A Dirichlet Problem for a Coupled System of Two Singularly Perturbed Convection-Diffusion Ordinary Differential Equations

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Master of Science is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Contents

Abstract

A Dirichlet problem for a system of two singularly perturbed convection-diffusion ordinary differential equations is examined where the two singular perturbation parameters can be of a different order. A finite difference numerical method whose solutions converge pointwise independently of the singular perturbation parameters is constructed. A full theoretical analysis is provided which shows that the numerical method is robust. This is done over a piecewise uniform fitted mesh involving two transition points.

The first differential equation has only one dependent variable while the second equation has two dependent variables. The solution to the first differential equation is present in the second differential equation and this introduces coupling which is examined in this thesis.

The solution of the first differential equation is decomposed into regular and singular components. The numerical solution is decomposed in an analogous manner. The convergence of the numerical method is analysed separately over each component. Sharp weighted derivative estimates for each of these components are examined as these are necessary for the analysis of the second differential equation.

The solution of the second differential equation is decomposed into regular, singular and coupling components. Again the numerical solution is decomposed analogously and the convergence of the numerical method is analysed separately over each component.

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Notation

u_1, u_2 the solutions of the singularly perturbed differential equations.

v_1, v_2 the regular components and w_1, w_2 the singular components of u_1 and u_2 respectively.

z the coupling component.

U_k, V_k, W_k $(k = 1, 2)$ the numerical approximations for u_k, v_k and w_k respectively.

Z the numerical approximation to the coupling component z.

ε_1, ε_2 the small parameters.

C is a constant independent of the small parameters ε_1, ε_2 and N the number of mesh points.

$\bar{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2\}$.

$\bar{\bar{\varepsilon}} = \max\{\varepsilon_1, \varepsilon_2\}$.

The differential operators $L_k = \varepsilon_k \frac{d^2}{dx^2} + a_{k,k}(x)\frac{d}{dx}$, $k = 1, 2$.

The transition parameters used in the analysis

$$\sigma_2 = \min\{\max\{\frac{8\varepsilon_1 \ln N}{\alpha_1}, \frac{8\varepsilon_2 \ln N}{\alpha_2}\}, \frac{1}{2}\}, \quad \sigma_1 = \min\{\frac{4\varepsilon_1 \ln N}{\alpha_1}, \frac{4\varepsilon_2 \ln N}{\alpha_2}, \frac{\sigma_2}{2}\}.$$

The domain $\bar{\Omega} = [0, 1]$ and

$$\Omega = (0, 1).$$

The mesh step $h_i = x_i - x_{i-1}$ and

$$\bar{h}_i = (h_{i+1} + h_i)/2.$$

The mesh steps $H_1 = \frac{4\sigma_1}{N}$,

$$H_2 = \frac{4(\sigma_2 - \sigma_1)}{N},$$

$$H_3 = \frac{2(1 - \sigma_2)}{N}.$$

An arbitrary non–uniform mesh $\bar{\Omega}^N = \{x_i : 0 = x_0 < ... < x_N = 1\}$ and

$$\Omega^N = \{x_i : i = 1, ..., N - 1\}.$$

The forward difference operator $D^+Y_i = \frac{Y_{i+1} - Y_i}{h_{i+1}}$.

The central difference operator $\delta^2 Y_i = (\frac{Y_{i+1} - Y_i}{h_{i+1}} - \frac{Y_i - Y_{i-1}}{h_i})/\bar{h}_i$.

The difference operators $L_k^N \equiv \varepsilon_k \delta^2 + a_{k,k}(x_i)D^+$, $k = 1, 2$.

A one transition point mesh using a single transition point σ:

$$\bar{\Omega}_\sigma^N = \{x_i\}, \quad x_i = \begin{cases} 2\sigma i/N & if \quad i \leq N/2 \\ x_{i-1} + 2(1-\sigma)/N & if \quad i > N/2 \end{cases}.$$

A two transition point mesh using the transition points σ_1 and σ_2:

$$\bar{\Omega}_{\varepsilon_1,\varepsilon_2}^N = \{x_i\}, \quad x_i = \begin{cases} 4\sigma_1 i/N & if \quad i \leq N/4 \\ x_{i-1} + 4(\sigma_2 - \sigma_1)/N & if \quad N/4 < i \leq N/2 \\ x_{i-1} + 2(1-\sigma_2)/N & if \quad i > N/2 \end{cases}.$$

Discrete barrier function:

$$B_{\gamma_k,\varepsilon_k}(x_i) = \begin{cases} 1 + \frac{\gamma_k h_1}{\varepsilon_k} & if \quad i = -1, \quad x_{-1} = -h_1 \\ 1 & if \quad i = 0 \\ \prod_{j=1}^{i}(1 + \frac{\gamma_k h_j}{\varepsilon_k})^{-1} & if \quad i > 0 \end{cases}.$$

The norms:

$$\begin{aligned} \| v \|_\infty &= \max_{x\in\bar{\Omega}}\{|v(x)|\}, \\ \| v \|_{\Omega_i} &= \max_{x\in\Omega_i}\{|v(x)|\}, \quad \Omega_i = [x_{i-1}, x_{i+1}], \\ \| v \|_{\Omega^N} &= \max_{1\leq j\leq N-1}\{|v_j|\}, \quad \Omega^N \text{an arbitrary mesh}, \\ \| v \|_1 &= (1,v)_1, \text{ where the discrete inner product } (w,v)_1 = \sum_{i=1}^{N-1} \bar{h}_i v_i w_i. \end{aligned}$$

Component pointwise errors:

$e_j = (V_1 - v_1)(x_i)$,

$\check{e}_j = (V_2 - v_2)(x_i)$,

$\hat{e}_j = (W_1 - w_1)(x_i)$,

$\bar{e}_j = (Z - z)(x_i)$.

Weighted Derivative error $E_j = D^+ e_j$.

Computed range of ε_1, ε_2:

$$S = \{(\varepsilon_1, \varepsilon_2) : 2^{-50} \leq \varepsilon_1 \leq 1, \quad 2^{-50} \leq \varepsilon_2 \leq 1\}.$$

Maximum pointwise errors:

$$
\begin{aligned}
E^N_{\varepsilon_1,\varepsilon_2}(u_k) &= \parallel U^N_k - u_k \parallel_{\Omega^N_{\varepsilon_1,\varepsilon_2}}, \quad k = 1, 2, \\
E^N_{\varepsilon_1}(u_k) &= \max_{\varepsilon_2} E^N_{\varepsilon_1,\varepsilon_2}(u_k), \quad k = 1, 2, \\
E^N(u_k) &= \max_{\varepsilon_1} E^N_{\varepsilon_1}(u_k), \quad k = 1, 2,
\end{aligned}
$$

where U^N_k are the numerical approximations to u_k ($k = 1, 2$) for a particular value of N.

Parameter–uniform orders of convergence $p^N = log_2 \frac{E^N}{E^{2N}}$.

Weighted derivative errors:

$$
\begin{aligned}
E^N_{\varepsilon_1,\varepsilon_2}(D^+U_1) &= \parallel \varepsilon_{x_k} D^+(U^N_1 - u_1) \parallel_{\Omega^N_{\varepsilon_1,\varepsilon_2}}, \\
E^N_{\varepsilon_1}(D^+U_1) &= \max_{\varepsilon_2} E^N_{\varepsilon_1,\varepsilon_2}(D^+U_1), \\
E^N(D^+U_1) &= \max_{\varepsilon_1} E^N_{\varepsilon_1}(D^+U_1).
\end{aligned}
$$

where $(\varepsilon_1, \varepsilon_2) \in S$ and $\varepsilon_{x_k} = \begin{cases} \tilde{\varepsilon} & if \quad x_k < \sigma_1 \\ \bar{\varepsilon} & if \quad \sigma_1 \leq x_k < \sigma_2 \\ 1 & if \quad x_k \geq \sigma_2 \end{cases}$ for $0 \leq k \leq N - 1$.

Computed maximum pointwise errors:

$$
\begin{aligned}
e^N_{\varepsilon_1,\varepsilon_2}(u_k) &= \parallel U^N_k - \bar{U}^{8192}_k \parallel_{\Omega^N_{\varepsilon_1,\varepsilon_2}}, \quad k = 1, 2, \\
e^N_{\varepsilon_1}(u_k) &= \max_{\varepsilon_2} e^N_{\varepsilon_1,\varepsilon_2}(u_k) \quad k = 1, 2, \\
e^N(u_k) &= \max_{\varepsilon_1} e^N_{\varepsilon_1}(u_k) \quad k = 1, 2,
\end{aligned}
$$

where U^N_k are the numerical approximations for u_k. $\bar{U}^{8192}_k$ are the interpolated values at the mesh points using U^{8192}_k.

Double–mesh differences:

$$
\begin{aligned}
D^N_{\varepsilon_1,\varepsilon_2}(u_k) &= \parallel U^N_k - \bar{U}^{2N}_k \parallel_{\Omega^N_{\varepsilon_1,\varepsilon_2}}, \quad k = 1, 2, \\
D^N_{\varepsilon_1}(u_k) &= \max_{\varepsilon_2} D^N_{\varepsilon_1,\varepsilon_2}(u_k) \quad k = 1, 2, \\
D^N(u_k) &= \max_{\varepsilon_1} D^N_{\varepsilon_1}(u_k) \quad k = 1, 2,
\end{aligned}
$$

where U^N_k are the numerical approximations to u_k and $\bar{U}^{2N}_k$ are the interpolated values at the N mesh points using the numerical solution U^{2N}_k.

The double mesh orders of convergence $p^N_d = log_2(\frac{D^N}{D^{2N}})$.

Introduction

This thesis considers a coupled system of two singularly perturbed second order, convection–diffusion differential equations. A singularly perturbed differential equation is characterised by a small parameter multiplying the highest derivative term of the differential equation. Linearisations of the Navier–Stokes equations frequently give rise to a set of coupled convection–diffusion partial differential equations [6].

Convection–diffusion can easily be illustrated by imagining a river which flows strongly and smoothly, where liquid pollution runs into the river at a certain point. The pollution diffuses slowly through the water but the rapid movement of the river swiftly convects the pollution down stream. Convection alone would carry the pollution along a one dimensional curve on the surface, diffusion gradually spreads that curve resulting in a long thin curved wedge shape. A second order ordinary differential equation can model convection–diffusion where the second derivative term corresponds to the diffusion and the first derivative term corresponds to the convection [7]. When the second derivative term is multiplied by a small parameter, the differential equation models a situation where the convection dominates and thus singularly perturbed problems can model convection dominated convection–diffusion processes.

Singularly perturbed differential equations are in the main, difficult or impossible to solve analytically. The numerical solutions of such convection–diffusion problems are of prime interest here. The mesh employed in the numerical method is a particular discretization of the continuous domain where the problem is defined. It is desirable that the numerical method used to generate approximations to the analytical solutions of the differential equations be robust. By this we mean that the numerical approximations converge pointwise to the analytic solution independently of the singular perturbation parameters.

The numerical approximations will not converge independently of the small parameters if we employ a uniform mesh [2]. We need to choose a fitted-mesh that insures the numerical solutions converge to the analytic solutions independent of the small parameters. The choice of our fitted method is an adaptation of the Shishkin mesh [6], [8], [9]. Here we use two transition points and the mesh consists of three regions. The first two regions are what we term the "layer" regions.

The work here is concerned with a system of singularly perturbed convection–diffusion ordinary differential equations. The pioneering work of Shishkin [8] in the area of singularly perturbed differential equations has been studied by a group of Irish mathematicians in the last decade. They have helped to make his work more accessible to the English speaking world through the publication of papers and books [2], [6] with Shishkin. The paper by Shishkin [9] in the area of singularly perturbed boundary–value problems for systems of elliptic and parabolic equations has been followed more recently by work in the area of systems of singularly perturbed reaction–diffusion differential equations by Matthews et al [5] and by Madden and Stynes [4] . Matthews et al presented error bounds for the numerical method when both small parameters were equal or when one of them was unity and provided numerical evidence that suggested the numerical method was robust when the parameters were different. Madden et al in fact verified that this is so. They also outlined and illustrated the coupling due to the two small parameters in the reaction–diffusion system.

Because of the further complexity of the convection–diffusion system we confine our attention to the situation when the first differential equation has only one dependent variable while the second equation has two dependent variables. The analysis here is mainly based on the work of Farrell et al [2]. The idea of the Shishkin mesh [2] is crucial. The decompositions of the solutions to the differential equations play a central role in the analysis. A stability technique due to Andreyev and Savin [1] is also used. This technique is discussed in appendix B for the numerical method employed here. The use of sharper weighted derivative estimates than those given in [2] is of interest. The weighted derivative results obtained in chapter two are akin to those of Kopteva–Stynes [3] who highlight a weighted derivative result for a Shishkin mesh with one transition point.

The analysis concentrates on the solution of the second differential equation in the system. The decomposition of the first equation into regular and singular components is given in section 1.2. Also in this section the solution obtained from the second equation is decomposed into what we call the second regular, the second singular and the coupling components. Separate bounds are then established for these components and their derivatives. The numerical approximation to the solution is decomposed in an analogous manner in section 1.3. The analysis then concentrates on each component in turn.

The analysis for the second singular component is analogous to that of the first singular component [2]. Next weighted derivative results for both the regular and singular components of the first equation are established in sections 2.1 and 2.2 respectively. The analysis for the second regular component in section 2.1 uses the results and techniques for the first regular component which are found in [2]. The weighted derivative result for the first regular component used in conjunction with the stability technique of Andreyev and Savin [1] completes the analysis for the second regular component.

The coupling component is a new feature in this work. The analysis for this component is based on two cases. In section 3.2 we consider the case when the second small parameter is smaller than the first small parameter. Away from the layer regions the coupling component and its numerical approximation are bounded separately in a similar manner to that of the singular components. The weighted derivative result for the first singular component is then used in conjunction with the stability technique of Andreyev and Savin [1] to complete the analysis for this case in the layer regions.

Next in sections 3.3 and 3.4 we consider the second case where the first small parameter is smaller than or equal to the second small parameter . In section 3.3 we recall the dependence of the coupling component on the ratio of the smaller parameter divided by the larger parameter from section 1.2. Similarly the numerical approximation to the coupling component depends on this ratio plus a quantity of order $N^{-1}\ln N$ where N is the number of mesh points. When the ratio is less than the order of N^{-1} the analysis is complete. Otherwise in section 3.4 the weighted derivative result for the first singular component and the stability technique of Andreyev and Savin [1] are used to complete the analysis for the second case and thus for the coupling component.

In chapters four and five we provide numerical and graphical evidence to validate and highlight various features from the analysis. Numerical data is provided in chapter four for a problem whose exact solutions are known. This is used to demonstrate the convergence of the numerical approximation to the continuous solutions. Some examples that justify the choice of the two transition point mesh are also shown. A problem with variable coefficients is also outlined to reflect the fact that exact errors are generally not available. In chapter five we illustrate the decomposition of the solution from the second equation into its second regular, second singular and coupling components. The “double–layer” effect on the solution due to both the coupling and singular components is also demonstrated.

Chapter 1

Bounds for the solutions and the numerical method

We start by stating the convection–diffusion problem due for consideration in this thesis. First the continuous solutions of the system of differential equations are decomposed into components. Bounds are then established for each of these components and their derivatives. A finite difference numerical method is constructed which generates approximations to the continuous solutions.

1.1 Statement of problem

Consider the following system of singularly perturbed ordinary differential equations:

$$L_1 u_1 \equiv \varepsilon_1 u_1''(x) + a_{1,1}(x) u_1'(x) = f_1(x), \quad x \in (0,1) = \Omega, \tag{1.1a}$$

$$\varepsilon_2 u_2''(x) + a_{2,2}(x) u_2'(x) + a_{2,1}(x) u_1'(x) = f_2(x), \quad x \in \Omega, \tag{1.1b}$$

where $u_1(0)$, $u_1(1)$, $u_2(0)$ and $u_2(1)$ are given.

The coefficients $a_{1,1}(x)$ and $a_{2,2}(x)$ are chosen to be strictly positive for all $x \in \bar{\Omega}$. The functions $f_2 \in C^2(\Omega)$ and $a_{1,1}$, $a_{2,1}$, $a_{2,2}$, $f_1 \in C^3(\Omega)$. We devise a numerical scheme for all $0 < \varepsilon_1 \leq 1$ and $0 < \varepsilon_2 \leq 1$. In the analysis we let $\alpha_1 = \min\{a_{1,1}\}$ and $\alpha_2 = \min\{a_{2,2}\}$ and we consider the cases

$$\frac{\varepsilon_2}{\alpha_2} \leq \frac{\varepsilon_1}{\alpha_1} \tag{1.2}$$

and

$$\frac{\varepsilon_1}{\alpha_1} \leq \frac{\varepsilon_2}{\alpha_2} \tag{1.3}$$

separately. The main result of the thesis is the establishment of the following error bounds:

$$|U_{j,i} - u_j(x_i)| \leq CN^{-1}(\ln N)^j, \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}, \quad j = 1, 2, \tag{1.4}$$

where $U_{j,i}$ ($j = 1, 2$ and $0 \leq i \leq N$) are the numerical solutions to (1.26) using mesh (1.24), also C is independent of ε_1, ε_2 and N.

