a gauge choice. As before the set of perturbation equations in this gauge reduces to one constraint equation defining ϱ, a first order system in A, D, K and G, and we can decouple a second order ordinary differential equation for D.

There is some gauge freedom in ξ left, and we will use this to gauge away k. Let us formalise this: let ξ satisfy gauge conditions $L_1\xi = 0$; let k satisfy perturbation field equation $L_2 k = 0$; and let k transform as $k \to k + L_3\xi$. If $L_2\big(L_3\xi\big) = 0$, subject to $L_1\xi = 0$, (that is to say, k and $L_3\xi$ solve the same equation), there is a gauge in which $k = 0$.

We give here the operators L_1, L_2 and L_3, in the x, r coordinate system, and after a Mellin transform over r such that $\xi = r^{s-1}\xi(x; s)$:

$$\begin{aligned}
L_1\xi =& x\left(-2 + x - x^2\,\lambda\right)\,\xi_s''(x) + \left(-2 + 2x - 3x^2\,\lambda + 2s\,\left(1 - x + x^2\,\lambda\right)\right)\,\xi_s'(x)\\
&(1 - s)\,\left(-x\,\lambda + s\,\left(-1 + x\,\lambda\right)\right)\,\xi_s(x),\\
L_2\xi =& m_1(x)\xi_s'''(x) + p_1(x)\,\xi_s''(x) + q_1(x)\,\xi_s'(x) + r_1(x)\,\xi_s(x),\\
L_3\xi =& (s + x\,\lambda - s\,x\,\lambda)\,\xi_s(x) + \left(1 - x + x^2\,\lambda\right)\,\xi_s'(x),
\end{aligned}$$

where

$$\begin{aligned}
m_1(x) =& x(\lambda x^2 - x + 2)(1 + s - x + x(3 - s + x)\lambda),\\
p_1(x) =& 6 + 5(-2 + x)x - 4x(-3 + x + 3x^2)\lambda + x^3(24 + 7x)\lambda^2 + s^2(-1 + x\lambda)(4 - 3x + 3x^2\lambda))\\
&+ s(2 + x(1 - 16\lambda + x(-3 + \lambda(16 + 6x - 17x\lambda + 3x^2\lambda)))),\\
q_1(x) =& (2 - s)\left(-2s\,(1 + s)\,(1 + s)\,(-1 + 3s)\,x + (1 - 3s)\,x^2\right) - 2x\left(s^3 - 3s^2 - s + 3\right)\\
&- \left(x - (-3 + s)\,s\,(-4 + 3\,s)\,x\right)\\
&+ \left(-2x\left(5 - 9s + s^2\right)\,x^2\right)\,\lambda + (-1 + s)\,x^3\,\left(-30 - 8\,x + s\,(19 - 3\,s + 3\,x)\right)\,\lambda^2,\\
r_1(x) =& (1 - n)\big[\left((3 - n)\,n\,(-4 + n - x)\,x^2\,\lambda^2\right)\\
&+ \left((2 - n)(1 + n)(n - x) + 2\,\left(3 + n^3\,x + x(5 + x) + 2s(1 + x^2) - n^2\,\left(1 + x(4 + x)\right)\right)\,\lambda\right)\big].
\end{aligned}$$

We find that there is indeed a gauge in which $k = 0$. A direct consequence of $k = 0$ is that $D = 0$. Thus we may transform to a gauge in which $p_A = k_A{}^A = k = D = 0$, and to remain in this gauge we are allowed further transformations provided

$$\begin{aligned}
\left[r^2(\xi/r^2)_{,A}\right]^{|A} &= 0, \quad (\text{preserves}\, k_A{}^A = 0)\\
2\xi/r^2 + 2v^A r^2(\xi/r^2)_{,A} &= 0. \quad (\text{preserves}\, k = 0)
\end{aligned}$$

The only ξ which satisfies both these constraints is $\xi = 0$. Therefore there is no remaining gauge freedom and so the remaining perturbation variables are gauge invariant.

The remaining perturbation equations are greatly simplified and can be easily solved. We find a one parameter family of solutions,

$$k_{AB} = \begin{pmatrix} 2g_0\lambda r^{s+1}x^s & 2g_0\lambda r^s x^{s+1} \\ 2g_0\lambda r^s x^{s+1} & 2g_0\lambda r^{s-1}x^{s+2} \end{pmatrix}, \quad T_{AB} = \begin{pmatrix} \frac{\lambda g_0(s+x)r^{s-1}x^{s-1}}{4\pi} & \frac{\lambda g_0(s+x)r^{s-2}x^s}{4\pi} \\ \frac{\lambda g_0(s+x)r^{s-2}x^s}{4\pi} & \frac{\lambda g_0(s+x)r^{s-3}x^{s+1}}{4\pi} \end{pmatrix},$$
$$T_A = \begin{pmatrix} \frac{-\lambda g_0 r^{s-1}x^s}{8\pi} \\ \frac{-\lambda g_0 r^{s-2}x^{s+1}}{8\pi} \end{pmatrix}, \quad T^1 = T^2 = k = 0.$$

To match continuously with the empty Minkowski region preceding $x = 0$, we require $Re(s) \geq 1$. These solutions will evolve without divergence through the rest of the spacetime.

Note the perturbations will vanish if $g_0 = 0$, that is to say if we had considered only metric perturbations, and no matter perturbations, we would have returned an empty sector. Note also the sector is empty in the Minkowski limit $\lambda \to 0$.

6.1.3 $l = 0$ mode, even parity, Vaidya

When $l = 0$, then $Y_{,a} = 0$ and our bare perturbations are

$$h_{\mu\nu} = \begin{pmatrix} h_{AB} & 0 \\ 0 & r^2 K \gamma_{ab} \end{pmatrix}, \qquad \Delta t_{\mu\nu} = \begin{pmatrix} \Delta t_{AB} & 0 \\ 0 & r^2 \Delta t^1 \gamma_{ab} \end{pmatrix}.$$

We will use the coordinates (v, r, θ, ϕ), where v is the null coordinate of the background. Thus the metric of the background is $\bar{g}_{\mu\nu}dx^\mu dx^\nu = -(1 - \frac{\lambda v}{r})dv^2 + 2dv\,dr + r^2\gamma_{ab}dx^a dx^b$.

Since the matter tensor has the form (6.1.1), and $\tilde{\ell}_\mu$ has no angular dependence, we find $\Delta t^1 = 0$. We can describe the ingoing radial null geodesic of the perturbed spacetime as $\tilde{\ell}_\mu + \delta\ell_\mu = -\nabla_\mu V$ where $V = v + \Gamma(v, r)$ is the null coordinate of the perturbed spacetime. Our unknowns therefore are $h_{AB}, K, \delta\rho$ and Γ.

We cannot construct gauge invariants in the $l = 0$ sector and thus we exploit the remaining gauge freedom to set some variables equal to zero. The perturbation variables are gauge dependent as

$$\begin{aligned} h_{AB} &\to h_{AB} - (\xi_{A|B} + \xi_{B|A}), \\ K &\to K - 2v^A\xi_A, \\ \Delta t_{AB} &\to \Delta t_{AB} - \tilde{t}_{AB|C}\xi^C - \tilde{t}_{CB}\xi^C{}_{|A} - \tilde{t}_{CA}\xi^C{}_{|B}. \end{aligned} \tag{6.1.12}$$

As before, we transform into a gauge in which $K = h^A{}_A = 0$. To do this we choose ξ_A such that (in the v, r coordinate system where $\dot{\theta}$ and θ' denote differentiation w.r.t. v and r respectively), $\xi_v + (1 - \frac{\lambda v}{r})\xi_r = \frac{1}{2} rK$, $\dot{\xi_r} = \frac{1}{2} h^A{}_A$. Then we are free to make further gauge transformations which preserve this condition, provided

$$\begin{aligned} \xi_v + (1 - \tfrac{\lambda v}{r})\xi_r &= 0, \\ \dot{\xi_r} &= 0. \end{aligned} \tag{6.1.13}$$

Now we look at the field equations in this gauge (we will let $h_{vv} = A$, $h_{vr} = h_{rv} = B$ and $h_{rr} = C$). The first is $\dot{C} = 0$, and thus $C = C(r)$. When we perform a gauge transformation on this quantity subject to (6.1.12) and (6.1.13), we find $C \to C - 2\xi_r'$, but since ξ_r is an arbitrary function of r, this means we can choose a gauge in which $C = 0$ (and thus $B = 0$ since $h^A{}_A = 0$). Thus we have transformed to a gauge in which $K = h^A{}_A = B = C = 0$, and to remain in this gauge we are allowed further gauge transformations of the form $\xi_A = c_0(-(1 - \frac{\lambda v}{r})\delta^v_A + \delta^r_A)$, with c_0 an arbitrary constant.

The remaining perturbation equations in this gauge are

$$\begin{aligned} rA'' + 2A' &= 0, \\ rA' + A + \lambda\Gamma' &= 0, \\ \tfrac{1}{r^2}(1 - \tfrac{\lambda v}{r})(rA' + A) + \tfrac{1}{r}\dot{A} - \tfrac{2\lambda}{r^2}\dot{\Gamma} &= 8\pi\delta\rho. \end{aligned}$$

That $\tilde{\ell}_\mu + \delta\ell_\mu$ must be null and geodesic gives $\Gamma' = 0$, and hence

$$\Gamma = \alpha(v), \qquad A = \frac{\beta(v)}{r}, \qquad \delta\rho = \frac{1}{8\pi r^2}(\dot{\beta} - 2\lambda\dot{\alpha}).$$

Further gauge transformations give

$$\begin{aligned} A &\to A - 2\tfrac{\lambda}{r}c_0, \\ \Delta t_{AB} &\to \Delta t_{AB}, \end{aligned}$$

and thus these remaining perturbation quantities cannot be gauged away.

What we have shown here is essentially a uniqueness result: all the above perturbations can be generated by a perturbation in the mass function and the null vector, thus a spherically symmetric perturbation of Vaidya spacetime is again Vaidya spacetime. This is because there is a 'Birkhoff-type' theorem for spherical null dust, which states that the metric and matter tensors for spherically symmetric null dust are given by

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{m(v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2,$$

$$t_{\mu\nu}dx^\mu dx^\nu = \frac{\dot{m}(v)}{8\pi r^2}\ell_\mu \ell_\nu.$$

Before perturbation $m(v) = \lambda v$ and $\ell_\mu = -\partial_\mu v$, and after a perturbation $m(v) = \lambda v + \beta(v) - 2\lambda c_0$ and $\ell_\mu = -\partial_\mu(v + \alpha(v))$, but the essential features remain the same. Therefore there cannot be divergence on the Cauchy horizon or elsewhere due to a perturbation, as these features are not present in the background.

6.2 Odd parity perturbations

6.2.1 $l \geq 2$ modes, odd parity, Vaidya

In this sector there are gauge invariant perturbation objects k_A, L_A and L, as defined in (4.2.11b). Since Vaidya is a spherically symmetric dust solution, therefore $\bar{t}^a{}_a = 0$, and the bare and gauge invariant matter perturbations match without the need for a Regge-Wheeler gauge.

The matter tensor of the perturbed spacetime is

$$t_{\mu\nu} = \bar{\rho}(\bar{\ell}_\mu + \delta\ell_\mu)(\bar{\ell}_\nu + \delta\ell_\nu), \tag{6.2.1}$$

where $\delta\ell_\mu = \Lambda(x,r)S_a$. To linear order we see $\Delta t = 0$, and since $\Delta\bar{t} - \Delta t = 0$, the vanishing of L is gauge invariant. Also $L_A = \Delta t_A = \bar{\rho}\bar{\ell}_A \Lambda$, thus

$$L_A = \left(-\frac{\lambda\Lambda}{8\pi r}, -\frac{\lambda x \Lambda}{8\pi r^2}\right)$$

is gauge invariant.

We solve the perturbation field equation $k^A{}_{|A} = 0$ as in Minkowski spacetime, to give $k^A = (\gamma_{,r}/r, -\gamma_{,x}/r)$. We write the Mellin transformation of γ as $r^s K(x;s)$. For $\bar{\ell}_\mu + \delta\bar{\ell}_\mu$ to be a null geodesic, we find the same constraint on Λ as that from the conservation equation (4.2.14); that is $x\Lambda_{,x} = r\Lambda_{,r}$.

Now when we look at the second perturbation field equation, we can decouple and get an equation in $\Lambda(x,r)$, (we have not taken a Mellin transform of Λ)

$$x\Lambda_{,x} = (s-1)\Lambda,$$

and an equation in K,

$$\begin{aligned} x(\lambda x^2 - x + 2)K''' - x[6 - 6x + 8\lambda x^2 + s(-4 + 3x - 3x^2\lambda)]K'' \\ -[x(-6 + l + l^2 + 10\lambda x) + s(-4 + 9x - 13\lambda x^2) + s^2(2 - 3x + 3\lambda x^2)]K' \\ +s[l + l^2 + (s-1)(2 - s - 4\lambda x + s\lambda x)]K = 0. \end{aligned} \tag{6.2.2}$$

Alternatively, we have a second order inhomogeneous equation in K,

$$\begin{aligned} x(\lambda x^2 - x + 2)K'' + [4 - 4x + 5x^2\lambda - 2s\left(1 - x + x^2\lambda\right)]K' \\ +[-2 + l + l^2 + s\,(3 - 4x\lambda) + s^2\,(-1 + x\lambda)]K = \frac{-2r^{1-s}x\lambda}{s}\Lambda. \end{aligned} \tag{6.2.3}$$

Using this, and given that in this spacetime ϵ_{AB} is as given in (5.2.2), we find

$$\Pi = (l^2 + l - 2)r^{s-4}K - \tfrac{2}{s}r^{-3}x\lambda\Lambda, \tag{6.2.4}$$

and we can write the scalars entirely in terms of K, or combinations of K and Λ. Solving for Λ we find

$$\Lambda = c_0 x^{s-1} r^{s-1}.$$

We can solve the homogeneous part of equation (6.2.3) to find indicial exponents $0, s-1$, and the solution due to the inhomogeneous term is a series beginning at $s+1$, thus a full solution would be (with $\bar{k} = 0$ if $s \notin \mathbb{Z}$)

$$K|_{\mathcal{N}} = k_5 \sum_{m=0}^{\infty} A_m x^{m+s-1} + k_6 \left\{ \bar{k} \ln x \sum_{m=0}^{\infty} A_m x^{m+s-1} + \sum_{m=0}^{\infty} B_m x^m \right\} + k_7 \sum_{m=0}^{\infty} C_m x^{m+s+1},$$

since $Re(s) - 1 > 0$ due to the restriction $Re(s) > 3$, which came from considerations of the region of flat spacetime preceding $x = 0$ (see Proposition 5.2).

Alternatively, we take the third order homogeneous equation in K and find three indicial exponents $0, s-1, s+1$. This gives the same solution as above, only we can use the inhomogeneous equation to clarify where the logarithmic terms are.

Now we use this solution to measure the five scalars, and we find

$$\begin{aligned}
\delta\Psi_0 &= k_5 O(x^{s+1}) + k_6 \left\{O(x^{s-1}) + O(1)\right\} + k_7 O(x^{s+1}), \\
\delta\Psi_1 &= k_5 O(x^{s-1}) + k_6 \left\{O(x^{s-2}) + O(1)\right\} + k_7 O(x^{s}), \\
\delta\Psi_2 &= k_5 O(x^{s-1}) + k_6 \left\{O(x^{s-2}) + O(1)\right\} + k_7 O(x^{s}), \\
\delta\Psi_3 &= k_5 O(x^{s-2}) + k_6 \left\{O(x^{s-3}) + O(1)\right\} + k_7 O(x^{s-1}), \\
\delta\Psi_4 &= k_5 O(x^{s-3}) + k_6 \left\{O(x^{s-3}\ln x) + O(1)\right\} + k_7 O(x^{s-1}).
\end{aligned}$$

As in Minkowski spacetime, the threshold will be stable for

$$Re(s) > 3, \quad s \in \mathbb{C}.$$

A quick comparison with (5.2.5), (5.2.6) shows the scalars will match continuously across $x = 0$ for $k_6 = k_4 \neq 0$.

Now we can go up to the Cauchy horizon, $x = x_c$. Here, the third order homogeneous equation in K has indicial exponents

$$0, \quad 1, \quad \frac{4\lambda(s-1)}{\sqrt{1-8\lambda}(1-\sqrt{1-8\lambda})} \equiv \sigma.$$

Since the homogeneous part of the inhomogeneous equation for K has indices $0, \sigma$, a general solution near the Cauchy horizon is

$$\begin{aligned}
K|_{CH} &= k_8 \sum_{m=0}^{\infty} A_m (x-x_c)^{m+\sigma} + k_9 \left\{ \hat{k} \ln(x-x_c) \sum_{m=0}^{\infty} A_m (x-x_c)^{m+\sigma} \right. \\
&\quad \left. + \sum_{m=0}^{\infty} B_m (x-x_c)^m \right\} + k_{10} \sum_{m=0}^{\infty} C_m (x-x_c)^{m+1},
\end{aligned}$$

since $\sigma > 0$. We use this solution to calculate the scalars near $x = x_c$, and we find

$$\begin{aligned}
\delta\Psi_0 &= k_8 O(\zeta^{\sigma-2}) + k_9 \left\{O(\zeta^{\sigma-2}\ln\zeta) + O(1)\right\} + k_{10} O(1), \\
\delta\Psi_1 &= k_8 O(\zeta^{\sigma-1}) + k_9 \left\{O(\zeta^{\sigma-2}\ln\zeta) + O(1)\right\} + k_{10} O(1), \\
\delta\Psi_2 &= k_8 O(\zeta^{\sigma}) + k_9 \left\{O(\zeta^{\sigma-1}) + O(1)\right\} + k_{10} O(1), \\
\delta\Psi_3 &= k_8 O(\zeta^{\sigma}) + k_9 \left\{O(\zeta^{\sigma-1}) + O(1)\right\} + k_{10} O(1), \\
\delta\Psi_4 &= k_8 O(\zeta^{\sigma}) + k_9 \left\{O(\zeta^{\sigma}\ln\zeta) + O(1)\right\} + k_{10} O(\zeta),
\end{aligned}$$

where $\zeta = x - x_c$. We note, since $0 < \lambda < 1/8$,

$$\begin{aligned} 1 - 8\lambda + 16\lambda^2 &> 1 - 8\lambda, \\ 1 - 4\lambda &> \sqrt{1-8\lambda}, \\ 4\lambda &> \sqrt{1-8\lambda} - 1 + 8\lambda, \\ \frac{4\lambda}{\sqrt{1-8\lambda}(1-\sqrt{1-8\lambda})} &> 1. \end{aligned}$$

If $Re(s) > 3$, then $Re(s) - 1 > 2$, and using the above this means

$$\frac{4\lambda(s-1)}{\sqrt{1-8\lambda}(1-\sqrt{1-8\lambda})} > 2.$$

Thus $\sigma - 2 > 0$ if $Re(s) > 3$, and therefore all the scalars given above will be finite on $\zeta = x - x_c = 0$.

Thus every mode of the perturbation which is finite on the axis and the threshold will evolve onto and through the Cauchy horizon without divergence.

On the second future similarity horizon, $x = x_e$, there are indices

$$0, \quad 1, \quad -\frac{4\lambda(s-1)}{\sqrt{1-8\lambda}(1+\sqrt{1-8\lambda})} \equiv \varsigma.$$

Since $\varsigma < 0$ when $Re(s) > 3$, the second future similarity horizon is always unstable.

We arrive at the following result:

Proposition 6.2. *The Cauchy horizon which forms in the collapse of self-similar Vaidya spacetime is stable with respect to odd parity metric and matter perturbations of multipole mode $l \geq 2$. The following second future similarity horizon is unstable with respect to same.*

6.2.2 $l = 1$ mode, odd parity, Vaidya

In this sector, L_A and L are as in $l \geq 2$, and both are gauge invariant. k_A however is not gauge invariant. It transforms as

$$k_A \to k_A - r^2(\xi/r^2)_{,A}\,.$$

If we choose ξ such that

$$\left[r^2\left(\xi/r^2\right)_{,A}\right]^{|A} = k_A{}^{|A},$$

we will have transformed into a gauge in which k_A has no divergence, and thus we recover the second perturbation equation (4.2.13b). If we write k_A as $k^A = (\gamma_{,r}/r, -\gamma_{,x}/r)$, and Mellin transform γ to $r^s K(x;s)$, the first perturbation equation (4.2.13a) again gives two equations in Λ and K. The expression for Π reduces to (see equation (6.2.4))

$$\Pi = -\frac{2x\lambda\Lambda}{r^3 s} = -2d_0\lambda x^s r^{s-4}, \tag{6.2.5}$$

since the same equations for Λ as in the $l \geq 2$ case hold.

For $l = 1$, $S_{(a:b)} = 0$ and thus $S_{a:b}$ is antisymmetric. As $w^a w^b$ is symmetric for any vector w^a, then $w^a w^b S_{a:b} = 0$, and thus the angular part of $\delta\Psi_{0,4} = 0$ for $l = 1$. $\delta\Psi_2 = -\frac{i}{2}Y^m_{l=1}\Pi$ is non-vanishing and gauge invariant,

$$\begin{aligned}
\Pi - \overline{\Pi} &= \epsilon^{AB}\left(r^{-2}(k_A - \overline{k}_A)\right)_{|B}, \\
&= \epsilon^{AB}\left[r^{-2}r^2\left(\xi/r^2\right)_{,A}\right]_{|B}, \\
&= \epsilon^{AB}\left[\left(\xi/r^2\right)_{,A,B} - \tilde{\Gamma}^C{}_{AB}\left(\xi/r^2\right)_{,C}\right] = 0,
\end{aligned}$$

since the term in square brackets is symmetric. The remaining scalars are not gauge invariant, since

$$\delta\Psi_1 - \delta\overline{\Psi}_1 = \tfrac{1}{r}\tilde{w}^a S_a\left(\xi/r^2\right)_{,A}\tilde{l}^A, \qquad \delta\Psi_3 - \delta\overline{\Psi}_3 = \tfrac{1}{r}\tilde{w}^{*a} S_a\left(\xi/r^2\right)_{,A}\tilde{m}^A.$$

Thus the only gauge invariant scalar to measure in the odd parity $l = 1$ sector is $\delta\Psi_2$. As $\delta\Psi_2 = \Pi$, we see this scalar will evolve through the spacetime without divergence; and so all $l = 1$ odd parity modes remain finite throughout spacetime.

Chapter 7

Perturbations of LTB spacetime

We consider metric and matter perturbations of the self-similar spherically symmetric timelike dust model described in §2.4.4. We use the formalism and definitions set out in Chapter 4.

With regards to coordinates, we will initially use the co-moving coordinates t, r, with line element

$$ds^2 = -dt^2 + R'^2 dr^2 + R^2 d\Omega^2,$$

as in this coordinate system the perturbed matter tensor is sparse (in particular $T_{rr} = 0$), which means that there are more homogeneous equations. Then we take the set of perturbation field equations, and perform a change of dependent variable

$$(t, r) \to (y = \tfrac{t}{r}, r). \tag{7.0.1}$$

Finally we take a Mellin transform over r, resulting in a set of coupled, linear ODE's.

For the background be self-similar we must have $R = rG(y)$, with $G(y) = (1 - \mu y)^{2/3}$. When $0 < \mu < \mu_* \approx 0.638014$, the axis is given by $y = -\infty$, the past null cone of the origin is the negative root of $G - yG' + y = 0$, and the (distinct) future null cones of the origin are given by the positive roots of $G - yG' - y = 0$. The Cauchy horizon is the first positive root.