1.2 Decompositions and Bounds

A decomposition of the solution of problem (1.1a)

The solution u_1 of (1.1a) can be written as the sum of two components v_1, w_1 [2], thus $u_1 = v_1 + w_1$ and $v_1(0)$ is suitably chosen (see [2]) where

$$\begin{aligned} L_1 v_1 &= f_1, \quad v_1(0) = A, \qquad\qquad v_1(1) = u_1(1), && (1.5a)\\ L_1 w_1 &= 0, \quad w_1(0) = u_1(0) - v_1(0), \quad w_1(1) = 0, && (1.5b) \end{aligned}$$

and A is bounded independently of ε_1.

We have the following bounds on these components:

$$\begin{aligned} \| v_1^{(k)} \|_\infty &\le C(1 + \varepsilon_1^{2-k}), \quad k = 0, 1, 2, 3, && (1.6a)\\ | w_1(x) | &\le Ce^{-\alpha_1 x/\varepsilon_1}, && (1.6b)\\ | w_1^{(k)}(x) | &\le C\varepsilon_1^{-k} e^{-\alpha_1 x/\varepsilon_1}, \quad k = 1, 2, 3, && (1.6c) \end{aligned}$$

where we define the norm $\| v \|_\infty = \max_{x \in \bar{\Omega}} \{|v(x)|\}$.

Note from [2] that the choice of v_1 depends on the functions $a_{1,1}$, $f_1 \in C^2(\Omega)$.

A decomposition of the solution of problem (1.1b)

The solution u_2 of (1.1b) is decomposed into regular, singular and coupling components. This can be viewed as $u_2 = v_2 + w_2 + z$ where v_2 and w_2 are the regular and singular components respectively of u_2. These are similar to v_1 and w_1 the regular and singular components respectively of u_1. The coupling component z is the new feature.

We define the differential operator L_2 by $L_2 v = \varepsilon_2 v'' + a_{2,2} v'$ for any $v \in C^2$. The second equation in the system (1.1) can be rewritten as:

$$L_2 u_2 = -a_{2,1} u_1' + f_2 = -a_{2,1} v_1' - a_{2,1} w_1' + f_2.$$

Now we have the decomposition of $u_2 = v_2 + w_2 + z$ where,

$$\begin{aligned} L_2 v_2 &= f_2 - a_{2,1} v_1', \quad v_2(0) = B, \qquad\qquad v_2(1) = u_2(1), && (1.7a)\\ L_2 w_2 &= 0, \quad w_2(0) = u_2(0) - v_2(0) - z(0), \quad w_2(1) = 0, && (1.7b)\\ L_2 z &= -a_{2,1} w_1', \quad z(0) = D, \qquad\qquad z(1) = 0, && (1.7c) \end{aligned}$$

and $v_2(0)$ is suitably chosen in a similar way to $v_1(0)$ [2], also B and D are both bounded independently of the the small parameters ε_1 and ε_2. When inequality

(1.3) holds we set $z(0) = 0$ so that the minimum principle [2] used in this case is applicable. When inequality (1.2) holds we further decompose (1.7c) below and then at the start of lemma 2 it is shown that $z(0)$ is bounded independent of the two small parameters. In a similar manner to v_1, we see from (1.7a) that v_2 depends on the functions $a_{2,2}$, $f_2 - a_{2,1}v_1' \in C^2(\Omega)$. Since v_1 depends on $a_{1,1}, f_1 \in C^2(\Omega)$, then it is necessary that the functions $a_{1,1}$, $f_1 \in C^3(\Omega)$ for the decomposition of u_2 to be valid as outlined above.

A further decomposition of the solution of (1.7c)

The coupling component z is further decomposed in an analogous way to that done for the regular component v_2 where

$$z = z_0 + \varepsilon_2 z_1 + \varepsilon_2^2 z_2 \tag{1.8}$$

and z_0, z_1 and z_2 are the solutions of the problems:

$$\begin{aligned} a_{2,2}z_0' &= -a_{2,1}w_1', & z_0(1) = 0, & & (1.9\text{a})\\ a_{2,2}z_1' &= -z_0'', & z_1(1) = 0, & & (1.9\text{b})\\ L_2 z_2 &= -z_1'', & z_2(0) = 0, & \quad z_2(1) = 0. & (1.9\text{c}) \end{aligned}$$

Bounds on the coupling component z and its derivatives

Lemma 1

Assume that (1.2) holds. If z is the solution of (1.7c), then $|z(x)| \leq Ce^{-\frac{\alpha_1 x}{2\varepsilon_1}}$ where $x \in \bar{\Omega}$.

Proof

The proof for the bound of the coupling component is carried out in two stages. First we show that $z(0)$ is bounded independently of the small parameters ε_1 and ε_2. Once this is done then $|z(x)|$ is bounded by applying a minimum principle [2].

The bound for $|z(0)|$ is established by bounding $|z_0(x)|$ and $|z_1(x)|$ in (1.9) and we note that $z_2(0) = 0$. Rearrange and integrate (1.9a) to give

$$z_0(x) = -\int_0^x \frac{a_{2,1}(t)}{a_{2,2}(t)} d(w_1(t)).$$

Then using (1.6b) the bound $|z_0(x)| \leq C$ is obtained where C is independent of the small parameters. Similarly integrating (1.9b) gives

$$z_1(x) = -\int_0^x \frac{1}{a_{2,2}(t)} d(z_0'(t))$$

which provides the bound $|z_1(x)| \leq \max_{x\in\Omega}\{|z_0'(x)|\} \leq \frac{C}{\varepsilon_1}$ where (1.9a) and (1.6c) are used. Using (1.8) it is clear that $|z(0)|$ is bounded independent of the small parameters where $\frac{\varepsilon_2}{\varepsilon_1} \leq C$ is noted when (1.2) holds.

A bound can now be obtained for $|z(x)|$. Recall from (1.7c) that $L_2 z = -a_{2,1}w_1'$. Then $|L_2 z| \leq \frac{Ce^{-\alpha_1 x/\varepsilon_1}}{\varepsilon_1}$ is obtained from (1.6c) and we let

$$\Psi(x) = \frac{4C_1 e^{-\frac{\alpha_1 x}{2\varepsilon_1}}}{\alpha_1\alpha_2} \pm z,$$

where $C_1 = \max\{\alpha_1\alpha_2 \,|z(0)|\,, C\}$. Recall that $-a_{2,2} \leq -\alpha_2$, then $\Psi(0) \geq 0$, $\Psi(1) \geq 0$ and

$$L_2\Psi = \frac{2C_1}{\alpha_2\varepsilon_1}(\frac{\varepsilon_2\alpha_1}{2\varepsilon_1} - a_{2,2})e^{-\frac{\alpha_1 x}{2\varepsilon_1}} + \frac{Ce^{-\frac{\alpha_1 x}{\varepsilon_1}}}{\varepsilon_1} \pm L_2 z \leq 0$$

where we also use the fact that $\frac{\varepsilon_2\alpha_1}{2\varepsilon_1} \leq \frac{\alpha_2}{2}$ as a consequence of (1.2). Apply the minimum principle (4.2). Thus $\Psi \geq 0$ on $\bar{\Omega}$ and hence $|z| \leq Ce^{-\frac{\alpha_1 x}{2\varepsilon_1}}$ where C is independent of the small parameters. ■

Lemma 2

Assume that (1.2) holds. If z is the solution of (1.7c), then for $x \in \bar{\Omega}$,

a) $|z'(x)| \leq \frac{C}{\varepsilon_1}$,
b) $|z''(x)| \leq \frac{C}{\varepsilon_1^2}$,
c) $|z'''(x)| \leq \frac{C}{\varepsilon_1^2\varepsilon_2}$.

Proof (a - c)

Now the bounds for the derivatives of z are established. We assume $a_{2,1}, a_{2,2} \in C^3(\Omega)$. Rearrange (1.9a) to give $z_0' = -\frac{a_{2,1}w_1'}{a_{2,2}}$, successively differentiating this equation and using (1.6c) the bounds for the derivatives of z_0, over $\bar{\Omega}$ are obtained as follows:

$$|z_0'(x)| \leq C\,|w_1'(x)| \leq \frac{Ce^{-\frac{\alpha_1 x}{\varepsilon_1}}}{\varepsilon_1}, \tag{1.10a}$$

$$|z_0''(x)| \leq C(|w_1'(x)| + |w_1''(x)|) \leq \frac{Ce^{-\frac{\alpha_1 x}{\varepsilon_1}}}{\varepsilon_1^2}, \tag{1.10b}$$

$$|z_0'''(x)| \leq C(|w_1'(x)| + |w_1''(x)| + |w_1'''(x)|) \leq \frac{Ce^{-\frac{\alpha_1 x}{\varepsilon_1}}}{\varepsilon_1^3}, \tag{1.10c}$$

$$\left|z_0^4(x)\right| \leq C(|w_1'(x)| + |w_1''(x)| + |w_1'''(x)| + \left|w_1^{(4)}(x)\right|) \leq \frac{Ce^{-\frac{\alpha_1 x}{\varepsilon_1}}}{\varepsilon_1^4}. \tag{1.10d}$$

The following bounds for the derivatives of z_1, the solution of (1.9b) which hold on $\bar{\Omega}$ follow from (1.10).

$$|z_1'| \le C|z_0''| \le \frac{Ce^{-\frac{\alpha_1 x}{\varepsilon_1}}}{\varepsilon_1^2}, \tag{1.11a}$$

$$|z_1''| \le C(|z_0''| + |z_0'''|) \le \frac{Ce^{-\frac{\alpha_1 x}{\varepsilon_1}}}{\varepsilon_1^3}, \tag{1.11b}$$

$$|z_1'''| \le C(|z_0''| + |z_0'''| + \left|z_0^{(4)}\right|) \le \frac{Ce^{-\frac{\alpha_1 x}{\varepsilon_1}}}{\varepsilon_1^4}. \tag{1.11c}$$

Next bounds for the derivatives of z_2 the solution to (1.9c) are established. We consider $\varepsilon_2 z_2'' + a_{2,2} z_2' = -z_1''$ with boundary conditions $z_2(0) = z_2(1) = 0$. Integrating once gives the derivative

$$z_2'(x) = \frac{K_{\varepsilon_1,\varepsilon_2} e^{-\frac{A(x)}{\varepsilon_2}}}{\int_0^1 e^{-\frac{A(t)}{\varepsilon_2}}\, dt} - \int_0^x \frac{e^{-(\frac{A(x)-A(t)}{\varepsilon_2})} z_1''(t)dt}{\varepsilon_2},$$

where

$$K_{\varepsilon_1,\varepsilon_2} = \frac{\int_0^1 \int_0^s e^{-(\frac{A(s)-A(t)}{\varepsilon_2})} z_1''(t)dtds}{\varepsilon_2},$$

$$A(x) = \int_0^x a_{2,2}(t)dt.$$

Use (1.11b) to obtain

$$\begin{aligned}
|K_{\varepsilon_1,\varepsilon_2}| &\le \int_0^1 \int_{t=0}^s \Big(\frac{e^{-\frac{\alpha_2}{\varepsilon_2}(s-t)}}{\varepsilon_2}\Big)\Big(\frac{e^{-\frac{\alpha_1 t}{\varepsilon_1}}}{\varepsilon_1^3}\Big)dtds \\
&= \int_0^1 \frac{e^{(\frac{\alpha_2}{\varepsilon_2} - \frac{\alpha_1}{\varepsilon_1})t}}{\alpha_2 \varepsilon_1^3}\Big[\int_{s=t}^1 \frac{\alpha_2 e^{-\frac{\alpha_2 s}{\varepsilon_2}}}{\varepsilon_2} ds\Big]dt \\
&\le C\int_0^1 \frac{e^{-\frac{\alpha_1 t}{\varepsilon_1}}}{\varepsilon_1^3} dt \le \frac{C}{\varepsilon_1^2}.
\end{aligned} \tag{1.12}$$

Note also $\int_0^1 e^{-\frac{A(t)}{\varepsilon_2}} dt \ge \int_0^1 e^{-\frac{Mt}{\varepsilon_2}} dt = \varepsilon_2(1 - e^{-M/\varepsilon_2}) \ge \varepsilon_2(1 - e^{-M})$ where $M = \max\{a_{2,2}\}$ thus

$$\int_0^1 e^{-\frac{A(t)}{\varepsilon_2}} dt \ge C\varepsilon_2. \tag{1.13}$$

Combine (1.12) and (1.13) to bound the first term of $|z_2'(x)|$ and then use (1.11b) to bound the integral term thus $|z_2'(x)| \le \frac{C}{\varepsilon_1^2 \varepsilon_2}$. It is easy to see that $|z_2''(x)| \le \frac{C}{\varepsilon_1^2 \varepsilon_2^2}$

and $|z_2'''(x)| \leq \frac{C}{\varepsilon_1^2\varepsilon_2^3}$ by using the differential equation $\varepsilon_2 z_2'' + a_{2,2}z_2' = -z_1''$. Now these bounds are combined with those of (1.10) and (1.11) and we use (1.8) to obtain the bounds on the derivatives of z as stated in the lemma. ■

Lemma 3

Assume that (1.3) holds. If z is the solution of (1.7c), then $|z(x)| \leq \frac{C\varepsilon_1 e^{-\frac{\alpha_2 x}{2\varepsilon_2}}}{\varepsilon_2}$ where $x \in \bar{\Omega}$.

Proof

The proof is considered in two parts. When (1.3) holds recall $z(0) = z(1) = 0$.

(a) First assume

$$\frac{\varepsilon_1}{\varepsilon_2} \leq \frac{\alpha_1}{2 \parallel a_{2,2} \parallel_\infty}. \tag{1.14}$$

Let

$$\Psi(x) = \frac{2C\varepsilon_1(e^{-\frac{\alpha_2 x}{2\varepsilon_2}} - e^{-\frac{\alpha_1 x}{\varepsilon_1}})}{\alpha_1^2\varepsilon_2} \pm z,$$

then $\Psi(0) = 0$ where we recall that $|L_2 z| \leq \frac{Ce^{-\alpha_1 x/\varepsilon_1}}{\varepsilon_1}$ and we use (1.3) to establish $\Psi(1) \geq 0$. Note that $-a_{2,2} \leq -\alpha_2$ and that $\frac{\varepsilon_1\|a_{2,2}\|_\infty}{\varepsilon_2} \leq \frac{\alpha_1}{2}$ as a consequence of (1.14). Then

$$\begin{aligned}
L_2\Psi(x) &= \frac{2C\varepsilon_1([\frac{\varepsilon_2\alpha_2^2}{4\varepsilon_2^2} - \frac{a_{2,2}\alpha_2}{2\varepsilon_2}]e^{-\frac{\alpha_2 x}{2\varepsilon_2}} - [\frac{\varepsilon_2\alpha_1^2}{\varepsilon_1^2} - \frac{a_{2,2}\alpha_1}{\varepsilon_1}]e^{-\frac{\alpha_1 x}{\varepsilon_1}})}{\alpha_1^2\varepsilon_2} \pm L_2 z \\
&\leq \frac{2C}{\alpha_1\varepsilon_1}(\frac{\varepsilon_1 a_{2,2}}{\varepsilon_2} - \alpha_1)e^{-\frac{\alpha_1 x}{\varepsilon_1}} + \frac{Ce^{-\frac{\alpha_1 x}{\varepsilon_1}}}{\varepsilon_1} \leq 0.
\end{aligned}$$

Use minimum principle (4.2) to give $\Psi \geq 0$ over $\bar{\Omega}$ and hence

$$|z| \leq \frac{C\varepsilon_1 e^{-\frac{\alpha_2 x}{2\varepsilon_2}}}{\varepsilon_2}. \tag{1.15}$$

(b) Second assume

$$\frac{\alpha_1}{2 \parallel a_{2,2} \parallel_\infty} \leq \frac{\varepsilon_1}{\varepsilon_2}, \tag{1.16}$$

then using (1.3), we have

$$\frac{\alpha_1}{\varepsilon_1} = \frac{M\alpha_2}{\varepsilon_2}, \quad 1 \leq M \leq \frac{2 \parallel a_{2,2} \parallel_\infty}{\alpha_2}. \tag{1.17}$$

Again recall $|L_2 z| \leq \frac{Ce^{-\alpha_1 x/\varepsilon_1}}{\varepsilon_1}$ and let

$$\Psi(x) = \frac{4MCe^{-\frac{\alpha_1 x}{2M\varepsilon_1}}}{\alpha_1\alpha_2} \pm z.$$

Then $\Psi(0) \geq 0$ and $\Psi(1) \geq 0$, also

$$L_2\Psi = \frac{2C}{\varepsilon_1\alpha_2}(\frac{\varepsilon_2\alpha_1}{2\varepsilon_1 M} - a_{2,2})e^{-\frac{\alpha_1 x}{2\varepsilon_1 M}} \pm L_2 z \leq -\frac{C}{\varepsilon_1}e^{-\frac{\alpha_1 x}{2\varepsilon_1 M}} + \frac{C}{\varepsilon_1}e^{-\frac{\alpha_1 x}{\varepsilon_1}} \leq 0,$$

where $-a_{2,2} \leq -\alpha_2$ and (1.17) justify the first inequality and $\frac{1}{M} \leq 1$ gives the second inequality. Use minimum principle (4.2) to give $|z| \leq Ce^{-\frac{\alpha_1 x}{2\varepsilon_1 M}} = Ce^{-\frac{\alpha_2 x}{2\varepsilon_2}}$. Combine (1.16) and (1.3) to imply $\frac{\alpha_1}{2||a_{2,2}||_\infty} \leq \frac{\varepsilon_1}{\varepsilon_2} \leq \frac{\alpha_1}{\alpha_2}$ and then we can conclude that $|z| \leq \frac{C\varepsilon_1 e^{-\frac{\alpha_2 x}{2\varepsilon_2}}}{\varepsilon_2}$. Combine this with (1.15) to obtain the desired result. ■

Lemma 4

Assume that (1.3) holds. If z is the solution of (1.7c), then for $x \in \bar{\Omega}$,

a) $|z'(x)| \leq \frac{Ce^{-\frac{\alpha_2 x}{2\varepsilon_2}}}{\varepsilon_2}$,

b) $|z''(x)| \leq C(\frac{e^{-\frac{\alpha_2 x}{2\varepsilon_2}}}{\varepsilon_2^2} + \frac{e^{-\frac{\alpha_1 x}{\varepsilon_1}}}{\varepsilon_1^2})$,

c) $|z'''(x)| \leq \frac{C}{\varepsilon_2}(\frac{e^{-\frac{\alpha_2 x}{2\varepsilon_2}}}{\varepsilon_2^2} + \frac{e^{-\frac{\alpha_1 x}{\varepsilon_1}}}{\varepsilon_1^2})$.