Some aspects of the following analysis are similar to previous chapters, so we will avoid unnecessary detail.

7.1 Even parity perturbations

We will consider only modes $l \geq 2$. We will assume the perturbed matter tensor remains that of a null dust,

$$\bar{t}_{\mu\nu} + \Delta t_{\mu\nu} = (\bar{\rho} + \delta\rho)(\bar{u}_\mu + \delta u_\mu)(\bar{u}_\nu + \delta u_\nu).$$

If we decompose as before, $\delta\rho = \varrho Y$ and $\delta u_\mu = \zeta_A Y + \zeta Y_{,a}$, then for $\bar{u}_\mu + \delta u_\mu$ to be a unit, future pointing, timelike geodesic of the perturbed spacetime, as the conservation equation $\nabla^\mu t_{\mu\nu} = 0$ implies must be the case, we must have $\delta u_\mu = \gamma_{,A} Y + \gamma Y_{,a}$ for some scalar γ. Additional, we must have

$$h_{tt} = -2\gamma_{,t}. \tag{7.1.1}$$

Including this from the beginning results in third order perturbation field equations; if we omit it, then this equation is recovered naturally from the field equations. We will omit it for now.

Using the Regge-Wheeler gauge, we may write the gauge invariant matter objects as

$$T_{AB} = \begin{pmatrix} \varrho + 2\tilde{\rho}\gamma_{,t} & \tilde{\rho}\gamma_{,r} \\ \tilde{\rho}\gamma_{,r} & 0 \end{pmatrix}, \quad T_A = \begin{pmatrix} \tilde{\rho}\gamma \\ 0 \end{pmatrix}, \quad T^1 = T^2 = 0.$$

Next we calculate the full set of perturbed field equations as given in (4.2.12). We use the equation $k_A{}^A = 0$ to remove $k_{rr} = R'^2 k_{tt}$, and we use the tt component of (4.2.12a) to define ϱ in terms of the other perturbation variables.

Thus we have a set of five second order, coupled, linear, partial differential equations in the four unknowns $\{k_{tt}, k_{tr}, k, \gamma\}$, and the two dependent variables t, r. (Had we removed k_{tt} with $k_{tt} = -2\gamma_{,t}$, these would be third order in γ.) Let us consider these as a set of equations, rather than components of tensors and so on. We make the change of coordinates (7.0.1), and then perform a Mellin transform over r, reducing the problem to five second order ordinary differential equations in y, the similarity variable, and parameterised by s, the transform parameter. The Mellin transforms of our unknowns can be written

$$k_{tt} = r^s A(y; s), \quad k_{tr} = r^s B(y; s), \quad k = r^s K(y; s), \quad \gamma = r^{s+1} H(y; s),$$

thus the four unknowns of our set of ODE's are $\{A, B, K, H\}$.

The future pointing ingoing and outgoing radial null geodesic tangents of the background spacetime in t, r coordinates are

$$\tilde{\ell}^A = \frac{1}{\sqrt{2}R'}\left(R', 1\right), \quad \tilde{n}^A = \frac{-1}{\sqrt{2}R'}\left(-R', 1\right)$$

respectively, since we restrict $R' > 0$ to avoid the shell crossing singularity occurring before the shell-focussing singularity. The $\mathcal{M}^2$ portion (i.e. neglecting the angular part) of the perturbed Weyl scalars then becomes

$$\delta\Psi_0 = \frac{1}{r^2}\left(k_{tt} + \frac{k_{tr}}{R'}\right), \qquad \delta\Psi_4 = \frac{1}{r^2}\left(k_{tt} - \frac{k_{tr}}{R'}\right).$$

After a change of coordinates, and Mellin transform, we may write each mode of these scalars in terms of A and B. We define a new variable $D = A + B/(G - yG')$, and the scalars' modes simplify to

$$\delta\Psi_0 = r^{s-2}D, \qquad \delta\Psi_4 = r^{s-2}(2A - D),$$

and $\delta P_{-1} = |\delta\Psi_0\delta\Psi_4|^{1/2}$. We must find solutions to the set of ODE's and use them to evaluate these modes on the relevant surfaces.

We can write this set of second order ODE's as a first order linear system $Y' = M(y)Y$ (see below), where a prime denotes differentiation w.r.t. y, and $Y = (A, D, K, H)^T$. We note that one of the equations in the system is $H' = -A/2$, and thus we have recovered (7.1.1), since $\partial/\partial t = \frac{1}{r}\partial/\partial y$. Due to its length, we give the components of the matrix M on the next page.

Examining the leading order coefficient matrix near the axis reveals that the axis is an irregular singular point, with multiple zero eigenvalues, and a number of off-diagonal entries in its Jordan normal form; all of which make the system methods described in §1.3.3 very unattractive. In any case, we anticipate that the system methods would break down when eigenvalues of leading matrices differ by integers, suggesting we would at some stage need to decouple an equation in one variable, and use its solution as an inhomogeneous term to integrate the other equations, as in the perturbed Vaidya spacetime.

Unfortunately, unlike the first order system derived for the Vaidya background, it can be shown that setting any one of the unknowns to zero results in a totally trivial solution. Thus the best we can do is decouple for each variable a fourth order ordinary differential equation. We will sketch the decoupling of an equation in H.

$$
\begin{aligned}
M_{11} = &G^4\left(l+l^2-2s^2+2sG'\right)-G^3y\left(l+l^2+2\left(s^2+s-1\right)-4\left(1-l-l^2-\tfrac{s}{2}+s^2\right)G'\right.\\
&\left.+\left(-1+8s\right)G'^2\right)+G^2y^2\left(l+l^2+2s+\left(-6+3l+3l^2+4s+2s^2\right)G'\right.\\
&\left.+\left(-14+6l+6l^2+s-2s^2\right)G'^2+2\left(-2+5s\right)G'^3\right)+y^4G'\left(-2+l+l^2+\right.\\
&\left.\left(-4+l+l^2\right)G'+\left(-4+l+l^2\right)G'^2+\left(-5+l+l^2\right)G'^3-G'^4\right)\\
&-Gy^3\left(-2+l+l^2+2\left(-2+l+l^2+s\right)G'+3\left(-3+l+l^2+s\right)G'^2+\right.\\
&\left.\left(-14+4l+4l^2-s\right)G'^3+4\left(-1+s\right)G'^4\right)\\
&/G\left(2G^2\left(1+s\right)-2G\left(1+s\right)yG'+y^2G'^2\right)\left(G-yG'-y\right)\left(G-yG'+y\right)\\
M_{12} = &G^4\left(l+l^2-2s^2\right)-2G^3\left(-2+2l+2l^2+s-2s^2\right)yG'+y^4G'^2\left(-4+l+l^2\right.\\
&\left.+\left(-5+l+l^2\right)G'^2\right)+Gy^3G'\left(-2\left(-2+l+l^2+s\right)+\left(14-4l-4l^2+s\right)G'^2\right)\\
&+G^2y^2\left(l+l^2+2s+\left(-14+6l+6l^2+s-2s^2\right)G'^2\right)\\
&/G\left(2G^2\left(1+s\right)-2G\left(1+s\right)yG'+y^2G'^2\right)\left(G-yG'-y\right)\left(G-yG'+y\right)\left(yG'-G\right)\\
M_{13} = &\left(G-yG'\right)\left(2G^3sG'+\left(-2+l+l^2\right)y^3\left(1+G'^2\right)+G^2y\left(l+l^2+2\left(-1+s+s^2\right)\right.\right.\\
&\left.\left.-5sG'^2\right)+Gy^2G'\left(-2\left(-2+l+l^2-s\right)+3sG'^2\right)\right)\\
&/G\left(2G^2\left(1+s\right)-2G\left(1+s\right)yG'+y^2G'^2\right)\left(G-yG'-y\right)\left(G-yG'+y\right)\\
M_{14} = &-2G'^2\left(G^3\left(-1+s\right)-2G^2\left(-2+s\right)yG'+y^3G'\left(2+G'^2\right)+Gy^2\left(-3-s+\right.\right.\\
&\left.\left.\left(-4+s\right)G'^2\right)\right)\\
&/G\left(2G^2\left(1+s\right)-2G\left(1+s\right)yG'+y^2G'^2\right)\left(G-yG'-y\right)\left(G-yG'+y\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{21} = &\left(G-yG'\right)\left(G-y\left(1+G'\right)\right)\left(y^2\left(-2+l+l^2-2G'\right)G'\left(1+G'\right)+G^2\left(l+l^2\right.\right.\\
&\left.\left.+2s+2sG'\right)-Gy\left(-2+l+l^2+2\left(-2+l+l^2+s\right)G'+\left(-3+2s\right)G'^2\right)\right)\\
&/G\left(G-yG'+y\right)\left(2G^2\left(1+s\right)-2G\left(1+s\right)yG'+y^2G'^2\right)\\
M_{22} = &-2G^4\left(l+l^2-2s^2\right)-y^4\left(2\left(-4+l+l^2\right)-G'\right)G'^3\left(1+G'\right)+2G^3y\left(l+l^2+2s\right.\\
&\left.+2\left(-2+2l+2l^2+s-2s^2\right)G'-\left(1+s\right)G'^2\right)+2G^2y^2G'\left(4-3l-3l^2-4s\right.\\
&\left.+\left(13-6l-6l^2-2s+2s^2\right)G'+2\left(1+s\right)G'^2\right)\\
&+Gy^3G'^2\left(6l+6l^2+4\left(-4+s\right)+\left(-26+8l+8l^2\right)G'-\left(3+2s\right)G'^2\right)\\
&/2G\left(2G^2\left(1+s\right)-2G\left(1+s\right)yG'+y^2G'^2\right)\left(G-yG'\right)\left(G-yG'+y\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{23} = & \left(G - y\,G'\right)^2\,\left(-2\,G^2\,s\,\left(1+s-G'\right) - \left(-2+l+l^2\right)\,y^2\,\left(1+G'\right)\right.\\
& \left.+G\,y\,\left(-2+l+l^2-2\,s\,G'-3\,s\,G'^2\right)\right)\\
& /G\,\left(G-y\,G'+y\right)\,\left(2\,G^2\,(1+s)-2\,G\,(1+s)\,y\,G'+y^2\,G'^2\right)\\
M_{24} = & -2\,G'^2\,\left(-G+y\,G'\right)\,\left(-\left(G^2\,(-1+s)\right)+y^2\,G'\,\left(2+G'\right)\right.\\
& \left.+G\,y\,\left(-3-s+(-3+s)\,G'\right)\right)\\
& /G\,\left(G-y\,G'+y\right)\,\left(2\,G^2\,(1+s)-2\,G\,(1+s)\,y\,G'+y^2\,G'^2\right)
\end{aligned}
$$

$$
\begin{aligned}
M_{31} = & \left(-\left(y^2\,\left(-2+l+l^2-2\,G'\right)\,G'\,\left(1+G'\right)\right)-G^2\,\left(l+l^2+2\,s+2\,s\,G'\right)\right.\\
& \left.+G\,y\,\left(-2+l+l^2+2\,\left(-2+l+l^2+s\right)\,G'+(-3+2\,s)\,G'^2\right)\right)\\
& /G\,\left(2\,G^2\,(1+s)-2\,G\,(1+s)\,y\,G'+y^2\,G'^2\right)\\
M_{32} = & \left(G\,\left(l+l^2+2\,s\right)-\left(l^2+l-4\right)\,y\,G'\right)/G\,\left(2\,G^2\,(1+s)-2\,G\,(1+s)\,y\,G'+y^2\,G'^2\right)\\
M_{33} = & \left(-2\,G^2\,s\,G'+\left(-2+l+l^2\right)\,y^2\,G'-G\,y\,\left(-2+l+l^2-3\,s\,G'^2\right)\right)\\
& /G\,\left(2\,G^2\,(1+s)-2\,G\,(1+s)\,y\,G'+y^2\,G'^2\right)\\
M_{34} = & \left(2\,G'^2\,\left(-\left(G\,(3+s)\right)+2\,y\,G'\right)\right)/G\,\left(2\,G^2\,(1+s)-2\,G\,(1+s)\,y\,G'+y^2\,G'^2\right)
\end{aligned}
$$

$$
M_{41} = -1/2 \qquad M_{42} = 0 \qquad M_{43} = 0 \qquad M_{44} = 0.
$$

This system can be written as four first order equations,

$$
h_1(A,D,K,H,A') = 0, \;\; h_2(A,D,K,H,D') = 0, \;\; h_3(A,D,K,H,K') = 0, \;\; h_4(A,H') = 0.
$$

We solve the first equation for $D = f_1(A,K,H,A')$ and substitute this into the other three equations, giving

$$
h_5(A,K,H,A',K',H',A'') = 0, \quad h_6(A,K,H,A',K') = 0, \quad h_4(A,H') = 0.
$$

Combining h_5 and h_6 to remove K' means we can solve for $K = f_2(A,H,A',H',A'')$, and we are left with two equations,

$$
h_7(A,H,A',H',A'',H'',A''') = 0, \quad h_4(A,H') = 0.
$$

Finally we remove A for a fourth order ODE* in H,

$$h_0(H, H', H'', H''', H'''') = 0.$$

The other variables can be calculated when the solutions for H are found, as

$$A = g_1(H'), \quad D = g_2(H, H', H'', H'''), \quad K = g_3(H, H', H'', H'''),$$

and thus we can write the scalars $\delta\Psi_{0,4}$ in terms of H and its derivatives.