Proof

a) To establish bounds for the derivatives of z it suffices to use the bound for z derived in lemma 3. We show that $|z'(x)| \leq \frac{Ce^{-\frac{\alpha_2 x}{2\varepsilon_2}}}{\varepsilon_2}$. First

$$\left|\int_x^1 a_{22}(t)z'(t)\,dt\right| = \left|a_{22}(t)z(t)\,|_x^1 - \int_x^1 a'_{22}(t)z(t)\,dt\right| \leq Ce^{-\frac{\alpha_2 x}{2\varepsilon_2}}. \tag{1.18}$$

By the mean value theorem, there exists a point $\xi \in (1-\varepsilon_2, 1)$ such that $z'(\xi) = \frac{z(1)-z(1-\varepsilon_2)}{\varepsilon_2} = \frac{-z(1-\varepsilon_2)}{\varepsilon_2}$ where $z(1) = 0$ from (1.7c). Then

$$|\,\varepsilon_2 z'(\xi)\,| \leq |\,z(1-\varepsilon_2)\,| \leq Ce^{-\frac{\alpha_2(1-\varepsilon_2)}{2\varepsilon_2}} \leq Ce^{-\frac{\alpha_2}{2\varepsilon_2}}. \tag{1.19}$$

Using (1.6) and (1.3) we have

$$\int_\xi^1 a_{2,1}(t)w_1'(t)\,dt \leq Ce^{-\frac{\alpha_1\xi}{\varepsilon_1}} \leq Ce^{-\frac{\alpha_2\xi}{\varepsilon_2}} \leq Ce^{-\frac{\alpha_2(1-\varepsilon_2)}{\varepsilon_2}} = Ce^{-\frac{\alpha_2}{\varepsilon_2}}. \tag{1.20}$$

Integrating (1.7c), that is $\varepsilon_2 z'' + a_{22}z' = -a_{21}w_1'$ from $t = \xi$ to $t = 1$ yields

$$\varepsilon_2 z'(1) = \varepsilon_2 z'(\xi) - \int_\xi^1 a_{22}(t)z'(t)\,dt - \int_\xi^1 a_{2,1}(t)w_1'(t)\,dt. \tag{1.21}$$

Then use (1.18), (1.19) and (1.20) to obtain $| \varepsilon_2 z'(1) | \leq Ce^{-\frac{\alpha_2}{2\varepsilon_2}} \leq Ce^{-\frac{\alpha_2 x}{2\varepsilon_2}}$ for all $x \leq 1$. Then take this bound, replace ξ by x in (1.21), then use (1.18) and (1.20) again to give

$$| \varepsilon_2 z'(x) | \leq \; Ce^{-\frac{\alpha_2 x}{2\varepsilon_2}}.$$

b) − c) Finally we find bounds for z'' and z''' by simply using the differential equation $\varepsilon_2 z'' + a_{2,2} z' = -a_{2,1} w_1'$. ■

In a similar way to that used for v_1 and w_1 [2] we find bounds for v_2 and w_2 which are simply stated as follows:

Lemma 5

Let v_2 be the solution of (1.7a) and w_2 be the solution of (1.7b) then we have the following bounds on these components and their derivatives.

$$\begin{aligned}
\| v_2^{(k)} \|_\infty &\leq C(1 + \varepsilon_2^{2-k}), \quad k = 0, 1, 2, 3, \\
| w_2(x) | &\leq Ce^{-\alpha_2 x/\varepsilon_2}, \\
| w_2^{(k)}(x) | &\leq C\varepsilon_2^{-k} e^{-\alpha_2 x/\varepsilon_2}, \quad k = 1, 2, 3.
\end{aligned}$$

Lemmas 1, 2, 3 and 4 can now be combined to give bounds on u_2 and its derivatives which we simply state.

Lemma 6

If u_2 is the solution of (1.1) then we have the following bounds:

a) $\| u_2 \|_\infty \leq C$,

b) $\| u_2' \|_\infty \leq C\varepsilon_2^{-1}$,

c) $\| u_2'' \|_\infty \leq C(\varepsilon_1^{-2} + \varepsilon_2^{-2})$,

d) $\| u_2''' \|_\infty \leq C\varepsilon_2^{-1}(\varepsilon_1^{-2} + \varepsilon_2^{-2})$.

1.3 Numerical method and discrete solution decomposition

A finite difference numerical method is constructed which will be used to generate approximations to the solutions of (1.1). We use standard upwinding on a Shishkin-type piecewise-uniform mesh using the transition points

$$\begin{aligned}
\sigma_2 &= \min\{\max\{\frac{8\varepsilon_1 \ln N}{\alpha_1}, \frac{8\varepsilon_2 \ln N}{\alpha_2}\}, \frac{1}{2}\}, \qquad (1.22) \\
\sigma_1 &= \min\{\frac{4\varepsilon_1 \ln N}{\alpha_1}, \frac{4\varepsilon_2 \ln N}{\alpha_2}, \frac{\sigma_2}{2}\}.
\end{aligned}$$

We specifically identify the five possibilities for the transition points as follows:

$$\text{(i).} \quad \sigma_1 = \frac{1}{4}, \qquad \sigma_2 = \frac{1}{2}, \tag{1.23a}$$

$$\text{(ii).} \quad \sigma_1 = \frac{4\varepsilon_2 \ln N}{\alpha_2}, \qquad \sigma_2 = \frac{1}{2}, \tag{1.23b}$$

$$\text{(iii).} \quad \sigma_1 = \frac{4\varepsilon_2 \ln N}{\alpha_2}, \qquad \sigma_2 = \frac{8\varepsilon_1 \ln N}{\alpha_1}, \tag{1.23c}$$

$$\text{(iv).} \quad \sigma_1 = \frac{4\varepsilon_1 \ln N}{\alpha_1}, \qquad \sigma_2 = \frac{8\varepsilon_2 \ln N}{\alpha_2}, \tag{1.23d}$$

$$\text{(v).} \quad \sigma_1 = \frac{4\varepsilon_1 \ln N}{\alpha_1}, \qquad \sigma_2 = \frac{1}{2}. \tag{1.23e}$$

These five possibilities correspond to the following values of the small parameters:

$$\begin{array}{lll}
\text{(i).} & \varepsilon_1 \geq \dfrac{\alpha_1}{16 \ln N}, & \varepsilon_2 \geq \dfrac{\alpha_2}{16 \ln N}, \\
\text{(ii).} & \varepsilon_1 \geq \dfrac{\alpha_1}{16 \ln N}, & \varepsilon_2 \leq \dfrac{\alpha_2}{16 \ln N}, \\
\text{(iii).} & \varepsilon_j \leq \dfrac{\alpha_j}{16 \ln N}, & j = 1,2 \text{ and } \dfrac{\varepsilon_1}{\alpha_1} \geq \dfrac{\varepsilon_2}{\alpha_2}, \\
\text{(iv).} & \varepsilon_j \leq \dfrac{\alpha_j}{16 \ln N}, & j = 1,2 \text{ and } \dfrac{\varepsilon_1}{\alpha_1} \leq \dfrac{\varepsilon_2}{\alpha_2}, \\
\text{(v).} & \varepsilon_1 \leq \dfrac{\alpha_1}{16 \ln N}, & \varepsilon_2 \geq \dfrac{\alpha_2}{16 \ln N}.
\end{array}$$

The piecewise–uniform mesh which is illustrated in figure 1.1 is then given by

$$\bar{\Omega}^N_{\varepsilon_1,\varepsilon_2} = \{x_i\}, \quad x_i = \begin{cases} 4\sigma_1 i/N & if \quad i \leq N/4 \\ x_{i-1} + 4(\sigma_2 - \sigma_1)/N & if \quad N/4 < i \leq N/2 \\ x_{i-1} + 2(1 - \sigma_2)/N & if \quad i > N/2 \end{cases}. \tag{1.24}$$

We use the notation $h_i = x_i - x_{i-1}$ and $\bar{h}_i = (h_{i+1} + h_i)/2$. On the intervals $[0, \sigma_1]$, $[\sigma_1, \sigma_2]$ and $[\sigma_2, 1]$ we define the mesh widths $H_1 = \frac{4\sigma_1}{N}$, $H_2 = \frac{4(\sigma_2 - \sigma_1)}{N}$ and $H_3 = \frac{2(1-\sigma_2)}{N}$ respectively. In chapters two and three the domain $\bar{\Omega}$ is divided into the intervals $[0, \sigma_1)$, $[\sigma_1, \sigma_2)$ and $[\sigma_2, 1]$. These are called the inner layer region, the outer layer region and the regular region respectively.

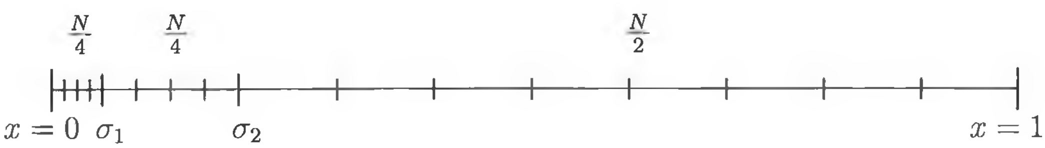

Figure 1.1: The piecewise uniform mesh for two coupled convection–diffusion differential equations.

Note, by design, $H_1 \le H_2$ since $\sigma_1 \le \frac{\sigma_2}{2}$. As a consequence note that $\sigma_2 - \sigma_1 \ge \frac{\sigma_2}{2}$, $H_2 = \frac{4(\sigma_2-\sigma_1)}{N} \ge \frac{2\sigma_2}{N}$ and $H_3 = \frac{2(1-\sigma_2)}{N} \ge \frac{1}{N}$. We also use the following in Chapters 2 and 3:

$$
\begin{aligned}
\frac{1}{H_2} &\le \frac{CN}{\sigma_2}, && && (1.25a)\\
\frac{1}{H_3} &\le N, && && (1.25b)\\
\frac{\alpha_1 H_k}{\varepsilon_1} &\ge \frac{16\ln N}{N}, \text{ if } \sigma_k = \frac{4k\varepsilon_1 \ln N}{\alpha_1}, && \text{k} = 1 \text{ or } 2, && (1.25c)\\
\frac{\alpha_2 H_2}{\varepsilon_2} &\ge \frac{16\ln N}{N}, \text{ if } \sigma_2 = \frac{8\varepsilon_2 \ln N}{\alpha_2}. && && (1.25d)
\end{aligned}
$$

For any mesh function $\{Y_i\}$ note that the forward difference operator is given by $D^+Y_i = \frac{Y_{i+1}-Y_i}{h_{i+1}}$ when $0 \le i \le N-1$ and the standard central difference operator is given by $\delta^2 Y_i = (\frac{Y_{i+1}-Y_i}{h_{i+1}} - \frac{Y_i - Y_{i-1}}{h_i})/\bar{h}_i$ when $1 \le i \le N-1$. We then define the discrete operators, $L_k^N \equiv \varepsilon_k \delta^2 + a_{k,k}(x_i)D^+$, where $k = 1, 2$. When $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$ the difference scheme corresponding to problem (1.1) is then:

$$
\begin{aligned}
L_1^N U_{1,i} &\equiv \varepsilon_1 \delta^2 U_{1,i} + a_{1,1}(x_i)D^+U_{1,i} = f_1(x_i), && (1.26a)\\
L_2^N U_{2,i} &\equiv \varepsilon_2 \delta^2 U_{2,i} + a_{2,2}(x_i)D^+U_{2,i} = f_2(x_i) - a_{2,1}(x_i)D^+U_{1,i}, && (1.26b)\\
U_{j,0} &= u_j(0), \quad U_{j,N} = u_j(1), \quad j = 1, 2,
\end{aligned}
$$

where $U_{j,i}$ is the corresponding numerical approximation to $u_j(x_i)$ $(j = 1, 2)$ at each mesh point $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$.

The discrete solutions U_1 and U_2 are decomposed in an analogous fashion to the continuous solutions. We first decompose the solution of (1.26a). Let $U_1 = V_1 + W_1$, where

$$
\begin{aligned}
L_1^N V_1 &= f_1(x_i), \quad V_1(0) = v_1(0), \quad V_1(1) = v_1(1), && (1.27a)\\
L_1^N W_1 &= 0, \quad W_1(0) = w_1(0), \quad W_1(1) = 0. && (1.27b)
\end{aligned}
$$

Next the solution of (1.26b) is decomposed. Let $U_2 = V_2 + W_2 + Z$, where

$$
\begin{aligned}
L_2^N V_2 &= f_2(x_i) - a_{2,1}(x_i)D^+V_1, \quad V_2(0) = v_2(0), \quad V_2(1) = v_2(1), && (1.28a)\\
L_2^N W_2 &= 0, \quad W_2(0) = w_2(0), \quad W_2(1) = 0, && (1.28b)\\
L_2^N Z &= -a_{2,1}(x_i)D^+W_1, \quad Z(0) = z(0) = D, \quad Z(1) = 0, && (1.28c)
\end{aligned}
$$

and recall from (1.7c) that $z(0) = 0$ when (1.3) holds.

Chapter 2

The error bounds

We start by establishing the pointwise convergence of the numerical method for the uniform mesh case. Once this is done the remainder of the chapter then provides the analysis for the non–uniform mesh cases. The error bounds (1.4) for these cases are obtained by proving the component-wise errors for $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$ as follows:

$$
\begin{aligned}
|(V_1 - v_1)(x_i)| &\leq CN^{-1}, && \text{(2.1a)}\\
|(W_1 - w_1)(x_i)| &\leq CN^{-1}\ln N, && \text{(2.1b)}\\
|(V_2 - v_2)(x_i)| &\leq CN^{-1}(\ln N)^2, && \text{(2.1c)}\\
|(W_2 - w_2)(x_i)| &\leq CN^{-1}\ln N, && \text{(2.1d)}\\
|(Z - z)(x_i)| &\leq CN^{-1}(\ln N)^2, && \text{(2.1e)}
\end{aligned}
$$

where v_1, w_1, v_2, w_2 and z are the solutions of (1.5a), (1.5b), (1.7a), (1.7b) and (1.7c) respectively, V_1, W_1, V_2, W_2 and Z are the solutions of (1.27a), (1.27b), (1.28a), (1.28b) and (1.28c) respectively.

The bounds for the first regular component (2.1a) mainly follow those of [2]. Weighted derivative results for the first components are established which are needed in this chapter for the second regular component proof (2.1c) and in chapter three for the coupling component proof (2.1e). This chapter establishes the above bounds on the regular component errors (2.1a) and (2.1c) and the singular component errors (2.1b) and (2.1d).

The notation $\tilde{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2\}$ and $\bar{\varepsilon} = \max\{\varepsilon_1, \varepsilon_2\}$ is adopted.

2.1 Uniform mesh case

Note that for any fixed values of the small parameters ε_1 and ε_2, if N is sufficiently large i.e. $\ln N \geq \frac{\alpha_j}{16\varepsilon_j}$, $j = 1, 2$ then the piecewise-uniform mesh is a uniform mesh where $\sigma_1 = \frac{1}{4}$ and $\sigma_2 = \frac{1}{2}$. First we note the truncation error where the bounds on the derivatives given in lemma 6 are used in conjunction with the facts that $\varepsilon_1^{-1} \leq C \ln N$ and $\varepsilon_2^{-1} \leq C \ln N$. Then at each mesh point

$$
\begin{aligned}
\left|L_2^N(U_2-u_2)\right| &\leq \left|(L_2-L_2^N)u_2\right| + \left|L_2u_2 - L_2^NU_2\right|, \qquad (2.2)\\
\left|(L_2-L_2^N)u_2\right| &\leq \frac{2\varepsilon_2\bar{h}_i}{3} \parallel u_2^{(3)} \parallel_{\Omega_i} + \frac{a_{2,2}(x_i)h_i}{2} \parallel u_2^{(2)} \parallel_{\Omega_i}\\
&\leq CN^{-1}(\ln N)^2,
\end{aligned}
$$

where we define the norm

$$\parallel v \parallel_{\Omega_i} = \max_{x\in\Omega_i}\{|v(x)|\},\ \Omega_i = [x_{i-1}, x_{i+1}] \text{ and } 1 \leq i \leq N-1.$$

The second term of (2.2) is

$$\left|L_2^NU_2 - L_2u_2\right| = \left|-a_{2,1}(D^+U_1 - u_1')(x_i)\right| \leq CN^{-1}(\ln N)^2$$

where we use theorem 3.17 from [2] which gives $|\varepsilon_1(D^+U_1 - u_1')(x_i)| \leq CN^{-1}\ln N$. Then

$$\left|L_2^N(U_2-u_2)\right| \leq CN^{-1}(\ln N)^2.$$

We use the barrier function $\Psi_i = \frac{CN^{-1}(\ln N)^2(1-x_i)}{\alpha_2} \pm (U_2-u_2)(x_i)$ then $\Psi_0 \geq 0$ and $\Psi_N \geq 0$. Then $L_2^N\Psi_i = -\frac{Ca_{2,2}N^{-1}(\ln N)^2}{\alpha_2} \pm L_2^N(U_2-u_2) \leq 0$. Apply a discrete minimum principle [2] to give $\Psi_i \geq 0$ for all $0 \leq i \leq N$ and hence $|(U_2-u_2)(x_i)| \leq CN^{-1}(\ln N)^2$. Now that the uniform mesh case is completed we concentrate on the non-uniform mesh cases in the following sections.

2.2 Error bounds for the regular components

Error bounds for the first regular component

The next three lemmas are essentially lemmas 3.4, 3.13 (the regular component part) and 3.14 of [2]. Using [2] it is easy to check that they hold over our two transition point mesh. The positions of the transition points are not crucial in the proofs.

Lemma 7

If v_1, V_1 are the solutions of (1.5a), (1.27a) respectively then
$|(V_1 - v_1)(x_i)| \leq CN^{-1}(1-x_i)$ where $x_i \in \bar{\Omega}^N_{\varepsilon_1,\varepsilon_2}$.

Lemma 8

If v_1, V_1 are the solutions of (1.5a), (1.27a) respectively then
$|D^+v_1(x_i) - v_1'(x)| \leq CN^{-1}$ for all $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cup \{0\}$ and all $x \in [x_i, x_{i+1}]$.

Lemma 9

If v_1, V_1 are the solutions of (1.5a), (1.27a) respectively then
$|\varepsilon_1 D^+(V_1 - v_1)(x_i)| \leq CN^{-1}$ where $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cup \{0\}$.

The next lemma gives sharper weighted derivative estimates that are employed in bounding $|(V_2 - v_2)(x_i)|$. The proof of lemma 9 follows that in [2] where the difference equation (2.3) was integrated across the domain from $x = 1$ to $x = 0$. The proof of the next lemma then uses this result and further sharpens the weighted derivative estimates by integrating across the domain from $x = 0$ to $x = 1$.

Lemma 10

If the transition points (1.22) are such that $\sigma_1 = \frac{4\varepsilon_1 \ln N}{\alpha_1}$ or $\sigma_2 = \frac{8\varepsilon_1 \ln N}{\alpha_1}$ then

$$|D^+(V_1 - v_1)(x_i)| \leq CN^{-1} \ln N \times \begin{cases} \frac{1}{\tilde{\varepsilon}} & if \quad x_i < \sigma_1 \\ \frac{1}{\bar{\varepsilon}} & if \quad \sigma_1 \leq x_i < \sigma_2 \\ 1 & if \quad x_i \geq \sigma_2 \end{cases}, \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cup \{0\},$$

where v_1, V_1 are the solutions of (1.5a), (1.27a) respectively.

Proof

Using lemma 9 it is clear that the result holds over the inner layer region $[0, \sigma_1)$, as $\frac{1}{\varepsilon_1} \leq \frac{1}{\tilde{\varepsilon}}$. We start with the case where $\sigma_1 = \frac{4\varepsilon_1 \ln N}{\alpha_1}$, $\tilde{\varepsilon} = \varepsilon_1$ and $\bar{\varepsilon} = \varepsilon_2$. We prove the result in the following steps:

(i) First we show that the result holds at the point $x = \sigma_1$.
(ii) Then the result on the entire outer layer region $[\sigma_1, \sigma_2)$ is proved.
(iii) Next the result is verified at the point $x = \sigma_2$.
(iv) Finally the result is proved on the regular region $[\sigma_2, 1)$.

(i) Start with the point $x = \sigma_1$ and let $e_j = (V_1 - v_1)(x_j)$, then

$$L_1^N e_j = \varepsilon_1 \delta^2 e_j + a_{1,1}(x_i) D^+ e_j = \tau_j. \tag{2.3}$$

Use (1.5a) and (1.27a) to obtain $\left|L_1^N e_j\right| = \left|(L_1 - L_1^N) v_1\right|$. Then apply a standard truncation argument at each mesh point and use (1.6a) to obtain

$$\left|(L_1 - L_1^N) v_1\right| \leq \frac{2\varepsilon_1 \bar{h}_i \parallel v_1^{(3)} \parallel_{\Omega_i}}{3} + \frac{a_{2,2}(x_i) h_i \parallel v_1^{(2)} \parallel_{\Omega_i}}{2} \leq CN^{-1}. \tag{2.4}$$

Hence we can define

$$\parallel \tau \parallel_{\Omega^N_{\varepsilon_1,\varepsilon_2}} = \max_{1 \leq j \leq N-1} \{|\tau_j|\} \leq CN^{-1}. \tag{2.5}$$

Let $E_j = D^+e_j$ and rearrange (2.3) to give $E_j - E_{j-1} + \rho_j E_j = \frac{\rho_j \tau_j}{a_{1,1}(x_j)}$, where $\rho_j = \frac{a_{1,1}(x_j)h_j}{\varepsilon_1}$. Now apply lemma 17 from Appendix A where $k = 0$, $i = \frac{N}{4}$. Then use (2.5) and when $0 \leq j \leq \frac{N}{4} - 1$ we can take $\rho_j \geq \rho = \frac{\alpha_1 H_1}{\varepsilon_1}$ to obtain

$$\left|E_{\frac{N}{4}}\right| \leq \frac{|E_0|\,(1+\rho)^{-(\frac{N}{4}-1)}}{1+\rho_{\frac{N}{4}}} + CN^{-1}. \tag{2.6}$$

Also use lemma 16 from Appendix A which gives

$$(1+\rho)^{-(\frac{N}{4}-1)} = (1+\rho)^{-\frac{N}{4}}(1+\rho) = (1+\frac{16\ln N}{N})^{-\frac{N}{4}}(1+\frac{16\ln N}{N}) \leq CN^{-1}. \tag{2.7}$$

Note that since $\sigma_2 = \frac{8\varepsilon_2 \ln N}{\alpha_2}$ or $\frac{1}{2}$ then $\frac{1}{\sigma_2} \leq \frac{C}{\varepsilon_2}$. Note also that $\rho_{\frac{N}{4}} \geq \frac{\alpha_1(H_1+H_2)}{2\varepsilon_1} = \frac{2\sigma_2}{N\varepsilon_1}$ and thus $(1+\rho_{\frac{N}{4}})^{-1} \leq \frac{C\varepsilon_1 N}{\varepsilon_2}$. Combine this with the fact that $|E_0| \leq \frac{CN^{-1}}{\varepsilon_1}$ (from lemma 9) and use (2.7) with (2.6) to obtain

$$\left|E_{N/4}\right| \leq \frac{CN^{-2}}{\varepsilon_1(1+\rho_{\frac{N}{4}})} + CN^{-1} \leq \frac{CN^{-1}}{\varepsilon_2} = \frac{CN^{-1}}{\bar{\varepsilon}}. \tag{2.8}$$

Thus the result holds for the first transition point σ_1.

(ii) It is now easy to obtain the result over the entire outer layer region $[\sigma_1, \sigma_2)$. Using lemma 17 successively on $\frac{N}{4} < i < \frac{N}{2}$, $i = \frac{N}{2}$ and on $\frac{N}{2} < i < N-1$ it is clear that $|E_i| \leq \frac{CN^{-1}}{\varepsilon_2}$ over the outer layer region. When $\sigma_2 = \frac{1}{2}$ the lemma is completed over $\Omega^N_{\varepsilon_1,\varepsilon_2}$ since $\varepsilon_2^{-1} \leq C\ln N$.

(iii) Now we only need to consider $\sigma_2 = \frac{8\varepsilon_2 \ln N}{\alpha_2}$. The result is verified for this point. When $\frac{N}{4} < i < \frac{N}{2}$, use lemma 17 to give the result

$$\left|E_{\frac{N}{2}}\right| \leq \frac{\left|E_{\frac{N}{4}}\right|(1+\rho)^{-(\frac{N}{4}-1)}}{1+\rho_{\frac{N}{2}}} + CN^{-1}, \tag{2.9}$$

where $\rho_j \geq \rho = \frac{\alpha_1 H_2}{\varepsilon_1} \geq \frac{\alpha_2 H_2}{\varepsilon_2}$. Using (1.25d) we have $\rho > \frac{4\ln N}{N}$. Note that since $H_3 \geq \frac{1}{N}$ we have $\rho_{\frac{N}{2}} \geq \frac{\alpha_1(H_2+H_3)}{2\varepsilon_1} \geq \frac{CH_3}{2\bar{\varepsilon}} \geq \frac{C}{\bar{\varepsilon}N}$. Combine this result, the fact that $\rho > \frac{4\ln N}{N}$, lemma 16, (2.8) and (2.9), to obtain

$$\left|E_{\frac{N}{2}}\right| \leq \frac{CN^{-2}}{\bar{\varepsilon}(1+\rho_{\frac{N}{2}})} + CN^{-1} \leq CN^{-1}.$$

Thus the result holds for the second transition point σ_2.

(iv) Finally on the regular region $[\sigma_2, 1]$ using lemma 17 it is clear that $|E_i| \leq CN^{-1}$ $(i \geq \frac{N}{2})$. This completes the proof where $\sigma_1 = \frac{4\varepsilon_1 \ln N}{\alpha_1}$.

Finally when $\sigma_2 = \frac{8\varepsilon_1 \ln N}{\alpha_1}$ using lemma 9 it is easy to see that the result holds over $[0, \sigma_2)$. We note (2.9) holds where $\rho \geq \frac{\alpha_1 H_2}{\varepsilon_1}$ and we use (1.25c) to give $\rho > \frac{4 \ln N}{N}$. The proof is completed in a similar manner to the previous case. ■

Error bounds for the second regular component

The error bounds for the second regular component are easily established using the results just obtained for the first regular component and the stability technique by Andreyev and Savin [1]. In appendix B we check that the technique holds for the standard upwinding used in this thesis. We refer to this as the stability technique [1] in the rest of the text. The notation $\breve{e}_i = (V_2 - v_2)(x_i)$ is adopted.

Lemma 11

If the transition points (1.22) are such that $\sigma_1 = \frac{4\varepsilon_j \ln N}{\alpha_j}$, $j = 1$ or 2 then $|\breve{e}_i| = |(V_2 - v_2)(x_i)| \leq CN^{-1}(\ln N)^2$ at each mesh point $x_i \in \bar{\Omega}^N_{\varepsilon_1,\varepsilon_2}$, where v_2 and V_2 are the solutions of (1.7a) and (1.28a) respectively.

Proof

Start by considering the truncation error $\left|L_2^N(\breve{e})\right| \leq \left|L_2^N V_2 - L_2 v_2\right| + \left|(L_2 - L_2^N)v_2\right|$ where $L_2^N V_2 - L_2 v_2 = -a_{2,1}(D^+ V_1 - v_1')(x_i)$ and

$$\left|(D^+ V_1 - v_1')(x_i)\right| \leq \left|D^+(V_1 - v_1)(x_i)\right| + \left|(D^+ v_1 - v_1')(x_i)\right|. \qquad (2.10)$$

Use standard truncation error arguments and lemma 5 to obtain

$$\left|(L_2 - L_2^N)v_2\right| \leq \frac{2\varepsilon_2 \bar{h}_i \parallel v_2^{(3)} \parallel_{\Omega_i}}{3} + \frac{a_{2,2}(x_i) h_i \parallel v_2^{(2)} \parallel_{\Omega_i}}{2} \leq CN^{-1}. \qquad (2.11)$$

When $\sigma_1 = \frac{4\varepsilon_1 \ln N}{\alpha_1}$ or $\sigma_2 = \frac{8\varepsilon_1 \ln N}{\alpha_1}$, combine this result with (2.10) in conjunction with lemmas 8 and 10 to derive

$$\left|L_2^N \breve{e}(x_i)\right| \leq CN^{-1} \ln N \times \begin{cases} \frac{1}{\varepsilon} & if \quad x_i < \sigma_1 \\ \frac{1}{\varepsilon} & if \quad \sigma_1 \leq x_i < \sigma_2 \,, x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cup \{0\}. \\ 1 & if \quad x_i \geq \sigma_2 \end{cases}$$

Since $\breve{e}(0) = \breve{e}(1) = 0$, we can then use the stability technique [1] to establish the error bounds. Apply the result of the theorem in appendix B to obtain

$$
\begin{aligned}
\| \breve{e} \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}} &\leq C\sum_{i=1}^{N-1} \bar{h}_i \left| L_2^N \breve{e}(x_i) \right| \\
&\leq C(\sum_{i=1}^{\frac{N}{4}-1} H_1 \left| L_2^N \breve{e}(x_i) \right| + \sum_{i=\frac{N}{4}}^{\frac{N}{2}-1} H_2 \left| L_2^N \breve{e}(x_i) \right| + \sum_{i=N/2}^{N-1} H_3 \left| L_2^N \breve{e}(x_i) \right|) \\
&\leq CN^{-1}(\ln N)^2, \qquad (2.12)
\end{aligned}
$$

where $\frac{H_1}{\bar{\varepsilon}} \leq \frac{C\ln N}{N}$, $\frac{H_2}{\bar{\varepsilon}} \leq \frac{C\ln N}{N}$ and $H_3 \leq \frac{C}{N}$.

When $\sigma_1 = \frac{4\varepsilon_2 \ln N}{\alpha_2}$ and $\sigma_2 = \frac{1}{2}$ then $\varepsilon_1^{-1} \leq C\ln N$. Now using lemma 9 it is easy to see that $|D^+(V_1 - v_1)(x_i)| \leq CN^{-1}\ln N$ on $\Omega^N_{\varepsilon_1,\varepsilon_2}$. Substitute this result and lemma 8 into (2.10) to obtain $|(D^+V_1 - v_1')(x_i)| \leq CN^{-1}\ln N$ on $\Omega^N_{\varepsilon_1,\varepsilon_2}$. Combine this result with (2.11) to give $\left| L_2^N \breve{e}(x_i) \right| \leq CN^{-1}\ln N$, and in a similar manner to (2.12) we obtain $\| \breve{e} \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}} \leq CN^{-1}(\ln N)^2$. ■

Barrier Functions

First we introduce notation for discrete functions that are used as barrier functions in bounding the errors for the singular components in this chapter and the coupling component in the next chapter. When $k = 1, 2$ we let

$$
B_{\gamma_k,\varepsilon_k}(x_i) = \begin{cases} 1 + \frac{\gamma_k h_1}{\varepsilon_k} & if \quad i = -1 \quad x_{-1} = -h_1 \\ 1 & if \quad i = 0 \\ \prod_{j=1}^{i}(1 + \frac{\gamma_k h_j}{\varepsilon_k})^{-1} & if \quad i > 0 \end{cases} . \qquad (2.13)
$$

We note some properties of this barrier function that are subsequently used. Note

$$
\begin{aligned}
B_{\gamma_k,\varepsilon_k}(\sigma_1) &= (1 + \frac{\gamma_k H_1}{\varepsilon_k})^{-\frac{N}{4}}, && (2.14a) \\
B_{\gamma_k,\varepsilon_k}(\sigma_2) &= (1 + \frac{\gamma_k H_1}{\varepsilon_k})^{-\frac{N}{4}}(1 + \frac{\gamma_k H_2}{\varepsilon_k})^{-\frac{N}{4}}. && (2.14b)
\end{aligned}
$$

Note the following when $0 \leq i \leq N - 1$:

$$
\begin{aligned}
D^+ B_{\gamma_k,\varepsilon_k}(x_i) &= -\frac{\gamma_k}{\varepsilon_k} B_{\gamma_k,\varepsilon_k}(x_{i+1}), \\
D^- B_{\gamma_k,\varepsilon_k}(x_i) &= -\frac{\gamma_k}{\varepsilon_k} B_{\gamma_k,\varepsilon_k}(x_i), \\
\delta^2 B_{\gamma_k,\varepsilon_k}(x_i) &= \frac{\gamma_k^2 h_{i+1}}{\varepsilon_k^2 \bar{h}_i} B_{\gamma_k,\varepsilon_k}(x_{i+1}).
\end{aligned}
$$