Axis.

We consider first the axis, $y = -\infty$. We make the transformation $y = -1/w$ to put the axis at $w = 0$, and then the transformation $w = z^3$ to ensure integer exponents in the series expansions about the axis of the coefficients of the differential equation. We find $z = 0$ is a regular singular point of the fourth order ODE in H, which we will write as

$$\sum_{j=0}^{4} z^j \left[h_j + O(z)\right] \frac{d^j H}{dz^j} = 0, \qquad h_j \neq 0. \tag{7.1.2}$$

We may write this fourth order equation as a 4-d system, by letting [9]

$$\mathcal{A} = H, \quad \mathcal{B} = zH', \quad \mathcal{C} = z^2 H'', \quad \mathcal{D} = z^3 H'''.$$

Then if $\mathcal{Z} = (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})^T$, we have the system $d\mathcal{Z}/dz = \frac{1}{z}\left(\sum_{i=0}^{\infty} N_i z^i\right)\mathcal{Z}$, where

$$N_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -h_0/h_4 & -h_1/h_4 & -h_2/h_4 & 3 - h_3/h_4 \end{pmatrix}.$$

Thus $z = 0$ is again a regular singular point, however the eigenvalues are

$$\{-3, 2, -4 - l - 3s, -3 + l - 3s\}. \tag{7.1.3}$$

These eigenvalues certainly differ by integers, and particularly when $s \in \mathbb{Z}$ we cannot

*Needless to say these manipulations were not carried out by hand. The coefficients in these differential equations are exceedingly long and ugly, and generating this equation in H tested Mathematica, and my computer, to its limits.

solve using system methods.

Thus we have no alternative but to use the method of Frobenius for this fourth order ODE. The indicial exponents near $z = 0$ are the same as the eigenvalues found above, (7.1.3). The ambiguity of the value of s complicates matters regarding the position of logarithmic terms, so we will begin by reminding ourselves of the two conditions the solutions must solve:

1. The solution must exist; that is we must be able to recover the original function from its Mellin transform. Our minimum stability requirement for this to hold is that an acceptable solution is one which does not diverge on the relevant surface.

2. The solution must be such that each mode of $\delta\Psi_{0,4}$ is finite on the axis.

Consider the indicial exponent -3. Regardless of the values of the other exponents, the corresponding solution will contain at least the series $\sum_{m=0}^{\infty} A_m z^{m-3}$. This is certainly not convergent, it diverges at $z = 0$ ($A_0 \neq 0$). Thus we must not consider this solution.

In examining requirement 2, we expand the coefficients of H and its derivatives in $\delta\Psi_{0,4}$ around $z = 0$, and we find the dominant term is

$$\delta\Psi_{0,4} \sim r^{s-2} z^4 H'.$$

Consider the indicial exponent $l - 3s - 3$. A solution due to this eigenvalue is

$$(\text{Logarithmic terms}) \times (\text{Series}) + \sum_{m=0}^{\infty} A_m z^{m+l-3s-3},$$

where the first portion of this solution depends on the other eigenvalues, and may not even be present. Near $z = 0$, we find $\delta\Psi_{0,4} \sim z^{l-6}$ due to the second term. We would certainly expect these scalars to be finite on the axis for the quadrupole and other modes with $l < 6$, thus we must rule out this solution.

Similarly for the indicial exponent $-4 - l - 3s$, we find $\delta\Psi_{0,4} \sim z^{-l-7}$ near $z = 0$. Thus we must also rule out this solution.

Finally for indicial exponent 2, we see the solution

$$H = \sum_{m=0}^{\infty} A_m z^{m+2} \tag{7.1.4}$$

is convergent (near $z = 0$) $\forall\ s$, and thus satisfies our minimum stability requirement. Further, the scalars $\delta\Psi_{0,4}$ will be finite on the axis for $Re(s) \geq 1/3$. Thus we have found a one-parameter family of solutions near the axis, as expected.

Past null cone

The past null cone, $y = y_p$, is the real, negative root of $G - yG' + y = 0$, where $G(y) = (1-\mu y)^{2/3}$ and

$$y_p = \frac{-\left(-27+\mu^3\right)}{108\,\mu} + \frac{\sqrt{\frac{2\,\mu}{9} + \frac{(-27+\mu^3)^2}{2916\,\mu^2} - \frac{3\,2^{\frac{1}{3}}}{(-27+2\,\mu^3+Q)^{\frac{1}{3}}} + \frac{(-27+2\,\mu^3+Q)^{\frac{1}{3}}}{3\,2^{\frac{1}{3}}\,\mu}}}{2}$$
$$-\frac{1}{2}\sqrt{\left(\frac{4\,\mu}{9} + \frac{(-27+\mu^3)^2}{1458\,\mu^2} + \frac{3\,2^{\frac{1}{3}}}{(-27+2\,\mu^3+Q)^{\frac{1}{3}}} - \frac{(-27+2\,\mu^3+Q)^{\frac{1}{3}}}{3\,2^{\frac{1}{3}}\,\mu}\right.}$$
$$\left. + \frac{-8 - \frac{4\left(-27+\mu^3\right)}{81} - \frac{\left(-27+\mu^3\right)^3}{19683\,\mu^3}}{4\sqrt{\frac{2\,\mu}{9} + \frac{(-27+\mu^3)^2}{2916\,\mu^2} - \frac{3\,2^{\frac{1}{3}}}{(-27+2\,\mu^3+Q)^{\frac{1}{3}}} + \frac{(-27+2\,\mu^3+Q)^{\frac{1}{3}}}{3\,2^{\frac{1}{3}}\,\mu}}}\right),$$

with $Q(\mu) = \sqrt{729 + 2808\,\mu^3 + 4\,\mu^6}$. There is only one real root (when $0 < \mu < \mu_*$), and it is parameterized by μ. Thus we may write

$$G - yG' + y = (y - y_p)F(y), \qquad F(y_p) \neq 0.$$

y_p is a very cumbersome surd, and is quite difficult to work with. Instead, we draw out the nature of the coefficients of the H-equation by using $G'(y_p) = (G(y_p) + y_p)/y_p$. We find that setting $G' = (G+y)/y$ makes each coefficient vanish, except for the coefficient of the highest derivative. Thus we may write the H-equation as

$$(G - yG' + y)[m_0 + O(y-y_p)]H^{(4)} + [n_0 + O(y-y_p)]H^{(3)} + [p_0 + O(y-y_p)]H^{(2)}$$
$$+[q_0 + O(y-y_p)]H' + [r_0 + O(y-y_p)]H = 0,$$

where m_0, n_0 etc. are the first nonzero terms in the series expansions about the past null cone.

We may write this in canonical form as

$$(y-y_p)^4 H^{(4)} + (y-y_p)^3\, b_1(y) H^{(3)} + (y-y_p)^2\, b_2(y) H'' + \ldots = 0.$$

If the series expansions of the b_i about the past null cone are denoted $b_i = \sum_{j=0}^{\infty} b_{i,j}\,(y -$

$y_p)^j$, then the first few terms in the expansions of the b_j about $y = y_p$ are

$$\begin{array}{lllll} b_{1,0} = n_0/(m_0 F_p) & b_{1,1} = \dots & & & \\ b_{2,0} = 0 & b_{2,1} = p_0/(m_0 F_p) & b_{2,2} = \dots & & \\ b_{3,0} = 0 & b_{3,1} = 0 & b_{3,2} = q_0/(m_0 F_p) & b_{3,3} = \dots & \\ b_{4,0} = 0 & b_{4,1} = 0 & b_{4,2} = 0 & b_{4,3} = r_0/(m_0 F_p) & b_{4,4} = \dots \end{array} \tag{7.1.5}$$

where $F_p = F(y_p)$. Therefore $y = y_p$ is a regular singular point of this ordinary differential equation, and the indicial equation for a fourth order ODE is

$$\lambda(\lambda-1)(\lambda-2)(\lambda-3) + b_{1,0}\lambda(\lambda-1)(\lambda-2) + b_{2,0}\lambda(\lambda-1) + b_{3,0}\lambda + b_{4,0} = 0.$$

Thus the indicial exponents are

$$\{\ 0, 1, 2, 3 - b_{1,0} \equiv \sigma\ \}.$$

To determine what exactly σ is, we note

$$F_p = F(y_p) = \lim_{y \to y_p} \frac{G - yG' + y}{y - y_p} = 1 - y_p G''(y_p),$$

using l'Hôpital's rule, and thus

$$\sigma = 3 - \left[\frac{7 - s + 2\left(\frac{y}{G} + \frac{G}{y}\right)}{1 - yG''}\right]_{y=y_p}. \tag{7.1.6}$$

We note that $\sigma = s$ in the limit $\mu \to 0$.

We may find the solutions due to these indicial exponents from the analysis in §1.3.2. Let us consider first the case $\sigma \notin \mathbb{Z}$. The analysis for a $\{0, 1, 2\}$ set of indicial exponents was carried out in detail in §1.3.2, and the general solution contained three logarithmic terms, (1.2.5), each multiplied by a constant. For the fourth order ODE in H we are considering here, these constants are

$$\begin{aligned} \lim_{\lambda \to 1} \left[(\lambda - 1)A_1(\lambda)\right] &= \frac{(b_{3,1} + b_{4,1})}{(2b_{1,0} - 2)}, \\ \lim_{\lambda \to 0} \left[\frac{d}{d\lambda}(\lambda^2 A_1)\right] &= \frac{b_{4,1}}{(2 - b_{1,0})}, \\ \lim_{\lambda \to 0} \left[\lambda^2 A_2\right] &= \frac{b_{4,1}(b_{3,1} + b_{4,1})}{(2 - b_{1,0})(2b_{1,0} - 2)}, \end{aligned}$$

where we have set $A_0 = 1$. Crucially, since [(7.1.5)] $b_{3,1} = b_{4,1} = 0$, each of these terms vanish, and thus when $\sigma \notin \mathbb{Z}$, we have a general solution

$$H|_{y=y_p} = h_1 \sum_{m=0}^{\infty} A_m (y - y_p)^{m+2} + h_2 \sum_{m=0}^{\infty} B_m (y - y_p)^{m+1} + h_3 \sum_{m=0}^{\infty} C_m (y - y_p)^m + h_4 \sum_{m=0}^{\infty} D_m (y - y_p)^{m+\sigma}, \quad (7.1.7)$$

with each series linearly independent. Our minimum stability requirement for these solutions will be satisfied for $Re(\sigma) > 0$.