When $k = 1, 2, m = 1, 2$ and $0 \leq i \leq N-1$ we will use

$$
\begin{aligned}
L_m^N B_{\gamma_k,\varepsilon_k}(x_i) &= \frac{\gamma_k}{\varepsilon_k}(\frac{\varepsilon_m \gamma_k h_{i+1}}{\varepsilon_k \bar{h}_i} - a_{m,m}) B_{\gamma_k,\varepsilon_k}(x_{i+1}) \\
&= \frac{\gamma_k}{\varepsilon_k}(\frac{2\varepsilon_m \gamma_k h_{i+1}}{\varepsilon_k (h_{i+1} + h_i)} - a_{m,m}) B_{\gamma_k,\varepsilon_k}(x_{i+1}). \qquad (2.15)
\end{aligned}
$$

Also note when $\gamma_k \leq \alpha_k$

$$
e^{-\frac{\alpha_k x_i}{\varepsilon_k}} \leq e^{-\frac{\gamma_k x_i}{\varepsilon_k}} \leq B_{\gamma_k,\varepsilon_k}(x_i), \quad k = 1, 2. \qquad (2.16)
$$

Next we develop two inequalities that will be used in the proof of lemma 12 where $\gamma_1 = \frac{\alpha_1}{4}$. When $\frac{N}{4} \leq i < \frac{N}{2}$, $h_{\frac{N}{4}} = H_1$ otherwise $h_i = H_2$ and $h_{i+1} = H_2$ then

$$
\begin{aligned}
e^{\frac{\alpha_1 H_2}{2\varepsilon_1}} B_{\gamma_1,\varepsilon_1}(x_{i+1}) &= e^{\frac{\alpha_1 H_2}{2\varepsilon_1}} (1 + \frac{\alpha_1 h_i}{4\varepsilon_1})^{-1} (1 + \frac{\alpha_1 h_{i+1}}{4\varepsilon_1})^{-1} B_{\gamma_1,\varepsilon_1}(x_{i-1}) \\
&\geq e^{\frac{\alpha_1 H_2}{2\varepsilon_1}} (1 + \frac{\alpha_1 H_1}{4\varepsilon_1})^{-1} (1 + \frac{\alpha_1 H_2}{4\varepsilon_1})^{-1} B_{\gamma_1,\varepsilon_1}(x_{i-1}) \\
&\geq e^{\frac{\alpha_1 H_2}{2\varepsilon_1}} (1 + \frac{\alpha_1 H_2}{4\varepsilon_1})^{-2} B_{\gamma_1,\varepsilon_1}(x_{i-1}) \\
&\geq B_{\gamma_1,\varepsilon_1}(x_{i-1}), \qquad N/4 \leq i < N/2, \qquad (2.17)
\end{aligned}
$$

where (2.16) justifies the final inequality. In a similar manner to (2.17) we obtain

$$
e^{\frac{\alpha_1 H_1}{2\varepsilon_1}} B_{\gamma_1,\varepsilon_1}(x_{i+1}) \geq B_{\gamma_1,\varepsilon_1}(x_{i-1}), \qquad 1 \leq i < N/4, \qquad (2.18)
$$

where $h_i = h_{i+1} = H_1$. Also note that

$$
B_{\gamma_k,\varepsilon_k}(x_{N-1}) \leq ... \leq B_{\gamma_k,\varepsilon_k}(x_{\frac{N}{2}}) \leq ... \leq B_{\gamma_k,\varepsilon_k}(x_{\frac{N}{4}}), \quad k = 1, 2. \qquad (2.19)
$$

2.3 Error bounds for the singular components

In this section the singular components are bounded. Weighted derivative estimates for the first singular component are also established in this section. These are used in chapter three in connection with the coupling component.

The proof for the singular component in [2] uses a single transition point but this does not directly apply here where we use two transition points. There are four cases to be considered due to the transition points (1.22). In our "two transition point" proof three of the four cases are merged. The fourth case follows easily. The proof is also structured so that the proof of the second singular component follows identically.

Error bounds for the first singular component

Lemma 12

If the transition points (1.22) are such that $\sigma_1 = \frac{4\varepsilon_j \ln N}{\alpha_j}$, $j = 1$ or 2 then $|(W_1 - w_1)(x_i)| \le CN^{-1}(\ln N)$ at each mesh point $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$, where w_1 and W_1 are the solutions of (1.5b) and (1.27b) respectively.

Proof

We start by considering the cases where $\sigma_k = \frac{4k\varepsilon_1 \ln N}{\alpha_1}$, $k = 1$ or $k = 2$. The stages of the proof are as follows:

(i) A bound is established for $|w_1|$ over the interval $[\sigma_k, 1]$, $k = 1, 2$.

(ii) Next $|W_1|$ is bounded over the same interval. The triangle inequality then provides the bound for $|W_1 - w_1|$ on this interval.

(iii) A barrier function argument provides the bound for $|W_1 - w_1|$ over the interval $[0, \sigma_k)$, $k = 1, 2$.

(i) Use (1.6b) to obtain

$$|w_1| \le CN^{-2}, \text{ for } x \in [\sigma_k, 1], \text{ if } \sigma_k = \frac{4k\varepsilon_1 \ln N}{\alpha_1}, \quad k = 1 \text{ or } 2. \tag{2.20}$$

(ii) To establish a similar bound for $|W_1|$, (2.13) is used to define the barrier function $\Psi_i = |W_1(0)|\, B_{\gamma_1,\varepsilon_1}(x_i) \pm W_1(x_i)$, $\Psi_0 \ge 0$ and $\Psi_N \ge 0$. Recall from (1.5b) that $|W_1(0)| = |w_1(0)|$ is bounded. Take $\gamma_1 = \frac{\alpha_1}{2}$ and use result (2.15) to obtain

$$\begin{aligned} L_1^N \Psi_i &= \frac{\gamma_1}{\varepsilon_1}[\frac{2\gamma_1 h_{i+1}}{h_{i+1} + h_i} - a_{1,1}]\, |W_1(0)|\, B_{\gamma_1,\varepsilon_1}(x_{i+1}) \pm L_1^N W_1(x_i) \\ &\le \frac{\gamma_1}{\varepsilon_1}(2\gamma_1 - a_{1,1})\, |W_1(0)|\, B_{\gamma_1,\varepsilon_1}(x_{i+1}) \pm 0 \le 0, \end{aligned}$$

where $\alpha_1 < a_{1,1}$. Then apply a discrete minimum principle [2] to obtain

$$|W_1(x_i)| \le |W_1(0)|\, B_{\gamma_1,\varepsilon_1}(x_i) = |w_1(0)|\, B_{\gamma_1,\varepsilon_1}(x_i), \quad 1 \le i \le N-1. \tag{2.21}$$

Now using (2.14) we get

$$B_{\gamma_1,\varepsilon_1}(\sigma_k) \le (1+\frac{\alpha_1 H_k}{2\varepsilon_1})^{-\frac{N}{4}} \le (1+\frac{8\ln N}{N})^{-\frac{N}{4}} \le CN^{-2}, \quad k = 1 \text{ or } 2, \tag{2.22}$$

where (1.25c) and lemma 16 from Appendix A are used. Then take (2.19) and (2.21) to give

$$|W_1(x_i)| \le CN^{-2}, \text{ for } x_i \in [\sigma_k, 1] \text{ if } \sigma_k = \frac{4k\varepsilon_1 \ln N}{\alpha_1}, \quad k = 1 \text{ or } 2. \tag{2.23}$$

(iii) It remains to prove the result over the interval $[0, \sigma_k)$, $k = 1, 2$. Use a standard truncation argument similar to (2.4) and use (1.6c) to obtain

$$\left|L_1^N(W_1 - w_1)(x_i)\right| \leq CN^{-1}\sigma_k\varepsilon_1^{-2}e^{-\frac{\alpha_1 x_{i-1}}{\varepsilon_1}}, \quad x_i \in [0, \sigma_k), \quad k = 1, 2. \tag{2.24}$$

Let

$$\Psi_i = \frac{8C\sigma_k\varepsilon_1^{-1}N^{-1}e^{\frac{\alpha_1 H_k}{2\varepsilon_1}}}{\alpha_1^2}B_{\gamma_1,\varepsilon_1}(x_i) + C'N^{-1} \pm (W_1 - w_1)(x_i),$$

then $\Psi_0 \geq 0$ and $\Psi_{\frac{N}{4}} \geq 0$ where $\gamma_1 = \frac{\alpha_1}{4}$. Then using (2.15) we have

$$\begin{aligned}
L_1^N\Psi_i &= \frac{2C\sigma_k e^{\frac{\alpha_1 H_k}{2\varepsilon_1}}}{\alpha_1\varepsilon_1^2 N}\left(\frac{h_{i+1}\alpha_1}{4h_i} - a_{1,1}\right)B_{\gamma_1,\varepsilon_1}(x_{i+1}) \pm L_1^N(W_1 - w_1)(x_i) \\
&\leq \frac{C\sigma_k}{\varepsilon_1^2 N}\left[-e^{\frac{\alpha_1 H_k}{2\varepsilon_1}}B_{\gamma_1,\varepsilon_1}(x_{i+1}) + e^{-\frac{\alpha_1 x_{i-1}}{\varepsilon_1}}\right] \\
&\leq \frac{C\sigma_k}{\varepsilon_1^2 N}\left[-B_{\gamma_1,\varepsilon_1}(x_{i-1}) + e^{-\frac{\alpha_1 x_{i-1}}{\varepsilon_1}}\right] \leq 0,
\end{aligned}$$

where the first inequality is justified using $-a_{1,1} \leq -\alpha_1$, $\frac{h_{i+1}}{h_i} \leq 2$ and (2.24). The second inequality is established using (2.18) when $k = 1$ and (2.17) when $k = 2$. The final inequality is obtained from (2.16).

We can now apply a discrete minimum principle [2] which implies $\Psi_i \geq 0$. Since $e^{\frac{\alpha_1 H_k}{2\varepsilon_1}} \leq e^{\frac{16\ln N}{N}} \leq C$ and $\frac{\sigma_k}{\varepsilon_1 N} \leq CN^{-1}\ln N$ then $|(W_1 - w_1)(x_i)| \leq CN^{-1}(\ln N)$ on $\Omega^N_{\varepsilon_1,\varepsilon_2}$ as required where $\sigma_k = \frac{4k\varepsilon_1\ln N}{\alpha_1}$, $k = 1$ or $k = 2$.

Finally if $\sigma_1 = \frac{4\varepsilon_2\ln N}{\alpha_2}$ then $\sigma_2 = \frac{1}{2}$ gives $\varepsilon_1^{-1} \leq C\ln N$. In a similar manner to (2.24) we note the truncation error $\left|L_1^N(W_1 - w_1)(x_i)\right| \leq C\varepsilon_1^{-2}N^{-1}e^{-\frac{\alpha_1 x_{i-1}}{\varepsilon_1}}$ over $\Omega^N_{\varepsilon_1,\varepsilon_2}$. Define the barrier function

$$\Psi_i = \frac{8Ce^{\frac{\alpha_1 H_3}{2\varepsilon_1}}}{\alpha_1^2}\varepsilon_1^{-1}N^{-1}B_{\gamma_1,\varepsilon_1}(x_i) \pm (W_1 - w_1)(x_i)$$

where $\gamma_1 = \frac{\alpha_1}{4}$. Apply a discrete minimum principle [2] over $\Omega^N_{\varepsilon_1,\varepsilon_2}$ in a similar manner to the previous case to complete the proof. ∎

We can now combine lemmas 7 and 12 to bound (1.4) when $j = 1$ as follows:

$$|(U_1 - u_1)(x_i)| \leq |(V_1 - v_1)(x_i)| + |(W_1 - w_1)(x_i)| \leq CN^{-1}\ln N, \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}.$$

The next lemma proves the sharp derivative estimate required for the coupling component in chapter three. The proof mirrors that of [2] where one transition point σ was used. In bounding $|\varepsilon D^+(W - w)|$ both $|\varepsilon D^+ w|$ and $|\varepsilon D^+ W|$ were point σ was used. In bounding $|\varepsilon D^+(W-w)|$ both $|\varepsilon D^+ w|$ and $|\varepsilon D^+ W|$ were bounded separately on the regular region $[\sigma, 1] = [x_{\frac{N}{2}}, 1]$. Then the result at the point $x = x_{\frac{N}{2}-1}$ was established. Finally the difference equation $L^N(W-w)(x_i) = \tau_i$ was integrated on the layer region from $x = x_{\frac{N}{2}-1}$ to $x = 0$ to establish the result over this region. The two transition point proof when $\sigma_2 = \frac{8\varepsilon_1 \ln N}{\alpha_1}$ uses these techniques from [2]. The proof when $\sigma_1 = \frac{4\varepsilon_1 \ln N}{\alpha_1}$ is similar.

Lemma 13

If the transition points (1.22) are such that $\sigma_1 = \frac{4\varepsilon_1 \ln N}{\alpha_1}$ or $\sigma_2 = \frac{8\varepsilon_1 \ln N}{\alpha_1}$ then

$$|D^+(W_1 - w_1)(x_i)| \leq CN^{-1} \ln N \times \begin{cases} \frac{1}{\tilde{\varepsilon}} & if \quad x_i < \sigma_1 \\ \frac{1}{\bar{\varepsilon}} & if \quad \sigma_1 \leq x_i < \sigma_2 \\ 1 & if \quad x_i \geq \sigma_2 \end{cases}, \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cup \{0\},$$

where w_1 and W_1 are the solutions of (1.5b) and (1.27b) respectively.

Proof

We start with the case where $\sigma_2 = \frac{8\varepsilon_1 \ln N}{\alpha_1}$, $\tilde{\varepsilon} = \varepsilon_2$ and $\bar{\varepsilon} = \varepsilon_1$. The result is established in the following stages:

(i) First on the regular region $[\sigma_2, 1]$,
(ii) Second at the point $x_{\frac{N}{2}-1} = \sigma_2 - H_2$,
(iii) Next on the entire outer layer region $[\sigma_1, \sigma_2)$ we integrate difference equation $L_1^N(W_1 - w_1)(x_i) = \tau_i$. This equation is equivalent to (2.28) on the outer layer region.
(iv) Fourth at the point $x_{\frac{N}{4}-1} = \sigma_1 - H_1$,
(v) Finally on the entire inner layer region $[0, \sigma_1)]$ the difference equation is integrated again to finish the proof.

(i) On the regular region recall (1.25b), combine this with (2.20) and (2.23) to obtain

$$\left|D^+(W_1 - w_1)(x_i)\right| \leq \frac{CN^{-2}}{H_3} \leq CN^{-1}, \quad x_i \in [\sigma_2, 1].$$

(ii) Next we show that the lemma holds at the point $x_{\frac{N}{2}-1} = \sigma_2 - H_2$. Start with $|\varepsilon_1 D^+ W_1(\sigma_2 - H_2)|$. The difference equation $L_1^N W_1(\sigma_2) = 0$ is used to obtain

$$\begin{aligned} \left|\varepsilon_1 D^+ W_1(x_{\frac{N}{2}-1})\right| &= \left|(\varepsilon_1 + a_{1,1}(\sigma_2)\bar{h}_{N/2}) D^+ W_1(\sigma_2)\right| = C\left|D^+ W_1(\sigma_2)\right| \\ &\leq CN^{-1}, \end{aligned} \tag{2.25}$$

where (2.23) and (1.25b) are used. Also note that $\left|\varepsilon_1 D^+ w_1(x_{\frac{N}{2}-1})\right| = |\varepsilon_1 w_1'(\eta)| \leq e^{-\frac{\alpha_1(\sigma_2 - H_2)}{\varepsilon_1}} \leq CN^{-1}$, $\eta \in (\sigma_2 - H_2, \sigma_2)$ where the mean value theorem and (1.6c) are used. Thus

$$\left|\varepsilon_1 D^+ w_1(x_{\frac{N}{2}-1})\right| \leq CN^{-1}. \tag{2.26}$$

Combine (2.25) and (2.26) to give

$$|\varepsilon_1 D^+(W_1 - w_1)(\sigma_2 - H_2)| \leq CN^{-1}. \tag{2.27}$$

(iii) The result now holds when $i = \frac{N}{2} - 1$ and now we show that the result holds over the outer layer region $[\sigma_1, \sigma_2)$. The notation $\hat{e}_i = (W_1 - w_1)(x_i)$ is adopted, then $|\varepsilon_1 D^+(W_1 - w_1)(\sigma_2 - H_2)| = \left|\varepsilon_1 D^+ \hat{e}_{\frac{N}{2}-1}\right|$. Recall from lemma 12 that $|\hat{e}_i| \leq CN^{-1}(\ln N)$ over $\bar{\Omega}^N_{\varepsilon_1,\varepsilon_2}$. Let $\hat{\tau}_i = L_1^N \hat{e}_i$ and note that $\bar{h}_j = H_2$ on $[\sigma_1, \sigma_2)$ then

$$\begin{aligned} \varepsilon_1 D^+ \hat{e}_j - \varepsilon_1 D^+ \hat{e}_{j-1} + \bar{h}_j a_{1,1} D^+ \hat{e}_j &= \bar{h}_j \tau_j, \\ \varepsilon_1 D^+ \hat{e}_j - \varepsilon_1 D^+ \hat{e}_{j-1} + H_2 a_{1,1}(x_j)(\hat{e}_{j+1} - e_j) &= H_2 \tau_j. \end{aligned} \tag{2.28}$$

The difference equation is now integrated by applying lemma 18 from Appendix A where $k = \frac{N}{2} - 1$ to obtain

$$\varepsilon_1 D^+ \hat{e}_{\frac{N}{2}-1-i} = \varepsilon_1 D^+ \hat{e}_{\frac{N}{2}-1} + a_{1,1}(x_{\frac{N}{2}-1}) \hat{e}_{\frac{N}{2}} - a_{1,1}(x_{\frac{N}{2}-1-i}) \hat{e}_{\frac{N}{2}-i}$$

$$- \sum_{j=\frac{N}{2}-i}^{\frac{N}{2}-1} H_2(a_{1,1}(x_j) - a_{1,1}(x_{j-1})) \hat{e}_j - \sum_{j=\frac{N}{2}-i}^{\frac{N}{2}-1} H_2 \tau_j. \tag{2.29}$$

Now we are in a position to bound $\left|\varepsilon_1 D^+ \hat{e}_{\frac{N}{2}-i}\right|$, ($i = 1$ to $i = \frac{N}{4}$) where (2.27) bounds the first term and lemma 12 bounds the next two terms. We use lemma 12 and the fact that $|a_{1,1}(x_j) - a_{1,1}(x_{j-1})| \leq H_2 a_{1,1}'(\eta_j) \leq CN^{-1}$, $\eta_j \in (x_{j-1}, x_j)$, to bound the first sum. For the second sum in (2.29) the bound in (2.24) is used in conjunction with the fact that if $Y > 0$ is bounded then $Y \sum_{j=1}^{\infty} e^{-(j-1)Y}$ is a bounded sum [2]. We apply this as follows where $1 \leq i \leq \frac{N}{4}$:

$$
\begin{aligned}
\left|\varepsilon_1 D^+ \hat{e}_{\frac{N}{2}-i}\right| &\leq CN^{-1}\ln N + C\sigma_2\varepsilon_1^{-2}N^{-1}H_2 \sum_{j=\frac{N}{2}-i}^{\frac{N}{2}-1} e^{-\frac{\alpha_1 x_{j-1}}{\varepsilon_1}} \\
&\leq CN^{-1}\ln N + C\sigma_2\varepsilon_1^{-1}N^{-1}\left(\frac{\alpha_1 H_2}{\varepsilon_1}\right) \sum_{j=\frac{N}{2}-i}^{\frac{N}{2}-1} e^{-\frac{\alpha_1 (j-1)H_2}{\varepsilon_1}} \\
&\leq CN^{-1}\ln N.
\end{aligned}
$$

The lemma is now complete over $[\sigma_1, 1]$.