Now we examine the scalars $\delta\Psi_{0,4}$ near the past null cone, and we find we can write

$$\delta\Psi_0 \sim c_1 H + c_2 H' + c_3 H'' + c_4 (y - y_p) H^{(3)}, \quad (7.1.8)$$

with a similar expression for $\delta\Psi_4$. The scalars are automatically finite on $y = y_p$ for the first three series in (7.1.7). For the fourth series, we find surprisingly that $c_3\sigma(\sigma - 1) + c_4\sigma(\sigma - 1)(\sigma - 2) = 0$ for both $\delta\Psi_0$ and $\delta\Psi_4$; that is the coefficient of the leading term, which goes like $(y - y_p)^{\sigma-2}$, vanishes exactly. Thus for finite scalars on the past null cone due to the fourth solution, we require only $Re(\sigma) > 1$.

Now let us consider $\sigma \in \mathbb{Z}$. Firstly if $\sigma < 0$, the minimum stability requirement is not met and we certainly cannot recover γ from H via the inverse Mellin transform; thus we consider $\sigma \geq 0$. Now we note an important point regarding the Frobenius method: if two indicial exponents differ by an integer, the solution corresponding to the lowest index may contain a logarithmic term; however if two indicial exponents are equal, the second solution *must* contain a logarithmic term.

If $\sigma = 0$, then there will be a solution which has leading term $\ln(y - y_p)$, which diverges at the past null cone, and thus the minimum stability requirement is not satisfied. If $\sigma = 1$, the corresponding leading term is $(y - y_p)\ln(y - y_p)$, which is finite in the limit $y \to y_p$. Thus we only consider $\sigma > 0$, when $\sigma \in \mathbb{Z}$.

When calculating the scalars $\delta\Psi_{0,4}$, we see from (7.1.8) that if $\sigma = 1$, then $H' \sim \ln(y - y_p)$, and thus we must discount $\sigma = 1$. Again, when $\sigma = 2$, we find $H'' \sim \ln(y - y_p)$, however for $\sigma \geq 3$ we have $\delta\Psi_{0,4} \sim O(1)$.

Thus for the scalars to be finite on the past null cone, we require s to be such that $Re(\sigma) > 1$, with the exception of $\sigma = 2$.

Cauchy Horizon

The situation on the Cauchy horizon is very similar to the past null cone: a fourth order ODE in H with $y = y_c$ as a regular singular point, series expansions about $y = y_c$ of the coefficients of the differential equation in the form (7.1.5), and indicial exponents $\{0, 1, 2, \bar{\sigma}\}$ where

$$\bar{\sigma} = 3 - \left[\frac{7 - s - 2\left(\frac{y}{G} + \frac{G}{y}\right)}{1 + yG''}\right]_{y=y_c}, \qquad \lim_{\mu \to 0} \bar{\sigma} = s. \tag{7.1.9}$$

The future similarity horizons, y_c and y_e, are given by

$$y_{c/e} = \frac{-\left(-27 - \mu^3\right)}{108\,\mu} + \frac{\sqrt{\frac{1}{4\,\mu^2} - \frac{11\,\mu}{54} + \frac{\mu^4}{2916} + \frac{3\,2^{\frac{1}{3}}}{\left(27+2\,\mu^3+\bar{Q}\right)^{\frac{1}{3}}} + \frac{\left(27+2\,\mu^3+\bar{Q}\right)^{\frac{1}{3}}}{3\,2^{\frac{1}{3}}\,\mu}}}{2}$$
$$\pm\frac{1}{2}\sqrt{\left(\frac{-4\,\mu}{9} + \frac{\left(-27 - \mu^3\right)^2}{1458\,\mu^2} - \frac{3\,2^{\frac{1}{3}}}{\left(27 + 2\,\mu^3 + \bar{Q}\right)^{\frac{1}{3}}} - \frac{\left(27 + 2\,\mu^3 + \bar{Q}\right)^{\frac{1}{3}}}{3\,2^{\frac{1}{3}}\,\mu}\right.}$$
$$\left. + \frac{8 + \frac{4\left(-27-\mu^3\right)}{81} - \frac{\left(-27-\mu^3\right)^3}{19683\,\mu^3}}{4\sqrt{\frac{1}{4\,\mu^2} - \frac{11\,\mu}{54} + \frac{\mu^4}{2916} + \frac{3\,2^{\frac{1}{3}}}{\left(27+2\,\mu^3+\bar{Q}\right)^{\frac{1}{3}}} + \frac{\left(27+2\,\mu^3+\bar{Q}\right)^{\frac{1}{3}}}{3\,2^{\frac{1}{3}}\,\mu}}}\right),$$

where $+$ is for y_e and $-$ is for y_c, and $\bar{Q} = \sqrt{729 - 2808\,\mu^3 + 4\,\mu^6}$.

When $\bar{\sigma} \notin \mathbb{Z}$, all the logarithmic terms in the general solution vanish as at the past null cone. The scalar $\delta\Psi_0$ can be written near $y = y_c$ as

$$\delta\Psi_0 \sim \bar{c}_1 H + \bar{c}_2 H' + \bar{c}_3 H'' + \bar{c}_4 (y - y_c) H^{(3)},$$

with a similar expansion for $\delta\Psi_4$. Again, the coefficient of the leading term due to the solution due to the indicial exponent $\bar{\sigma}$ vanishes, and we find the scalars will be finite on the Cauchy horizon iff $Re(\bar{\sigma}) > 1$, when $\bar{\sigma} \notin \mathbb{Z}$.

When $\bar{\sigma} \in \mathbb{Z}$, we find, for the same reasons as at the past null cone, we must rule out $\bar{\sigma} \leq 1$; when $\bar{\sigma} = 2$ the scalars diverge like $\ln(y - y_c)$; and when $\bar{\sigma} \geq 3$ the scalars are finite on the Cauchy horizon.

Let us consider first the clearer picture, when neither σ or $\bar{\sigma}$ are integers. Both σ and $\bar{\sigma}$ are parameterized by s and μ, and thus we can plot the line in the $Re(s), \mu$ parameter

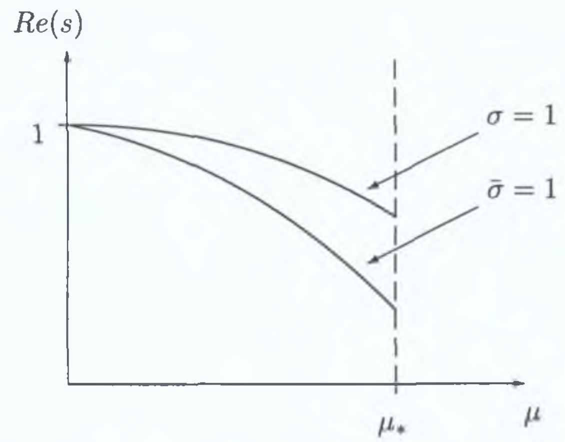


Figure 7.1: The lines $\sigma = 1$ and $\bar{\sigma} = 1$ plotted in the $Re(s), \mu$ parameter space for $0 < \mu < \mu_*$ (even parity perturbation).

space where $\sigma = 1$ and $\bar{\sigma} = 1$. We give this schematically in Figure 7.1 for $0 < \mu < \mu_*$.

We interpret this plot so: for every μ, if $Re(s)$ is such that the point $(Re(s), \mu)$ is above the line $\sigma = 1$, the scalars will be finite on the past null cone. Similarly, if $Re(s)$ is such that the point $(Re(s), \mu)$ is above the line $\bar{\sigma} = 1$, the scalars will be finite on the Cauchy horizon. As the $\bar{\sigma} = 1$ line is always below the $\sigma = 1$ line for $0 < \mu < \mu_*$, this means that *all* perturbations which are finite on the past null cone will be finite on the Cauchy horizon, when $\sigma, \bar{\sigma} \notin \mathbb{Z}$.

When $\sigma, \bar{\sigma} \in \mathbb{Z}$, the picture is a touch more intricate, due to the fact that $\sigma = 2$ or $\bar{\sigma} = 2$ will give a divergence in the scalars. Consider Figure 7.2, and let's choose a particular value for μ, μ_0 where $0 < \mu_0 < \mu_*$. The solid portion of the line $\mu = \mu_0$ represents all the allowable values (from the point of view of initial data) of $Re(s)$ for this μ_0, with the exception of where the line intersects $\sigma = 2$. We see that this line must intersect $\bar{\sigma} = 2$ at some point (μ_0, s^*), represented by the black dot in Figure 7.2.

This point represents a precise value of s for which, if we were to perform the inverse Mellin transform over the vertical contour in the complex plane of s given by $Re(s) = s^*$, the perturbation variables thus returned would generate finite scalars $\delta\Psi_{0,4}$ on the past null cone of the origin, but *diverging* scalars on the Cauchy horizon. However, we maintain this is not enough to conclude the Cauchy horizon is unstable, for the following reasons:

1. From our definition of $\bar{\sigma}$ (7.1.9), for $\bar{\sigma} = 2$, a real integer, we require $s = s^* \in \mathbb{R}$.

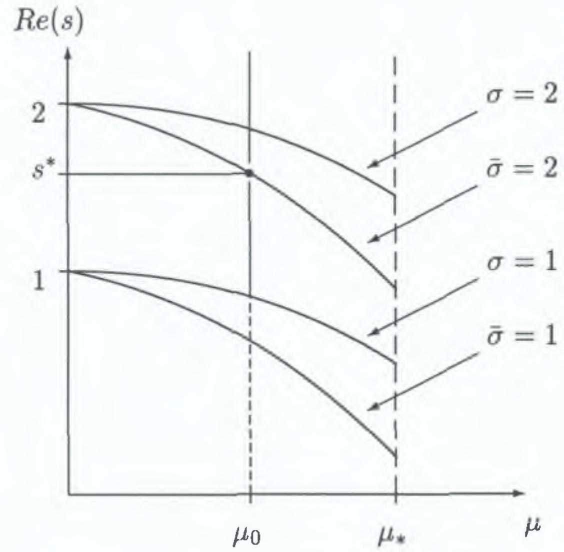


Figure 7.2: The lines $\sigma = 1, 2$ and $\bar{\sigma} = 1, 2$ plotted in the $Re(s), \mu$ parameter space for $0 < \mu < \mu_*$ (even parity perturbation).

Thus there is only a single, isolated point in the s complex plane at which s is such that $\bar{\sigma} = 2$, and it lies on the real axis. When performing the inverse Mellin transform, we must integrate over the contour $Re(s) = s^*$ in the complex plane, where $\varsigma_1 < s^* < \varsigma_2$, as in Figure 7.3. Thus the function γ is recovered as

$$\gamma(y, r) = \frac{1}{2\pi i} \int_{s^*-i\infty}^{s^*+i\infty} r^s H(y; s) ds.$$

A well known theorem in complex analysis (Cauchy's integral theorem), states that we may continuously deform the contour of integration if the region thus swept out does not contain any poles. From our solution for H when $s = s^*$ (and thus $\bar{\sigma} = 2$), we see that the integrand has no poles due to the value of y. That the integrand has no poles due to the value of s is a technically very difficult question to address fully, and is beyond the scope of this thesis, as mentioned in §3.2. However, some analysis in this direction was carried out by Nolan in Section 6 of [41], and there is evidence that no poles would be encountered in the general solution for H.

Thus when performing the inverse Mellin transform we may avoid the single, isolated point which makes the scalars diverge.