(iv) Next we check that $\left|\varepsilon_1 D^+ \hat{e}_{\frac{N}{4}-1}\right| \leq CN^{-1}\ln N$. Rearrange $L_1^N \hat{e}_{\frac{N}{4}} = \tau_{\frac{N}{4}}$ to obtain

$$
\begin{aligned}
\left|\varepsilon_1 D^+ \hat{e}_{\frac{N}{4}-1}\right| &\leq \left|\varepsilon_1 D^+ \hat{e}_{\frac{N}{4}}\right| \left(1 + \frac{\bar{h}_{\frac{N}{4}} a_{1,1}}{\varepsilon_1}\right) + \left|\bar{h}_{\frac{N}{4}} \tau_{\frac{N}{4}}\right| \qquad (2.30) \\
&\leq (CN^{-1}\ln N)\left(1 + \frac{C\ln N}{N}\right) + \left(\frac{C\sigma_2}{N}\right)\left(\frac{\sigma_2}{\varepsilon_1^2 N}\right) \leq CN^{-1}\ln N,
\end{aligned}
$$

where (2.24) is used to bound $\left|\tau_{\frac{N}{4}}\right|$ and $\bar{h}_{\frac{N}{4}} \leq \frac{C\sigma_2}{N}$.

(v) Finally over the inner layer region $[0, \sigma_1)$ we repeat the procedure just applied over the outer layer region to complete the proof. Note the truncation error using (2.24) and take H_1 instead of H_2.

When $\sigma_2 = \frac{4\varepsilon_1 \ln N}{\alpha_1}$ the proof is similar. First $|\varepsilon_1 D^+(W_1 - w_1)|$ is bounded over the interval $[\sigma_1, 1]$ using (2.20) and (2.23) in conjunction with (1.25a) and (1.25b). The result at the point $x_{\frac{N}{4}-1}$ is then established. Finally the difference equation is integrated on the inner layer region as described above. ■

The next lemma establishes the weighted derivative result for the fourth case which was not considered in lemma 13. The proof is akin to that of the uniform case in [2].

Lemma 14

If the transition points (1.22) are such that $\sigma_1 = \frac{4\varepsilon_2 \ln N}{\alpha_2}$ and $\sigma_2 = \frac{1}{2}$, then $|D^+(W_1 - w_1)(x_i)| \leq CN^{-1}(\ln N)^2$, at each mesh point $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$, where w_1 and W_1 are the solutions of (1.5b) and (1.27b) respectively.

Proof

We check that

$$|\varepsilon_1 \hat{e}_i| \leq CN^{-1} \ln N(1 - x_i). \tag{2.31}$$

Recall $\hat{e}_i = (W_1 - w_1)(x_i)$ then $\hat{e}_0 = \hat{e}_N = 0$ and in a similar manner to (2.24) note that $|L_1^N \hat{e}_i| \leq C\varepsilon_1^{-2}N^{-1}$. Let $\Psi = \frac{C\varepsilon_1^{-2}N^{-1}(1-x_i)}{\alpha_1} \pm \hat{e}_i$ then $\Psi(0) \geq 0$, $\Psi(1) = 0$ and $L_1^N \Psi_i \leq 0$. Apply a minimum principle [2] to establish (2.31) over $\Omega^N_{\varepsilon_1,\varepsilon_2}$. Then use $\hat{e}_N = 0$ and (2.31) to obtain

$$|\varepsilon_1 D^+ \hat{e}_{N-1}| = \left|\frac{-\varepsilon_1 \hat{e}_{N-1}}{1 - x_{N-1}}\right| \leq CN^{-1} \ln N.$$

We repeatedly apply lemma 18 in a similar way to that done in lemma 13 over the intervals $[\sigma_2, 1)$, $[\sigma_1, \sigma_2)$ and $[0, \sigma_1)$ where we also check $\left|\varepsilon_1 D^+ e_{\frac{kN}{4}-1}\right| \leq CN^{-1} \ln N$ $(k = 1, 2)$ in a similar way to (2.30). Thus $|\varepsilon_1 D^+ \hat{e}_i| \leq CN^{-1} \ln N$ and hence using $\varepsilon_1^{-1} \leq C \ln N$ we obtain the required result. ■

Error bounds for the second singular component

We have already remarked that proof of the second singular component follows the proof of the first singular component. The proof for lemma 12 used $\alpha_j, a_{j,j}, \gamma_j, w_j$, and W_j when $j = 1$. The proof for this lemma follows identically where $j = 2$ and we use lemma 5 to bound w_2 and the truncation error. Thus we simply state the lemma which follows.

Lemma 15

If the transition points (1.22) are such that $\sigma_1 = \frac{4\varepsilon_j \ln N}{\alpha_j}$, $j = 1$ or 2 then $|(W_2 - w_2)(x_i)| \leq CN^{-1} \ln N$ at each mesh point $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$, where w_2 and W_2 are the solutions of (1.7b) and (1.28b) respectively.

Error bounds for the coupling component

We now consider the coupling component and establish the error bound (2.1e) at each mesh point for the non–uniform mesh cases, that is:

$$|(Z-z)(x_i)| \leq CN^{-1}(\ln N)^2, \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2},$$

where z and Z are the solutions of (1.7c) and (1.28c) respectively.
First we decompose the discrete coupling component Z of (1.28c) and then derive the truncation error for the coupling component. The analysis is then considered in sections 2.5–2.7 as follows:

$$\text{Section 2.5:} \quad \frac{\varepsilon_2}{\alpha_2} \leq \frac{\varepsilon_1}{\alpha_1},$$

$$\text{Section 2.6:} \quad \frac{\varepsilon_1}{\alpha_1} \leq \frac{\varepsilon_2}{\alpha_2} \text{ and } \frac{\varepsilon_1}{\varepsilon_2} \leq CN^{-1},$$

$$\text{Section 2.7:} \quad \frac{\varepsilon_1}{\alpha_1} \leq \frac{\varepsilon_2}{\alpha_2} \text{ and } \frac{\varepsilon_2}{\varepsilon_1} \leq CN.$$

2.4 The decomposition of the discrete coupling component

Recall from (1.28c) that $L_2^N Z_i = -a_{2,1}(x_i)D^+W_1(x_i)$, where $Z(0) = z(0)$ and $Z(1) = 0$. We decompose Z_i where $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$ as follows:

$$\begin{aligned}
Z_i &= Y_{1,i} + Y_{2,i}, && && (2.32a)\\
L_2^N Y_{1,i} &= -a_{2,1}(x_i)D^+(W_1 - w_1)(x_i), && Y_1(0) = 0, && Y_1(1) = 0, (2.32b)\\
L_2^N Y_{2,i} &= -a_{2,1}(x_i)D^+w_1(x_i), && Y_2(0) = Z(0), && Y_2(1) = 0. (2.32c)
\end{aligned}$$

We note $|L_2^N Y_{2,i}| = C\,|w_1'(\eta)|$ for some $\eta \in (x_i, x_{i+1})$, then use (1.6c) to obtain

$$|L_2^N Y_{2,i}| \leq \frac{Ce^{-\frac{\alpha_1 \eta}{\varepsilon_1}}}{\varepsilon_1} \leq \frac{Ce^{-\frac{\alpha_1 x_i}{\varepsilon_1}}}{\varepsilon_1}, \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}. \tag{2.33}$$

It is easy to obtain a bound for $|Y_1|$ by applying the stability technique [1] to (2.32b) where the bounds from lemma 13 and lemma 14 are used. Thus

$$
\begin{aligned}
\| Y_1 \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}} &\le C\sum_{i=1}^{N-1} \bar{h}_i \left|L_2^N Y_1\right| \\
&\le CH_1 \sum_{i=1}^{\frac{N}{4}-1} \left|L_2^N Y_{1,i}\right| + CH_2 \sum_{i=\frac{N}{4}}^{\frac{N}{2}-1} \left|L_2^N Y_{1,i}\right| + CH_3 \sum_{i=\frac{N}{2}}^{N-1} \left|L_2^N Y_{1,i}\right| \\
&\le CN^{-1}(\ln N)^2, \qquad (2.34)
\end{aligned}
$$

where the bounds in (2.34) follow in a similar manner to the bounds in (2.12).

The truncation error for the coupling component

Let $\tilde{e}(x_i) = (Z - z)(x_i)$ and then note the truncation error for the coupling component $L_2^N \tilde{e} = L_2^N Z - L_2 z + (L_2 - L_2^N)z$. Also note

$$
\left|(L_2 - L_2^N)z\right| \le \frac{2\bar{h}_i \varepsilon_2}{3} \| z^{(3)} \|_{\Omega_i} + \frac{a_{2,2}(x_i)h_i}{2} \| z^{(2)} \|_{\Omega_i} . \qquad (2.35)
$$

When (1.2) holds use (2.35) and lemma 2 to give

$$
\left|(L_2 - L_2^N)z\right| \le \frac{C\bar{h}_i}{\varepsilon_1^2}, \qquad \frac{\varepsilon_2}{\alpha_2} \le \frac{\varepsilon_1}{\alpha_1}. \qquad (2.36)
$$

When (1.3) holds then use (2.35) and lemma 4 to give

$$
\left|(L_2 - L_2^N)z\right| \le C(\frac{\bar{h}_i}{\varepsilon_2^2} + \frac{\bar{h}_i e^{-\frac{\alpha_1 x_{i-1}}{\varepsilon_1}}}{\varepsilon_1^2}), \qquad \frac{\varepsilon_1}{\alpha_1} \le \frac{\varepsilon_2}{\alpha_2}. \qquad (2.37)
$$

Recall $\hat{e}_i = (W_1 - w_1)(x_i)$ and note that $\left|L_2^N Z - L_2 z\right| = C\left|D^+ W_1 - w_1')(x_i)\right|$ then

$$
\left|L_2^N Z - L_2 z\right| \le C\left|D^+\hat{e}_i\right| + C\left|(D^+ w_1 - w_1')(x_i)\right|, \qquad (2.38a)
$$

$$
\left|(D^+ w_1 - w_1')(x_i)\right| = 2\bar{h}_i \left|w_1''(\eta)\right| \le \frac{2\bar{h}_i e^{-\alpha_1 \eta/\varepsilon_1}}{\varepsilon_1^2} \le \frac{C\bar{h}_i e^{-\alpha_1 x_i/\varepsilon_1}}{\varepsilon_1^2}, \qquad (2.38b)
$$

for some $\eta \in (x_i, x_{i+1})$, where the mean value theorem and (1.6c) are used. We can now merge the second term of (2.38a) with (2.36) or (2.37), thus

$$
\left|L_2^N \tilde{e}\right| \le C(\left|D^+\hat{e}_i\right| + \frac{C\bar{h}_i}{\varepsilon_1^2}), \qquad \frac{\varepsilon_2}{\alpha_2} \le \frac{\varepsilon_1}{\alpha_1}, \qquad (2.39a)
$$

$$
\left|L_2^N \tilde{e}\right| \le C(\left|D^+\hat{e}_i\right| + \frac{\bar{h}_i}{\varepsilon_2^2} + \frac{\bar{h}_i e^{-\frac{\alpha_1 x_{i-1}}{\varepsilon_1}}}{\varepsilon_1^2}), \qquad \frac{\varepsilon_1}{\alpha_1} \le \frac{\varepsilon_2}{\alpha_2}. \qquad (2.39b)
$$

2.5 The case when $\frac{\varepsilon_2}{\alpha_2} \leq \frac{\varepsilon_1}{\alpha_1}$

Here $\sigma_1 = \frac{4\varepsilon_2 \ln N}{\alpha_2}$ and we start with the case where $\sigma_2 = \frac{8\varepsilon_1 \ln N}{\alpha_1}$. When $\sigma_2 = \frac{1}{2}$ the result easily follows. The proof is established using the following stages:
(i) Use lemma 1 to obtain

$$|z(x)| \leq CN^{-1}, \quad x \in [\sigma_2, 1]. \tag{2.40}$$

(ii) The bound $|Y_1| \leq CN^{-1} \ln N$ on $\Omega^N_{\varepsilon_1,\varepsilon_2}$ has already been obtained in (2.34).
(iii) The bound $|Y_2| \leq CN^{-1}$ is established on the regular interval $[\sigma_2, 1]$. Then the triangle inequality and (2.32a) can be used to bound $|Z|$ on this interval.
(iv) Finally $|Z - z|$ is bounded on the interval $[0, \sigma_2)$.

(iii) Here we use (2.13) to define the barrier function for $|Y_{2,i}|$ $(1 \leq i \leq N-1)$ as $\Psi_i = \frac{8C_1 B_{\gamma_1,\varepsilon_1}(x_{i-1})}{\alpha_1\alpha_2} \pm Y_{2,i}$ where (2.33) holds and we take $C_1 = \max\{\frac{\alpha_1\alpha_2|Z(0)|}{8}, C\}$. Then $\Psi_0 \geq 0$ and $\Psi_N \geq 0$. Choose $\gamma_1 = \frac{\alpha_1}{4}$, use (2.15) and (2.33) to give

$$L_2^N \Psi_i \leq \frac{2C_1}{\alpha_2\varepsilon_1}[\frac{\varepsilon_2\alpha_1 h_{i+1}}{2(h_{i+1}+h_i)\varepsilon_1} - a_{2,2}]B_{\gamma_1,\varepsilon_1}(x_i) + \frac{Ce^{-\frac{\alpha_1 x_i}{\varepsilon_1}}}{\varepsilon_1} \leq 0,$$

where $\frac{h_{i+1}}{h_i+h_{i+1}} \leq 1$, $\frac{\varepsilon_2\alpha_1}{2\varepsilon_1} \leq \frac{\alpha_2}{2}$ as a consequence of (1.2), $-a_{2,2} \leq -\alpha_2$ and (2.16) give the second inequality. Apply the discrete minimum principle [2] to obtain $\Psi_i \geq 0$ and hence

$$|Y_{2,i}| \leq CB_{\gamma_1,\varepsilon_1}(x_{i-1}), \quad 1 \leq i \leq N-1. \tag{2.41}$$

Since $\sigma_2 = \frac{8\varepsilon_1 \ln N}{\alpha_1}$, then (1.25c) and lemma 16 (Appendix A) are used in a similar way to (2.22) to give $B_{\gamma_1,\varepsilon_1}(x_{\frac{N}{2}-1}) = (1+\frac{\alpha_1 H_2}{4\varepsilon_1})B_{\gamma_1,\varepsilon_1}(\sigma_2) \leq (1+\frac{4\ln N}{N})^{-\frac{N}{4}} \leq CN^{-1}$. Then use (2.19) and (2.41) to obtain

$$|Y_{2,i}| \leq CN^{-1}, \quad x_i \in [\sigma_2, 1]. \tag{2.42}$$

Now combine (2.40), (2.34) and (2.42) to obtain

$$|\tilde{e}(x_i)| \leq CN^{-1}(\ln N)^2, \quad x_i \in [\sigma_2, 1]. \tag{2.43}$$

(iv) It remains to bound $|\tilde{e}(x_i)|$ over the interval $[0, \sigma_2)$. Using lemma 19 from Appendix B and (2.43) then

$$\| \tilde{e} \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}} \leq C \sum_{i=1}^{N/2-1} \bar{h}_i \left|L_2^N \tilde{e}_i\right| + CN^{-1}(\ln N)^2. \tag{2.44}$$

Substitute (2.39a) into (2.44), then use the stability technique [1] to obtain

$$\| \tilde{e} \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}} \leq C \sum_{i=1}^{\frac{N}{2}-1} (\bar{h}_i \left|D^+\hat{e}_i\right| + \frac{\bar{h}_i^2}{\varepsilon_1^2}) + CN^{-1}(\ln N)^2 \leq CN^{-1}(\ln N)^2,$$

where lemma 13 and the fact that $\frac{h_i}{\varepsilon_1} \leq CN^{-1}\ln N$ over the interval $[0, \sigma_2)$ are both used. This concludes case where $\sigma_2 = \frac{8\varepsilon_1 \ln N}{\alpha_1}$.

If $\sigma_2 = \frac{1}{2}$ then $\varepsilon_1^{-1} \leq C \ln N$. The result $\left|L_2^N \tilde{e}\right| \leq CN^{-1}(\ln N)^2$ is obtained by using lemma 14 and (2.39a). A barrier function argument over $\Omega^N_{\varepsilon_1,\varepsilon_2}$ completes the
proof where $\Psi_i = \frac{CN^{-1}(\ln N)^2(1-x_i)}{\alpha_2} \pm \tilde{e}_i$. Thus

$$|(Z - z)(x_i)| \leq CN^{-1}(\ln N)^2, \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}, \quad \frac{\varepsilon_2}{\alpha_2} \leq \frac{\varepsilon_1}{\alpha_1}. \tag{2.45}$$

2.6 The case when $\frac{\varepsilon_1}{\alpha_1} \leq \frac{\varepsilon_2}{\alpha_2}$ and $\frac{\varepsilon_1}{\varepsilon_2} \leq CN^{-1}$

Here $\sigma_1 = \frac{4\varepsilon_1 \ln N}{\alpha_1}$. We note the first two parts of the proof are already complete.

(i) Recall the dependence of $|z|$ on $\frac{\varepsilon_1}{\varepsilon_2}$ from lemma 3, thus $|z| \leq CN^{-1}$ on $\Omega^N_{\varepsilon_1,\varepsilon_2}$.
(ii) The bound $|Y_1| \leq CN^{-1}(\ln N)^2$ on Ω has been established in (2.34).

(iii) Here we show that $|Y_2|$ depends on $\frac{\varepsilon_1}{\varepsilon_2}$ in a similar manner to $|z|$ and hence $|Y_2| \leq CN^{-1}$ over $\Omega^N_{\varepsilon_1,\varepsilon_2}$. Then (2.32a) and the triangle inequality are used with part (ii) to bound $|Z|$ over $\Omega^N_{\varepsilon_1,\varepsilon_2}$. Recall from (1.28c) that $Y_2(0) = Z(0) = 0$ and note that $\frac{h_1}{\varepsilon_1} = \frac{16 \ln N}{\alpha_1 N}$ is bounded. The proof is considered in two parts in an analogous manner to that of lemma 4 where (1.14) and (1.16) were used.

(a) First assume that (1.14) holds. We then take $\gamma_1 = \alpha_1$ and $\gamma_2 = \frac{\alpha_2}{4}$ and let

$$\Psi_i = \frac{2C\varepsilon_1}{\alpha_1\varepsilon_2}[(1 + \frac{\gamma_1 h_1}{\varepsilon_1})B_{\gamma_2,\varepsilon_2}(x_{i-1}) - B_{\gamma_1,\varepsilon_1}(x_{i-1})] \pm Y_{2,i}, \quad 1 \leq i \leq N-1,$$

then $\Psi_0 \geq 0$ where (2.33) holds. We then use (2.13) and (1.3) to establish $\Psi_N \geq 0$. Thus

$$L_2^N \Psi_i = \frac{2C\varepsilon_1}{\alpha_1^2\varepsilon_2}[(1 + \frac{\gamma_1 h_1}{\varepsilon_1})L_2^N B_{\gamma_2,\varepsilon_2}(x_{i-1}) - L_2^N B_{\gamma_1,\varepsilon_1}(x_{i-1})] \pm L_2^N Y_{2,i}$$

$$\leq -\frac{2C\varepsilon_1}{\alpha_1\varepsilon_2}(\frac{2\varepsilon_2\alpha_1 h_{i+1}}{\varepsilon_1^2(h_{i+1}+h_i)} - \frac{a_{2,2}}{\varepsilon_1})B_{\gamma_1,\varepsilon_1}(x_i) + \frac{Ce^{-\frac{\alpha_1 x_i}{\varepsilon_1}}}{\varepsilon_1}$$

where (2.33) is noted and (2.15) is used to obtain $L_2^N B_{\gamma_2,\varepsilon_2}(x_i) \leq 0$. The next two inequalities are established using $-\frac{h_{i+1}}{h_i+h_{i+1}} \leq -\frac{1}{2}$ and the fact that $\frac{\varepsilon_1\|a_{2,2}\|_\infty}{\varepsilon_2} \leq \frac{\alpha_1}{2}$ as a consequence of (1.14). Then

$$\begin{aligned} L_2^N \Psi_i &\leq -\frac{2C}{\alpha_1\varepsilon_1}(\alpha_1 - \frac{\varepsilon_1 a_{2,2}}{\varepsilon_2})B_{\gamma_1,\varepsilon_1}(x_i) + \frac{Ce^{-\frac{\alpha_1 x_i}{\varepsilon_1}}}{\varepsilon_1} \\ &\leq -\frac{CB_{\gamma_1,\varepsilon_1}(x_i)}{\varepsilon_1} + \frac{Ce^{-\frac{\alpha_1 x_i}{\varepsilon_1}}}{\varepsilon_1} \leq 0, \end{aligned}$$

where the final inequality is justified using (2.16). The choice of γ_2 is needed when it is used in (2.51). The discrete minimum principle [2] holds, thus $\Psi_i \geq 0$ and

$$|Y_{2,i}| \leq CB_{\gamma_2,\varepsilon_2}(x_{i-1}), \quad \gamma_2 = \frac{\alpha_2}{4}, \quad 1 \leq i \leq N-1, \tag{2.46a}$$

$$|Y_{2,i}| \leq \frac{C\varepsilon_1}{\varepsilon_2}. \tag{2.46b}$$

(b) In this part we assume (1.16) holds and hence (1.17) holds. A barrier function for $|Y_{2,i}|$ when $1 \leq i \leq N-1$ and (2.33) holds is defined by

$$\Psi_i = \frac{8MC}{\alpha_1\alpha_2}(1 + \frac{\gamma_1 h_1}{\varepsilon_1})B_{\gamma_1,\varepsilon_1}(x_{i-1}) \pm Y_{2,i} \text{ then } \Psi_0 \geq 0 \text{ and } \Psi_N \geq 0.$$

Take $\gamma_1 = \frac{\alpha_1}{4M}$ where (1.17) implies $\frac{2\varepsilon_2\gamma_1}{\varepsilon_1} = \frac{\alpha_2}{2}$. Recall $-a_{2,2} \leq -\alpha_2$, then in a similar manner to part (a) we obtain

$$\begin{aligned} L_2^N \Psi_i &= \frac{8MC\gamma_1}{\alpha_1\alpha_2\varepsilon_1}(\frac{2\varepsilon_2\gamma_1 h_{i+1}}{\varepsilon_1(h_{i+1}+h_i)} - a_{2,2})B_{\gamma_1,\varepsilon_1}(x_i) + \frac{Ce^{-\frac{\alpha_1 x_i}{\varepsilon_1}}}{\varepsilon_1} \\ &\leq -\frac{C}{\varepsilon_1}B_{\gamma_1,\varepsilon_1}(x_i) + \frac{Ce^{-\frac{\alpha_1 x_i}{\varepsilon_1}}}{\varepsilon_1} \leq 0. \end{aligned}$$

A minimum principle [2] is then applied. Since $\frac{\alpha_1}{2\|a_{2,2}\|_\infty} \leq \frac{\varepsilon_1}{\varepsilon_2} \leq \frac{\alpha_1}{\alpha_2}$ we can also represent the barrier function in terms of $\frac{\varepsilon_1}{\varepsilon_2}$. Thus $\Psi_i \geq 0$ and

$$|Y_{2,i}| \leq CB_{\gamma_1,\varepsilon_1}(x_{i-1}), \quad \gamma_1 = \frac{\alpha_1}{4M}, \quad 1 \leq i \leq N-1, \tag{2.47a}$$

$$|Y_{2,i}| \leq \frac{C\varepsilon_1}{\varepsilon_2}. \tag{2.47b}$$

Combine (2.46b) and (2.47b) to obtain

$$|Y_{2,i}| \leq \frac{C\varepsilon_1}{\varepsilon_2}, \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}, \quad \frac{\varepsilon_1}{\alpha_1} \leq \frac{\varepsilon_2}{\alpha_2}. \tag{2.48}$$

Combine the result from lemma 3, (2.34) and (2.48) to give the result

$$|(Z-z)(x_i)| \leq CN^{-1}(\ln N)^2, \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}, \quad \frac{\varepsilon_1}{\alpha_1} \leq \frac{\varepsilon_2}{\alpha_2}, \quad \frac{\varepsilon_1}{\varepsilon_2} \leq CN^{-1}. \qquad (2.49)$$

2.7 The case when $\frac{\varepsilon_1}{\alpha_1} \leq \frac{\varepsilon_2}{\alpha_2}$ and $\frac{\varepsilon_2}{\varepsilon_1} \leq CN$

Here $\sigma_1 = \frac{4\varepsilon_1 \ln N}{\alpha_1}$ and we start with the case where $\sigma_2 = \frac{8\varepsilon_2 \ln N}{\alpha_2}$. When $\sigma_2 = \frac{1}{2}$ the result easily follows. The proof is similar that of section 3.2 which runs as follows:

(i) Note using lemma 3 that

$$|z| \leq CN^{-1}, \quad x \in [\sigma_2, 1]. \qquad (2.50)$$

(ii) The bound $|Y_1| \leq CN^{-1}(\ln N)^2$ on $\Omega^N_{\varepsilon_1,\varepsilon_2}$ has been established in (2.34).
(iii) The bound $|Y_2| \leq CN^{-1}$ is established on the regular interval $[\sigma_2, 1]$.
(iv) Finally $|Z-z|$ is bounded on $[0, \sigma_2)$.

(iii) On the regular interval the result for $|Y_2|$ is established in two parts in a similar manner to the last section where (1.14) and (1.16) are used.

(a) When (1.14) holds we bound the barrier function (2.46a) for $|Y_2|$. Using (2.13) and (1.25d) we note that $B_{\gamma_2,\varepsilon_2}(x_{\frac{N}{2}-1}) = (1 + \frac{\alpha_2 H_2}{4\varepsilon_2})B_{\gamma_2,\varepsilon_2}(x_{\frac{N}{2}}) \leq CB_{\gamma_2,\varepsilon_2}(x_{\frac{N}{2}})$. Then

$$B_{\gamma_2,\varepsilon_2}(x_{\frac{N}{2}}) \leq C(1 + \frac{\alpha_2 H_2}{4\varepsilon_2})^{-\frac{N}{4}} \leq C(1 + \frac{4\ln N}{N})^{-\frac{N}{4}} \leq CN^{-1}, \qquad (2.51)$$

where (1.25d) and lemma 14 are used. By using the barrier function (2.46a), (2.19) and (2.51) then $|Y_2| \leq CN^{-1}$.

(b) When (1.16) holds we note that $\frac{\varepsilon_2}{\varepsilon_1}$ is bounded and hence $\frac{H_2}{\varepsilon_1}$ is bounded. Note also that (1.17) implies $\frac{\alpha_1}{4M\varepsilon_1} = \frac{\alpha_2}{4\varepsilon_2}$. This result is used in conjunction with the barrier function (2.47a) for $|Y_2|$ to obtain $B_{\gamma_1,\varepsilon_1}(x_{\frac{N}{2}}) \leq CN^{-1}$ in a similar manner to (2.51). Thus $|Y_2| \leq CN^{-1}$.

Combining both parts we establish

$$|Y_{2,i}| \leq CN^{-1}, \quad x_i \in [\sigma_2, 1]. \qquad (2.52)$$

Combine (2.50), (2.52) and (2.34) to obtain

$$|\tilde{e}(x_i)| \leq CN^{-1}(\ln N)^2, \quad x_i \in [\sigma_2, 1]. \qquad (2.53)$$

(iv) It remains to bound $|\tilde{e}(x_i)|$ over the interval $[0, \sigma_2)$. Using lemma 19 and (2.53) then

$$\| \tilde{e} \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}} \leq C \sum_{i=1}^{N/2-1} \bar{h}_i \left|L_2^N \tilde{e}_i\right| + CN^{-1}(\ln N)^2. \tag{2.54}$$

Substitute (2.39b) into (2.54), then use the stability technique [1] to obtain

$$\| \tilde{e} \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}} \leq C \sum_{i=1}^{\frac{N}{2}-1} [\bar{h}_i \left|D^+(W_1 - w_1)(x_i)\right| + \frac{\bar{h}_i^2}{\varepsilon_2^2} + \frac{\bar{h}_i^2 e^{-\frac{\alpha_1 x_{i-1}}{\varepsilon_1}}}{\varepsilon_1^2}] + CN^{-1}(\ln N)^2.$$

Using lemma 13 the required bound is found for the first term and thus

$$\| \tilde{e} \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}} \leq C \sum_{i=1}^{N/4-1} \frac{\bar{h}_i^2}{\varepsilon_1^2} + C \sum_{i=N/4}^{N/2-1} \frac{\bar{h}_i^2}{\varepsilon_2^2}(1 + \frac{\varepsilon_2^2 e^{-\frac{\alpha_1 x_{i-1}}{\varepsilon_1}}}{\varepsilon_1^2}) + CN^{-1}(\ln N)^2.$$

We note that $\frac{\bar{h}_i}{\varepsilon_1} \leq \frac{C \ln N}{N}$ if $i < \frac{N}{4}$, $\frac{\bar{h}_i}{\varepsilon_2} \leq \frac{C \ln N}{N}$ if $i < \frac{N}{2}$ and $\frac{\varepsilon_2}{\varepsilon_1} \leq CN$. Also when $i \geq \frac{N}{4}$ then $e^{-\frac{\alpha_1 x_{i-1}}{\varepsilon_1}} \leq e^{-\frac{\alpha_1(\sigma_1 - H_1)}{\varepsilon_1}} = e^{\frac{\alpha_1 H_1}{\varepsilon_1}} e^{-\frac{\alpha_1 \sigma_1}{\varepsilon}} \leq C(e^{\frac{16 \ln N}{N}})(N^{-4})$ thus $\| \tilde{e} \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}} \leq CN^{-1}(\ln N)^2$. This completes the case where $\sigma_2 = \frac{8\varepsilon_2 \ln N}{\alpha_2}$.

If $\sigma_2 = \frac{1}{2}$ we use $\tilde{e}(0) = \tilde{e}(1) = 0$ and apply the stability technique [1] over $\Omega^N_{\varepsilon_1,\varepsilon_2}$ to (2.39b). Note that $\varepsilon_2^{-1} \leq \ln N$ and then the result

$$\| \tilde{e} \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}} \leq C[\sum_{i=1}^{N-1} \bar{h}_i \left|D^+ \hat{e}_i\right| + \frac{\bar{h}_i^2}{\varepsilon_2^2}(1 + \frac{\varepsilon_2^2 e^{-\frac{\alpha_1 x_{i-1}}{\varepsilon_1}}}{\varepsilon_1^2}) \leq CN^{-1}(\ln N)^2,$$

is obtained by similar reasoning to that used above. Thus

$$|(Z - z)(x_i)| \leq CN^{-1}(\ln N)^2, \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}, \quad \frac{\varepsilon_1}{\alpha_1} \leq \frac{\varepsilon_2}{\alpha_2}, \quad \frac{\varepsilon_2}{\varepsilon_1} \leq CN. \tag{2.55}$$

Now combine (2.45), (2.49) and (2.55) to obtain

$$|(Z - z)(x_i)| \leq CN^{-1}(\ln N)^2, \quad x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}. \tag{2.56}$$

Finally combine (2.56) with lemmas (11) and (15) to obtain (1.4) when $j = 2$ as follows when $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$,

$$\begin{aligned} |(U_2 - u_2)(x_i)| &\leq |(V_2 - v_2)(x_i)| + |(W_2 - w_2)(x_i)| + |(Z - z)(x_i)| \\ &\leq CN^{-1}(\ln N)^2. \end{aligned}$$

Chapter 3

Numerical Computations

In this chapter numerical results are presented which validate the theoretical results of the previous chapters. We consider a constant coefficient problem and a variable coefficient problem to illustrate the pointwise errors in the numerical approximations. The constant coefficient problem is chosen to illustrate the coupling due to the parameters ε_1 and ε_2. We also highlight the necessity of choosing a mesh with at least two transition points by providing counterexamples where only one transition point is taken. The weighted derivative results for u_1 from chapter two are also illustrated.