2. We stress that integrating over the contour given by *any* other value for $Re(s)$, other than those ruled out for the sake of initial data, will recover perturbation

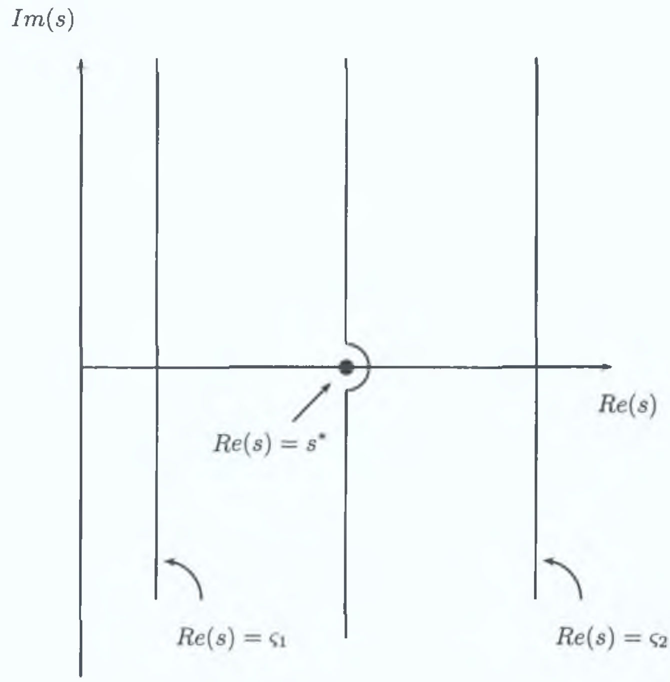


Figure 7.3: Integrating over a contour in the complex plane of s.

variables which generate finite scalars on the Cauchy horizon. The diverging mode corresponds to a single isolated point, that is a set of zero measure, in the s plane. This is not generic in any sense; to conclude an unstable Cauchy horizon we would be looking for an open region in the s plane in which the modes diverge.

3. Note that $\bar{\sigma} = 2$ lies between $\sigma = 1$ and $\sigma = 2$. Thus the value $Re(s) = s^*$ means non-integer exponents in the solution for H near the past null cone; that is the solution is non-analytic. From the point of view of critical collapse, we would restrict our initial data to only consider analytic perturbations, and thus would avoid the diverging mode altogether.

For these three reasons we conclude that the Cauchy horizon formed in the collapse of self-similar timelike dust is stable under even parity perturbations with $l \geq 2$.

SFSH

On the second future similarity horizon, denoted $y = y_e$, we find indicial exponents for the fourth order ODE in H as $\{0, 1, 2, \bar{\bar{\sigma}}\}$, where

$$\bar{\bar{\sigma}} = 3 - \left[\frac{7 - s - 2\left(\frac{y}{G} + \frac{G}{y}\right)}{1 + yG''} \right]_{y=y_e}, \qquad \lim_{\mu \to 0} \bar{\bar{\sigma}} = -1. \tag{7.1.10}$$

Again we find the scalars go like $(y - y_e)^{\bar{\bar{\sigma}}-1}$. We may write

$$\bar{\bar{\sigma}} - 1 = \alpha(\mu)s + \beta(\mu).$$

Our initial data confined $Re(s) > 0$, and it is easily found that for $0 < \mu < \mu_*$, both $\alpha(\mu)$ and $\beta(\mu)$ are always negative. Thus the scalars $\delta\Psi_{0,4}$ will diverge on the SFSH for *all* values of $Re(s)$ allowed by initial data (in contrast to the Cauchy horizon). Thus the second future similarity horizon is *unstable* under even parity perturbations with $l \geq 2$.

Proposition 7.1. *The Cauchy horizon formed in the collapse of the self-similar Lemaître-Tolman-Bondi spacetime is stable with respect to even parity metric and matter perturbations of multipole mode $l \geq 2$. The second future similarity horizon is unstable with respect to same.*

These perturbations model gravitational waves impinging on the background spacetime, and thus interesting features are expected for modes $l \geq 2$, for the reasons set out in Chapter 4. In the two preceding chapters, we included the low modes for completeness, however it was felt unnecessary to do so in this chapter. While the same reasoning applies for the odd parity sector, after a question at a recent conference from an expert in the field we decided to include a brief discussion of the odd parity $l = 1$ mode.

7.2 Odd parity perturbations

First we consider modes $l \geq 2$. As before we assume the perturbed matter tensor is that of a timelike dust,

$$\tilde{t}_{\mu\nu} + \Delta t_{\mu\nu} = \bar{\rho}(\bar{u}_\mu + \delta u_\mu)(\bar{u}_\nu + \delta u_\nu).$$

Writing $\delta u_\mu dx^\mu = \Lambda S_a dx^a$, we find

$$L_A = \bar{\rho}\bar{u}_A \Lambda, \quad L = 0, \tag{7.2.1}$$

and both are gauge invariant since $\bar{t}^a{}_a = 0$. There are two perturbation field equations,

$$- \left[r^4 (k^A/r^2)^{|C} - r^4 (k^C/r^2)^{|A} \right]_{|C} + (l-1)(l+2) k^A = 16\pi r^2 L^A, \tag{7.2.2}$$

$$k^A{}_{|A} = 16\pi L. \tag{7.2.3}$$

The second equation, using co-moving coordinates t, r, reads

$$\dot{\alpha} + \alpha \frac{\dot{R}'}{R'} + \beta' + \beta \frac{R''}{R'} = 0, \qquad \text{where} \qquad k^A = (\alpha, \beta),$$

and is solved by $k^A = \frac{1}{R'}(\gamma', -\dot{\gamma})$ for any function γ.

The vector perturbation field equation has two entries, of the form

$$f_1(\Lambda, \gamma, \ldots, \ddot{\gamma}') = 0, \qquad f_2(\gamma, \ldots, \dddot{\gamma}) = 0.$$

We may solve $f_1 = 0$ for M, and use this to write the scalars entirely in terms of γ. There is a further equation for Λ, that is the conservation equation (4.2.14), and to satisfy this we must have $\partial\Lambda/\partial t = 0$ (this is the same equation for Λ as found from requiring $\delta^t_A + \Lambda S_a$ be a timelike geodesic). Differentiating f_1 w.r.t. t means we can write down a fourth order equation in γ, however this is simply a derivative of f_2; thus the perturbation is completely specified by solving $f_2 = 0$ for γ, and then solving $f_1 = 0$ for Λ.

Now we use the fact that the background is self-similar to transform to similarity coordinates (y, r) where $y = t/r$ and $R = rG(y)$. We perform a Mellin transform of $f_2 = 0$ over r such that $\gamma = r^s K(y; s)$, and we have a third order ODE in K with regular singular points at the axis, past null cone and future null cones. The scalars are written in terms of r and K.

Axis

We make the coordinate transformation $y \to -1/z^3$, and we find $z = 0$ is a regular singular point with indicial exponents

$$\{0, 1 - l - 3s, 2 + l - 3s\}.$$

Considering $Re(s) \notin \mathbb{Z}$ first, we can calculate the leading behaviour of the scalars due to a general solution

$$\begin{aligned} K|_{axis} &= k_1 \sum_{m=0}^{\infty} A_m z^m + k_2 \sum_{m=0}^{\infty} B_m z^{m+2+l-3s} \\ &\quad + k_3 \left(\hat{k} \ln z \sum_{m=0}^{\infty} B_m z^{m+2+l-3s} + \sum_{m=0}^{\infty} C_m z^{m+1-l-3s} \right). \end{aligned}$$

Since, near the axis, $r \sim z^3$, we find the leading behaviour of the scalars to be (neglecting the logarithmic term for the moment),

$$\delta\Psi_n \sim k_1 z^{3s-8+2n} + k_2 z^{l-6+2n} + k_3 z^{-l-7+2n}, \qquad n = 0, \ldots 4.$$

Thus $\delta P_0 = \delta\Psi_2$ goes like $k_3 z^{-l-3}$, and we therefore must set $k_3 = 0$, which also removes the logarithmic term in the general solution.

For each mode of the scalars to be finite due to the k_1 solution we require $Re(s) \geq 4/3$, however for the minimum stability requirement to be satisfied by the k_2 solution we require $Re(s) \leq (l+2)/3$. For $l = 2$ this means only $Re(s) = 4/3$ satisfies both requirements, however this is too restrictive for the inverse Mellin transform, and for analysis on the past null cone, etc. Thus we set $k_2 = 0$, and we find a one-parameter family of solutions on the axis, with the restriction $Re(s) \geq 4/3$.

Past null cone

We may write the third order ODE for K near the past null cone as

$$\begin{aligned} (G - yG' + y)[m_0 + O(y-y_p)]K^{(3)} + [n_0 + O(y-y_p)]K'' + [p_0 + O(y-y_p)]K' \\ + [q_0 + O(y-y_p)]K = 0, \end{aligned}$$

with m_0, n_0 etc. constants, and in canonical form,

$$(y-y_p)^3 K^{(3)} + (y-y_p)^2 b_1 K'' + \ldots = 0,$$

with the first few terms in the expansions of the coefficients around $y = y_p$ as

$$\begin{array}{llll} b_{1,0} = n_0/(m_0 F_p) & b_{1,1} = \ldots & & \\ b_{2,0} = 0 & b_{2,1} = p_0/(m_0 F_p) & b_{2,2} = \ldots & \\ b_{3,0} = 0 & b_{3,1} = 0 & b_{3,2} = q_0/(m_0 F_p) & b_{3,3} = \ldots \end{array} \tag{7.2.4}$$

where $F_p = \lim_{y\to y_p}(G - yG' + y)/(y - y_p)$. Thus the indicial exponents are $\{0, 1, \sigma\}$, where

$$\sigma = 2 - n_0/(m_0 F_p) = \left.\frac{2(s-1)yG}{y^2 + 4yG + G^2}\right|_{y=y_p}, \qquad \lim_{\mu\to 0} \sigma = (s-1).$$

Let us consider $\sigma \notin \mathbb{Z}$ first. As 0 and 1 certainly differ by an integer, we use the analysis of §1.3.2 to find the coefficient of the logarithmic term in the solution due to the 0 indicial exponent as

$$\lim_{\lambda\to 0}\left[\lambda A_1\right] = -\frac{b_{3,1}}{1 + b_{1,0}} = 0,$$

but from (7.2.4) we see $b_{3,1} = 0$. Thus the general solution near the past null cone will be

$$K|_{y=y_p} = k_4 \sum_{m=0}^{\infty} A_m (y - y_p)^{m+1} + k_5 \sum_{m=0}^{\infty} B_m (y - y_p)^m + k_6 \sum_{m=0}^{\infty} C_m (y - y_p)^{m+\sigma}.$$

Using this solution we calculate the leading behaviour of the five scalars,

$$\begin{aligned}
\delta\Psi_0 &= k_4 O(1) + k_5 O(1) + k_6 O(y - y_p)^{\sigma-2},\\
\delta\Psi_1 &= k_4 O(1) + k_5 O(1) + k_6 O(y - y_p)^{\sigma-1},\\
\delta\Psi_2 &= k_4 O(1) + k_5 O(1) + k_6 O(y - y_p)^{\sigma},\\
\delta\Psi_3 &= k_4 O(1) + k_5 O(1) + k_6 O(y - y_p)^{\sigma},\\
\delta\Psi_4 &= k_4 O(1) + k_5 O(1) + k_6 O(y - y_p)^{\sigma}.
\end{aligned}$$

Thus when $\sigma \notin \mathbb{Z}$, we require $Re(\sigma) > 2$. We plot this in the $Re(s), \mu$ parameter space in Figure 7.4.

In considering the case $\sigma \in \mathbb{Z}$, it is sufficient to note that the only case when there *must* be a logarithmic term in the general solution is when $\sigma = 0, 1$, but these are not significant as the lines $\sigma = 0$ and $\sigma = 1$ will be below $\sigma = 2$ in Figure 7.4 (unlike in the even parity sector).

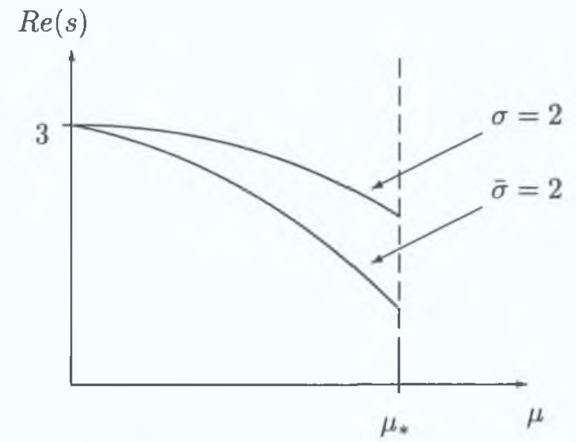


Figure 7.4: The lines $\sigma = 2$ and $\bar{\sigma} = 2$ plotted in the $Re(s), \mu$ parameter space for $0 < \mu < \mu_*$ (odd parity perturbation).

Future similarity horizons

On the Cauchy horizon, $y = y_c$, the indicial exponents are $\{0, 1, \bar{\sigma}\}$, with

$$\bar{\sigma} = \left.\frac{2(s-1)yG}{4yG - y^2 - G^2}\right|_{y=y_c}, \qquad \lim_{\mu\to 0} \bar{\sigma} = (s-1).$$

Again the logarithmic term vanishes, and again for $Re(\bar{\sigma}) > 2$ the scalars $\delta\Psi_{0,4}$ will be finite on $y = y_c$. We see from Figure 7.4 that all perturbations which satisfy the initial regularity conditions will remain finite on the Cauchy horizon.

The second future similarity horizon, $y = y_e$, has indicial exponents $\{0, 1, \bar{\bar{\sigma}}\}$, with

$$\bar{\bar{\sigma}} = \left.\frac{2(s-1)yG}{4yG - y^2 - G^2}\right|_{y=y_e}, \qquad \lim_{\mu\to 0} \bar{\sigma} = 0.$$

The scalars go like $(y - y_e)^{\bar{\sigma}-2}$, and if we write $\bar{\bar{\sigma}} - 2 = \alpha(\mu)s + \beta(\mu)$, we find $Re(s) > 0$ and $\alpha, \beta < 0$ for $0 < \mu < \mu_*$. Thus the second future similarity horizon is unstable.

Proposition 7.2. *The Cauchy horizon formed in the collapse of the self-similar Lemaître-Tolman-Bondi spacetime is stable with respect to odd parity metric and matter perturbations of multipole mode $l \geq 2$. The second future similarity horizon is unstable with respect to same.*

$l = 1$ mode

As in the $l \geq 2$ sector, the gauge invariant matter terms are given by (7.2.1). k_A is not gauge invariant, and we use this gauge freedom to recover (7.2.3). Thus we write $k^A = \frac{1}{R'}(\gamma', -\dot{\gamma})$.

Π however is gauge invariant, and hence the only Weyl scalar to measure in this sector is $\delta\Psi_2 = \Pi$ (see §6.2.2). Using the vector perturbation equation (7.2.2), we perform a Mellin transform over r such that $\gamma(y,r) = r^s K(y;s)$, and we find that Π reduces to

$$\Pi = \frac{2\Lambda G'^2}{r^3 s G^3}.$$

The conservation equation, $\left(r^2 L^A\right)_{|A} = 0$, again gives $\partial\Lambda/\partial t = 0$, and thus the only gauge invariant Weyl scalar reduces to

$$\delta\Psi_2 = \frac{2\Lambda(r) G'^2}{r^3 s G^3}.$$

This term will not diverge on the Cauchy horizon or elsewhere.

Chapter 8

Metric perturbations of generic self-similar spherically symmetric spacetimes

In this last chapter we will consider perturbations of general self-similar spherically symmetric (4-S) spacetimes; that is, we avoid specifying the background matter tensor, save for these conditions:

1. The background matter tensor must satisfy the dominant energy condition.
2. The background spacetime must be such that there are no trapped surfaces prior to $\mathcal{N}$, the past null cone of the singularity.
3. The singularity formed is at least locally naked.

Using the coordinates (x, r, θ, ϕ), where $x = v/r$ (v being the advanced Bondi coordinate) and r is the areal radius, and following the analysis of Chapter 2, we specify our background solution thus: the line element is given by

$$ds^2 = -2r^2 G e^{\psi} dx^2 + 2re^{\psi}(1 - 2xG) dx\, dr + 2xe^{\psi}(1 - xG)\, dr^2 + r^2 d\Omega^2, \qquad (8.0.1)$$

where $G = G(x)$, $\psi = \psi(x)$; the axis given by $r = 0$ or $x = -\infty$ will be regular for $F(-\infty) = \frac{1}{2}$ and $\psi(-\infty) = 0$, where $F = Ge^{-\psi}$; a singularity will form at the origin of coordinates if the spacetime is non-flat, and we may choose v such that the past null cone of the origin, $\mathcal{N}$, is given by $x = 0$; the singularity will be naked if there is a positive real root of $G = 1/x$, the first of which is the Cauchy horizon denoted $x = x_c$, and the second of which we will call the second future similarity horizon (SFSH) $x = x_e$.

As we are not specifying the background matter tensor, we cannot say anything meaningful about the perturbed matter tensor, as in the previous chapters. Thus we must consider source-free perturbations, that is a non-zero metric perturbation and a zero matter perturbation. This has important implications for even parity metric perturbations, which we will discuss at the appropriate time. First we will consider the odd parity perturbation.

8.1 Odd parity perturbations

We will first consider modes $l \geq 2$. As there is no matter perturbation, our only perturbation variable is k_A, which is gauge invariant. Using the scalar perturbation equation (4.2.13b), we can write k_A in terms of a scalar $\gamma(x,r)$ as

$$k^A = \left(\frac{\gamma_{,r}}{re^{\psi}}, -\frac{\gamma_{,x}}{re^{\psi}}\right).$$

The vector perturbation equation (4.2.13a) has two entries, and taking the Mellin transform of these equations over r such that $\gamma(x,r) = r^s K(x;s)$, we find that one equation is simply the derivative of the other. Thus we have a second order ordinary differential equation in K, which completely specifies the perturbation:

$$\begin{aligned}2x(1-xG)K'' + [4(s-2)xG - 2(s-2+x^2G')]K' \\ +[e^{\psi}(l^2+l-2) - 2s(s-3)G + 2sxG']K = 0.\end{aligned} \tag{8.1.1}$$

Given that the anti-symmetric tensor and null vectors may be written

$$\epsilon_{AB} = \begin{pmatrix} 0 & re^{\psi} \\ -re^{\psi} & 0 \end{pmatrix}, \quad l^A = \begin{pmatrix} xe^{-\psi}/r \\ -e^{-\psi} \end{pmatrix}, \quad n^A = \begin{pmatrix} (1-xG)/r \\ G \end{pmatrix},$$

we may write the Weyl scalars in terms of K as (using the ODE in K to simplify)

$$\begin{aligned}\delta\Psi_0 &= \frac{r^{s-4}e^{-2\psi}}{2(xG-1)}\left(2xsK' + K[e^{\psi}(l^2+l-2)x + 2s(1-s)] + (2sxK - 2x^2K')A\right), \\ \delta\Psi_1 &= r^{s-4}e^{-\psi}\left((6s + l(l+1)(2-s) - 4)K + (l^2+l-6)xK'\right), \\ \delta\Psi_2 &= r^{s-4}(l^2+l-2)K, \\ \delta\Psi_3 &= r^{s-4}\left((l(l+1)(s-2) + 2(s+2))GK + (l^2+l+2)(1-xG)K'\right),\end{aligned} \tag{8.1.2}$$

$$
\begin{aligned}
\delta\Psi_4 &= \frac{r^{s-4}}{2x}\left(2(xG-1)(s-2+xA)K'\right. \\
&\quad \left. + \left(e^{-\psi}(l^2+l-2)(xG-1) - 2sG(s-3+xA)\right) K\right),
\end{aligned}
$$

where $A = (2G + xG' + (-1+xG)\,\psi')$ and we have left out the angular dependence. Note $\delta\Psi_2 = 0$ for $l = 1$. Also (8.1.2) indicates that K itself has direct physical significance.

Near the axis, the second order ODE (8.1.1) has indicial exponents $2 + l - s$ and $1 - l - s$, and we find the scalars go like

$$
\delta\Psi_n = k_1\, r^{l-2} + k_2\, r^{-l-3}, \quad n = 0, \ldots, 4,
$$

where $k_{1,2}$ are the constants used to make a linear combination in the general solution. As before, we must set $k_2 = 0$ for a set of finite scalars on the axis.

Near the past null cone there are indicial exponents $\{0, s-1\}$, and to satisfy the minimum stability requirement we must have $Re(s) > 1$. We write the general solution for K near $x = 0$ as

$$
K|_{\mathcal{N}} = k_3 \sum_{m=0}^{\infty} A_m x^{m+s-1} + k_4 \left\{ \hat{k} \ln x \sum_{m=0}^{\infty} A_m x^{m+s-1} + \sum_{m=0}^{\infty} B_m x^m \right\},
$$

where $\hat{k} = 0$ if $s \notin \mathbb{Z}$ (thus we distinguish the $s \in \mathbb{Z}$ and $s \notin \mathbb{Z}$ cases with $\hat{k}$). We find the scalars have the following behaviour near $x = 0$,

$$
\begin{aligned}
\delta\Psi_0 &\sim k_3 x^s + k_4 \left\{x^{s-1} + O(1)\right\}, \\
\delta\Psi_1 &\sim k_3 x^{s-1} + k_4 \left\{x^{s-1} \ln x + O(1)\right\}, \\
\delta\Psi_2 &\sim k_3 x^{s-1} + k_4 \left\{x^{s-1} \ln x + O(1)\right\}, \\
\delta\Psi_3 &\sim k_3 x^{s-2} + k_4 \left\{x^{s-2}(s-1) \ln x + O(1)\right\}, \\
\delta\Psi_4 &\sim k_3 x^{s-3} + k_4 \left\{x^{s-3}(s-1)(s-2) \ln x + O(1)\right\}.
\end{aligned}
$$

If $s \notin \mathbb{Z}$ there are no logarithmic terms and thus we require $Re(s) > 3$, $s \notin \mathbb{Z}$. When $s \in \mathbb{Z}$, we see $s = 2, 3$ (we can only consider $s > 1$) will give scalars which diverge like $\ln x$ near $x = 0$. Thus the scalars will be finite on the past null cone for

$$
Re(s) > 3, \qquad s \in \mathbb{C}.
$$

Next we consider the Cauchy horizon, given by the first real root of $xG = 1$. Let us define, as before, $W(x) = xG - 1 = (x - x_c)h(x)$, with $h(x_c) \neq 0$; that is we consider only the case where the first and second future similarity horizons are distinct.

We may write the second order ODE for K in canonical form near $x = x_c$ as

$$(x - x_c)^2 K'' + (x - x_c)\left[\frac{(s-2)(1 - 2x_c G_c) + x_c^2 G_c'}{x_c h_c} + O(x - x_c)\right] K' + \big[O(x - x_c)\big] K = 0,$$

where a subscript $_c$ denotes the value of that function at x_c. There are two indicial exponents near $x = x_c$ given by

$$\left\{0, \frac{s-1}{x_c h_c} \equiv \sigma\right\},$$

since $x_c G_c = 1$ and

$$W' = xG' + G = (x - x_c)h' + h \quad \Rightarrow \quad G_c' = \frac{x_c h_c - 1}{x_c^2}. \tag{8.1.3}$$

To satisfy the minimum stability requirement we must have $Re(\sigma) > 0$, thus a general solution near $x = x_c$ is given by

$$K|_{CH} = k_5 \sum_{m=0}^{\infty} A_m (x - x_c)^{m+\sigma} + k_6 \left\{ \bar{k} \ln(x - x_c) \sum_{m=0}^{\infty} A_m (x - x_c)^{m+\sigma} + \sum_{m=0}^{\infty} B_m (x - x_c)^m \right\},$$

where $\bar{k} = 0$ if $\sigma \notin \mathbb{Z}$. The Weyl scalars go like

$$\begin{aligned}
\delta\Psi_0 &\sim k_5 (x - x_c)^{\sigma - 2} + k_6 \big\{(x - x_c)^{\sigma - 2} \ln(x - x_c) + O(1)\big\}, \\
\delta\Psi_1 &\sim k_5 (x - x_c)^{\sigma - 1} + k_6 \big\{(x - x_c)^{\sigma - 1} \ln(x - x_c) + O(1)\big\}, \\
\delta\Psi_2 &\sim k_5 (x - x_c)^{\sigma} + k_6 \big\{(x - x_c)^{\sigma} \ln(x - x_c) + O(1)\big\}, \\
\delta\Psi_3 &\sim k_5 (x - x_c)^{\sigma} + k_6 \big\{(x - x_c)^{\sigma} \ln(x - x_c) + O(1)\big\}, \\
\delta\Psi_4 &\sim k_5 (x - x_c)^{\sigma} + k_6 \big\{(x - x_c)^{\sigma} \ln(x - x_c) + O(1)\big\}.
\end{aligned}$$

In Chapter 3 we were able to prove two important inequalities,

$$W_c' > 0, \quad G_c' \leq 0.$$

From (8.1.3) above, we see $x_c h_c = 1 + x_c^2 G_c'$, and thus $x_c h_c \leq 1$. Also, $W_c' = h_c > 0$, and thus $0 < x_c h_c \leq 1$. Since $Re(s) > 3$ for a finite past null cone, $Re(s) - 1 > 2$ and thus $Re(\sigma) > 2$. Therefore all the scalars given above will be finite on the Cauchy horizon.

Finally we may consider the SFSH given by $x = x_e$. Here we let $W = xG - 1 = (x - x_e)\bar{h}(x)$, with $\bar{h}(x_e) \neq 0$. Writing the ODE for K in its canonical form near $x = x_e$ allows us to find two indicial exponents,

$$\left\{0, \frac{s-1}{x_e\bar{h}_e}\right\}.$$

Differentiating $xG - 1 = (x - x_e)\bar{h}$ with respect to x and evaluating at $x = x_e$ gives

$$\bar{h}_e = \frac{1}{x_e}(1 + x_e^2 G_e'),$$

and in Chapter 3 we showed that $1 + x_e^2 G_e' < 0$, thus $\bar{h}_e < 0$.

Thus our second indicial exponent is negative when $Re(s) > 3$, and thus the scalars formed from the solution due to this indicial exponent will diverge on the second future similarity horizon.

Proposition 8.1. *Let the $(\mathcal{M}, g)$ be self-similar, spherically symmetric, satisfy the dominant energy condition, and admit a Cauchy horizon. This Cauchy horizon is stable with respect to odd parity metric perturbations with $l \geq 2$. The following second future similarity horizon is unstable with respect to same.*

$l = 1$ modes

Again there is only one perturbation variable, k_A, which is not gauge invariant. We use the gauge freedom to set ${k_A}^{|A} = 0$, and proceed in the same manner as in the $l \geq 2$ sector. As shown in §6.2.2, there is, in general, only one gauge invariant Weyl scalar to measure in the odd parity $l = 1$ sector, that is $\delta\Psi_2$. From (8.1.2), we see for this particular background this scalar vanishes for $l = 1$. This underlies the importance, particularly in the $l = 1$ sector, of considering a non-zero matter perturbation. In each model studied before, both null and timelike dust, the $l = 1$ odd parity perturbation was given entirely in terms of the matter perturbation, and setting this equal to zero would return a zero perturbation, as in this case.

8.2 Comment on even parity perturbations

A problem arises when we try and repeat the analysis of the previous section for even parity perturbations. As we are not specifying the matter content of the background spacetime, we must consider vanishing matter perturbations. Then the only unknowns are the three entries of k_{AB} and the scalar k. However, there are now seven equations; this is because one perturbation equation was being used up in defining the perturbation in the energy density. These seven second order ODE's may be combined to remove perturbation variables one at a time, for example we use (4.2.12d) to remove one of the components of k_{AB} as before, then use (4.2.12a) to remove second derivatives of k, and so on. The point is that there are now enough equations to reduce the set of equations to

$$\Big(\text{Expression involving background terms, } l,\ s \text{ and so on}\Big) \times A = 0,$$

where A is a metric perturbation variable. Setting the first term $= 0$ would impose extra conditions on the background geometry, which would not hold in general. Thus $A = 0$ must hold in general. The vanishing of any one perturbation variable means the vanishing of the other perturbation variables. Therefore there is only a purely trivial solution: that is, without a matter perturbation the metric perturbation vanishes. We see this strong coupling of the metric and matter perturbation in the other spacetimes we have considered in this thesis.

In the self-similar Vaidya spacetime, there were variables $(A, D, K, G)^T = Y$, which we wrote as a first order system $Y' = MY$. The fourth entry in this system was the equation

$$G' - \frac{s}{x}G = -\frac{1}{x(1 - x\lambda)}D,$$

and G was a matter perturbation variable (perturbation in the null vector). Thus setting $G = 0$ means $D = 0$. Considering the other equations in the system results in a first order ODE in A or K, however if we also consider the equation defining $\delta\rho$ in terms of A, D, K, G with $\delta\rho = 0$, this results in $A = K = 0$, and thus there is only a trivial solution.

In the self-similar timelike dust spacetime, we have already mentioned how setting any one of the four perturbation variables to zero must result in a trivial solution; this is why we had to consider a fourth order ODE in H. All the other variables were expressed in terms of H, and since H was a matter perturbation variable (perturbation in the timelike vector), we find that a vanishing matter perturbation results in a vanishing metric perturbation.

However, there is another way to consider the perturbation equations: as a set of partial differential equations with inhomogeneous terms. A perfectly standard way to deal with inhomogeneities is first to find the homogeneous solution by setting the right hand side to zero, i.e. zero matter perturbation. Then once the homogenous solution has been found there are various methods for integrating the full equations, as in the $s \in \mathbb{Z}$ case in §6.1.1.

Why this method does not work in this case is not clear, and is an issue which merits further attention. Perhaps a way forward would be to consider a non-zero matter perturbation with certain restraints, for example that the perturbed matter tensor satisfies the dominant energy condition. Using the longitudinal gauge, this would place certain restrictions on the matter perturbation variables, and would perhaps be enough to make possible some analysis of this interesting problem. And this *is* an interesting problem: if we were to find that generic self-similar spherically symmetric spacetimes admitting a naked singularity were stable under full metric and matter perturbations, this would indeed be compelling evidence against the Cosmic Censorship hypothesis.

Chapter 9

Conclusions

The main finding of this thesis is that the Cauchy horizon formed in the collapse of a variety of self-similar, spherically symmetric spacetimes is stable with respect to the individual modes of a linear perturbation.

We have found that the self-similar Vaidya (Chapter 6) and Lemaître-Tolman-Bondi (Chapter 7) solutions are stable with respect to gauge invariant perturbations, of both even and odd parity, of the metric and matter tensors. Further, we have found that a general self-similar spherically symmetric spacetime whose matter content is unspecified (save for satisfying the dominant energy condition) is stable with respect to an infalling scalar field (Chapter 3) and gauge invariant metric perturbations of odd parity (Chapter 9). More precisely, we have shown stability with respect to individual modes of the field after a Mellin transform, however we believe there is good evidence to suggest that the finiteness of all of the modes implies finiteness of the field (see below).

These spacetimes, which contain naked singularities, do *not* exhibit on the Cauchy horizon the blue-sheet instability which was seen in the Reissner-Nordström spacetime and was perhaps naively expected here. These naked singularities are stable, making self-similar spherically symmetric spacetimes strong contenders for counter-examples to the Cosmic Censorship hypothesis.

Continuing past the Cauchy horizon we have found the second future similarity horizon is unstable with respect to the fields listed above. Thus the naked singularity which results from collapse in these models is stable but only for a finite period of time.

This research has highlighted a number of topics for future analysis, which we briefly outline here:

1. Firstly, it would be interesting to address the even parity perturbations of 4-S

spacetimes. It is possible that information regarding the constraint and evolution equations of the perturbed spacetime is lost through performing the Mellin transform. By considering instead the characteristics of the two dimensional evolution PDE's for the perturbation we could possibly find non-trivial solutions.

A sensible place to start would be in considering the odd parity perturbations of the metric tensor, as the equations in this case are more straightforward. Bounds on the solutions to these partial differential equations on relevant surfaces would perhaps yield some interesting results. For example, Nolan (unpublished) has the following result for a massless scalar field Φ in the self-similar Vaidya spacetime: if Φ has finite L^2 norm on $x = x_i \in (0, x_c)$, then Φ has finite L^2 norm on $x = x_c$ and $|\Phi(x = x_c, r)| < +\infty, \forall r > 0$.

2. The alternative to not specifying the background matter content is to specify a more realistic background (which admits a naked singularity). The backgrounds considered in this thesis were both dust solutions, as then the components of the metric tensor could be solved for in closed form. The next obvious choice is to consider a perfect fluid solution with non-vanishing pressure (analysis of this problem would be mostly numerical in nature).

   Gundlach [18] has made some investigations of self-similar perfect fluids from the point of view of critical collapse, however we would suggest the question is not fully settled for two reasons: Gundlach looked for stability in the approach to the singularity at $r = 0$ along homothetic lines, rather that testing for stability of the Cauchy horizon (if one forms) as we have in this thesis. Also, Harada [21, 26] has found that there is a self-similar solution, known as the Larson-Penston solution, which acts as an attractor for non-self-similar solutions and has a naked singularity which is stable with respect to the 'kink' mode (that is, a discontinuity in the derivatives of the density function). This Larson-Penston solution has no unstable modes whereas the critical solutions have exactly one, but perturbations of this solution have not been studied in the approach to the Cauchy horizon. If these perturbations are finite, we would have a strong counter-example to the CCH.

3. While spherically symmetric spacetimes are a rich source of naked singularities, another source is the cylindrically symmetric spacetimes. These are less extensively studied, however there are interesting issues raised by spacetimes of this type with regard to the Cosmic Censorship hypothesis. A perturbation analysis of these spacetimes would be informative, even a cylindrically symmetric perturbation would be

non-trivial as there are few uniqueness theorems in cylindrical symmetry, and in particular there is no cylindrical Birkhoff-type theorem.

With regards to non-cylindrically symmetric perturbations, there is an issue in defining the cylindrical equivalent to the spherical harmonics. While the solutions to the (z, ϕ) part to Laplace's equation and are in fact more straightforward than the spherical harmonics, their generating equations being of the form

$$y'' \pm m^2 y = 0,$$

where $y = y(z)$ or $y(\phi)$, we see that when we consider the general line element for a cylindrically symmetric spacetime

$$ds^2 = g_{AB}dx^A dx^B + \alpha^2(x^A)dz^2 + \beta^2(x^A)d\phi^2,$$

the fact that α and β are functions of the coordinates of the Lorentzian two-space may cause complications in attempting to decompose the perturbed metric tensor in terms of objects built from the cylindrical harmonics. A way forward would perhaps be to restrict the background spacetime and consider non-cylindrically symmetric perturbations of these backgrounds, rather than attempting to develop a formalism which applies to all cylindrically symmetric spacetimes. If we were able to derive Teukolsky-type equations for gauge invariant scalars, then this could motivate a simple decomposition over the (z, ϕ) coordinates of the form $e^{kz}e^{im\phi}$.

4. Finally, the unresolved issue of whether the finiteness of each individual mode guarantees that we may perform the inverse Mellin transform and recover the field could be pursued. Nolan has done some work in this direction in §6 of [41], by considering a massless scalar field propagating in Minkowski and Vaidya spacetime. An attempt of a similar analysis on the Mellin transform of gauge invariant perturbations would be technically very difficult indeed; perhaps looking at the scalar field propagating in generic 4-S spacetimes would be tractable.

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