Numerical evidence is also provided to illustrate the convergence of the component–wise errors (2.1c), (2.1d) and (2.1e) for the second regular, second singular and coupling components respectively. Since the solution of the constant coefficient problem is known the exact maximum pointwise errors are found. Generally the exact solution is not available so we consider a variable coefficient problem to outline how the errors from the numerical method are measured.

3.1 A constant coefficient problem

Example 1.
Consider the constant coefficient problem for $x \in \Omega$,

$$\begin{aligned} \varepsilon_1 u_1''(x) + 3u_1'(x) &= 15x^4, & (3.1a)\\ \varepsilon_2 u_2''(x) + 2u_2'(x) + 2.75u_1'(x) &= 0.6e^x, & (3.1b) \end{aligned}$$

where $u_1(0) = 0$, $u_1(1) = 0$, $u_2(0) = 0$ and $u_2(1) = 0$.
The corresponding discrete problem for $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$ is:

$$\begin{aligned} \varepsilon_1 \delta^2 U_{1,i} + 3D^+U_{1,i} &= 15x_i^4, & (3.2a)\\ \varepsilon_2 U_{2,i} + 2D^+U_{2,i} + 2.75D^+U_{1,i} &= 0.6e^{x_i}, & (3.2b) \end{aligned}$$

where $U_{1,0} = u_1(0)$, $U_{1,N} = u_1(1)$, $U_{2,0} = u_2(0)$ and $U_{2,N} = u_2(1)$.
The coefficients in (3.1) are chosen so that $\| u_1 \|_\infty$ and $\| u_2 \|_\infty$ are approximately one. The exact solutions u_1 and u_2 are known and thus the exact pointwise errors

of the numerical method for this problem are illustrated. The solutions $u_1(x)$ of (3.1a) and $u_2(x)$ of (3.1b) are illustrated in figures 4.1–4.4.

Errors from Equation (3.1)

First we illustrate the $\{\varepsilon_1, \varepsilon_2\}$-uniform convergence of the maximum pointwise errors $E^N(u_k)$ and the orders of convergence $p^N(u_k)$, $k = 1, 2$, for the numerical method in tables 3.1 and 3.2 where we define the transition points,

$$\begin{aligned} \sigma_2 &= \min\{\frac{1}{2}, \max\{\frac{4\varepsilon_1 \ln N}{\alpha_1}, \frac{4\varepsilon_2 \ln N}{\alpha_2}\}\}, \\ \sigma_1 &= \min\{\frac{\sigma_2}{2}, \frac{2\varepsilon_1 \ln N}{\alpha_1}, \frac{2\varepsilon_2 \ln N}{\alpha_2}\}, \end{aligned} \tag{3.3}$$

and $\bar{\Omega}^N_{\varepsilon_1,\varepsilon_2} = \{x_i\}, \quad x_i = \begin{cases} 4\sigma_1 i/N & if \quad i \leq N/4 \\ x_{i-1} + 4(\sigma_2 - \sigma_1)/N & if \quad N/4 < i \leq N/2\,. \\ x_{i-1} + 2(1 - \sigma_2)/N & if \quad i > N/2 \end{cases}$

The computed range of ε_1 and ε_2 is taken over the set

$$S = \{(\varepsilon_1, \varepsilon_2) : 2^{-50} \leq \varepsilon_1 \leq 1, \quad 2^{-50} \leq \varepsilon_2 \leq 1\}. \tag{3.4}$$

Note the transition points (3.3) differ from those used in the analysis, (1.23) by a factor of two. A full theoretical analysis holds using the transition points (3.3). The analysis is similar to that of chapters two and three for the regular and coupling components. The analysis for the singular component is more complex where similar mesh functions to those used in [2] are employed. The rate of convergence is better for smaller values of N when the transition points (3.3) are used and so we illustrate the error tables using these transition points. We also note that the error tables for u_1 and u_2 show $\{\varepsilon_1, \varepsilon_2\}$-uniform convergence when the transition points

$$\begin{aligned} \sigma_2 &= \min\{\frac{1}{2}, \max\{\frac{2\varepsilon_1 \ln N}{\alpha_1}, \frac{2\varepsilon_2 \ln N}{\alpha_2}\}\}, \\ \sigma_1 &= \min\{\frac{\sigma_2}{2}, \frac{\varepsilon_1 \ln N}{\alpha_1}, \frac{\varepsilon_2 \ln N}{\alpha_2}\}, \end{aligned} \tag{3.5}$$

are used but no analysis is available to prove that this is the case in general.

ε_1	Number of Mesh Points N					
	64	128	256	512	1024	2048
2^0	0.007838	0.003966	0.001996	0.001001	0.000501	0.000251
2^{-2}	0.054983	0.029258	0.015102	0.007672	0.003867	0.001942
2^{-4}	0.145435	0.093646	0.056039	0.031720	0.016389	0.008347
2^{-6}	0.167567	0.108315	0.064312	0.037516	0.021132	0.011678
2^{-8}	0.173777	0.112432	0.066790	0.038929	0.021911	0.012108
2^{-10}	0.175373	0.113490	0.067445	0.039292	0.022112	0.012219
2^{-12}	0.175774	0.113756	0.067609	0.039384	0.022162	0.012247
2^{-14}	0.175875	0.113823	0.067651	0.039407	0.022175	0.012254
2^{-16}	0.175900	0.113839	0.067661	0.039412	0.022178	0.012255
2^{-18}	0.175906	0.113843	0.067664	0.039414	0.022179	0.012256
2^{-20}	0.175908	0.113844	0.067664	0.039414	0.022179	0.012256
2^{-22}	**0.175908**	**0.113845**	**0.067664**	**0.039414**	**0.022179**	**0.012256**
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-30}	0.175908	0.113845	0.067664	0.039414	0.022179	0.012256
2^{-34}	0.175908	0.113843	0.067661	0.039413	0.022179	0.012256
2^{-38}	0.175908	0.113812	0.067617	0.039399	0.022176	0.012254
2^{-42}	0.175908	0.113184	0.067365	0.039070	0.022034	0.012177
2^{-46}	0.154189	0.089766	0.050589	0.027318	0.014354	0.007358
2^{-50}	0.130130	0.074975	0.042520	0.023660	0.012958	0.007018
$E^N(u_1)$	0.175908	0.113845	0.067664	0.039414	0.022179	0.012256
$p^N(u_1)$	0.628	0.751	0.778	0.830	0.856	0.875

Table 3.1: Exact errors $E^N_{\varepsilon_1}(u_1)$, computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(u_1)$ and orders of convergence $p^N(u_1)$ for the solution u_1 of (3.1a) on the piecewise uniform mesh $\Omega^N_{\varepsilon_1,\varepsilon_2}$ using the transition points (3.3) over the set (3.4).

We define the exact errors $E^N_{\varepsilon_1,\varepsilon_2}(u_k)$, $E^N_{\varepsilon_1}(u_k)$ and the computed parameter–uniform maximum pointwise errors $E^N(u_k)$ as follows:

$$\begin{aligned}
E^N_{\varepsilon_1,\varepsilon_2}(u_k) &= \| U^N_k - u_k \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}}, \quad k = 1, 2, \\
E^N_{\varepsilon_1}(u_k) &= \max_{\varepsilon_2} E^N_{\varepsilon_1,\varepsilon_2}(u_k) \quad k = 1, 2, \\
E^N(u_k) &= \max_{\varepsilon_1} E^N_{\varepsilon_1}(u_k) \quad k = 1, 2,
\end{aligned}$$

where U^N_k are the numerical approximations to u_k $(k = 1, 2)$ for a particular value N, ε_1 and ε_2. We also define the computed parameter–uniform orders of convergence to be $p^N = log_2 \frac{E^N}{E^{2N}}$.

Table 3.1 illustrates that the approximations U_1 converge $\{\varepsilon_1, \varepsilon_2\}$–uniformly to the exact solution u_1 of example 1 where the finite difference method was used over the mesh $\Omega^N_{\varepsilon_1,\varepsilon_2}$. Similarly table 3.2 exhibits the $\{\varepsilon_1, \varepsilon_2\}$–uniform convergence of the numerical approximations U_2 to u_2 of example one. The boldface in each table highlights the maximum error over ε_1 for each N value.

ε_1	Number of Mesh Points N					
	64	128	256	512	1024	2048
2^0	0.050392	0.029756	0.017750	0.010214	0.005748	0.003181
2^{-2}	0.070608	0.037731	0.021373	0.012493	0.007096	0.003946
2^{-4}	0.195129	0.126169	0.075772	0.042935	0.022203	0.011307
2^{-6}	0.224595	0.145498	0.086910	0.050760	0.028655	0.015862
2^{-8}	0.232548	0.150735	0.089964	0.052572	0.029660	0.016417
2^{-10}	0.234579	0.152072	0.090748	0.053034	0.029916	0.016558
2^{-12}	0.235090	0.152408	0.090958	0.053150	0.029981	0.016594
2^{-14}	0.235218	0.152492	0.091010	0.053179	0.029997	0.016603
2^{-16}	0.235250	0.152513	0.091023	0.053186	0.030001	0.016605
2^{-18}	0.235258	0.152518	0.091026	0.053188	0.030002	0.016605
2^{-20}	0.235260	0.152519	0.091027	0.053188	0.030002	0.016605
2^{-22}	**0.235260**	**0.152520**	**0.091028**	**0.053188**	**0.030002**	**0.016606**
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-30}	0.235260	0.152520	0.091027	0.053189	0.030002	0.016606
2^{-34}	0.235260	0.152519	0.091025	0.053188	0.030002	0.016606
2^{-38}	0.235260	0.152513	0.091004	0.053183	0.030002	0.016606
2^{-42}	0.235260	0.152215	0.091004	0.052953	0.029933	0.016570
2^{-46}	0.214928	0.126649	0.071979	0.039056	0.020581	0.011216
2^{-50}	0.172974	0.107959	0.063729	0.036550	0.020421	0.011216
$E^N(u_2)$	0.235260	0.152520	0.091028	0.053188	0.030002	0.016606
$p^N(u_2)$	0.625	0.745	0.775	0.826	0.853	0.874

Table 3.2: Exact errors $E^N_{\varepsilon_1}(u_2)$, computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(u_2)$ and orders of convergence $p^N(u_2)$ for the solution u_2 of (3.1b) on the piecewise uniform mesh $\Omega^N_{\varepsilon_1,\varepsilon_2}$ using the transition points (3.3) over the set (3.4).

3.2 Weighted derivative errors

The weighted derivative results for u_1 the solution of (3.1a) are similar to those of v_1 and w_1 the first regular and singular components respectively. Combining lemmas 10 and 13 we obtain the following weighted derivative result for u_1 where the parameters $\{\varepsilon_1, \varepsilon_2, N\}$ are such that $\sigma_1 = \frac{4\varepsilon_1 \ln N}{\alpha_1}$ or $\sigma_2 = \frac{8\varepsilon_1 \ln N}{\alpha_1}$. Thus

$$|D^+(U_1 - u_1)(x_i)| \leq CN^{-1} \ln N \times \begin{cases} \frac{1}{\tilde{\varepsilon}} & if \quad x_i < \sigma_1 \\ \frac{1}{\bar{\varepsilon}} & if \quad \sigma_1 \leq x_i < \sigma_2 \,, x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2} \cup \{0\} \\ 1 & if \quad x_i \geq \sigma_2 \end{cases}$$

is obtained where the notation, $\tilde{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2\}$ and $\bar{\varepsilon} = \max\{\varepsilon_1, \varepsilon_2\}$ is adopted.

ε_1	Number of Mesh Points N 64	128	256	512	1024	2048
2^{0}	0.051831	0.027704	0.014336	0.007294	0.003679	0.001848
2^{-2}	0.269121	0.172641	0.098784	0.052997	0.027471	0.013988
2^{-4}	0.419637	0.382270	0.298375	0.200231	0.115243	0.062003
2^{-6}	0.483264	0.434538	0.336369	0.230047	0.143769	0.084558
2^{-8}	0.508262	0.449127	0.347076	0.237143	0.148124	0.087093
2^{-10}	0.514738	0.452869	0.349822	0.238962	0.149240	0.087743
2^{-12}	0.516558	0.453811	0.350513	0.239420	0.149521	0.087906
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-18}	0.518498	0.454121	0.350740	0.239570	0.149613	0.087960
2^{-22}	0.518744	**0.454125**	**0.350743**	**0.239573**	**0.149615**	**0.087961**
2^{-26}	0.518808	0.454125	0.350743	0.239572	0.149615	0.087960
2^{-30}	0.518824	0.454115	0.350733	0.239565	0.149609	0.087957
2^{-34}	0.518829	0.453949	0.350581	0.239445	0.149524	0.087901
2^{-38}	**0.518830**	0.451304	0.348159	0.237533	0.148174	0.087018
2^{-42}	0.518830	0.410959	0.311572	0.208934	0.128156	0.074041
2^{-46}	0.518830	0.375328	0.247306	0.151519	0.088211	0.049640
2^{-50}	0.518830	0.375328	0.247306	0.151519	0.088211	0.049640
$E^N(D^+U_1)$	0.518830	0.454125	0.350743	0.239573	0.149615	0.087961
$p^N(D^+U_1)$	0.192	0.373	0.550	0.679	0.766	0.822

Table 3.3: Exact weighted derivative errors $E_{\varepsilon_1}^N(D^+U_1)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(D^+U_1)$ and orders of convergence $p^N(D^+U_1)$ for the solution u_1 of (3.1a) on the piecewise uniform mesh $\Omega_{\varepsilon_1,\varepsilon_2}^N$ using the transition points (3.3) over the set (3.4).

When $\sigma_1 = \frac{4\varepsilon_2 \ln N}{\alpha_2}$ and $\sigma_2 = \frac{1}{2}$ we combine lemma 9 where $\varepsilon_1^{-1} \leq C \ln N$ with lemma 14 to obtain $|D^+(U_1 - u_1)(x_i)| \leq CN^{-1}(\ln N)^2$. Using [2] we obtain $|D^+(U_1 - u_1)(x_i)| \leq CN^{-1}(\ln N)^2$ for the uniform–mesh case.

The weighted derivative errors which are akin to those of [3] are discussed in the analysis in chapter two and are defined as follows:

$$\begin{aligned}
E_{\varepsilon_1,\varepsilon_2}^N(D^+U_1) &= \| \varepsilon_{x_k} D^+(U_1^N - u_1) \|_{\Omega_{\varepsilon_1,\varepsilon_2}^N}, \\
E_{\varepsilon_1}^N(D^+U_1) &= \max_{\varepsilon_2} E_{\varepsilon_1,\varepsilon_2}^N(D^+U_1) \\
E^N(D^+U_1) &= \max_{\varepsilon_1} E_{\varepsilon_1}^N(D^+U_1).
\end{aligned}$$

where $(\varepsilon_1, \varepsilon_2) \in S$ and $\varepsilon_{x_k} = \begin{cases} \tilde{\varepsilon} & if \quad x_k < \sigma_1 \\ \bar{\varepsilon} & if \quad \sigma_1 \leq x_k < \sigma_2 \\ 1 & if \quad x_k \geq \sigma_2 \end{cases}$, $0 \leq k \leq N-1$.

The results in table 3.3 validate the $\{\varepsilon_1, \varepsilon_2\}$–uniform convergence of the weighted derivative errors for the solution u_1 of example 1 using the transition points (3.3) and the set (3.4).

Next we justify the choice of the constant 4 for the factor in σ_2 for the weighted derivatives by means of a counterexample. We illustrate this over the set S for $E^N(D^+U_1)$ taking the transition points

$$\begin{aligned} \sigma_2 &= \min\{\frac{1}{2}, 2\max\{\frac{\varepsilon_1 \ln N}{\alpha_1}, \frac{\varepsilon_2 \ln N}{\alpha_2}\}\}, \\ \sigma_1 &= \min\{\frac{\sigma_2}{2}, \frac{\varepsilon_1 \ln N}{\alpha_1}, \frac{\varepsilon_2 \ln N}{\alpha_2}\}. \end{aligned} \tag{3.6}$$

	Number of Mesh Points N					
ε_1	64	128	256	512	1024	2048
2^{0}	0.051831	0.027704	0.014336	0.007294	0.003679	0.001848
2^{-2}	0.269121	0.172641	0.098784	0.052997	0.027471	0.013988
2^{-4}	0.437506	0.318889	0.211011	0.129598	0.075559	0.042560
2^{-6}	0.587243	0.359946	0.237457	0.145583	0.084787	0.047724
2^{-8}	0.863551	0.651765	0.419621	0.207659	0.098850	0.049151
2^{-10}	0.953154	0.797238	0.638586	0.483270	0.318134	0.156323
2^{-12}	0.977213	0.839070	0.710204	0.603395	0.507821	0.404350
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-16}	0.984878	0.852664	0.734389	0.647015	0.586843	0.544976
2^{-20}	0.985360	0.853523	0.735932	0.649854	0.592190	0.555218
2^{-24}	0.985390	0.853576	0.736028	0.650032	0.592526	0.555865
2^{-28}	0.985392	0.853580	0.736034	0.650043	0.592547	0.555905
2^{-32}	**0.985392**	**0.853580**	**0.736035**	**0.650043**	**0.592548**	**0.555908**
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-50}	0.985392	0.853580	0.736035	0.650043	0.592548	0.555908
$E^N(D^+U_1)$	0.985392	0.853580	0.736035	0.650043	0.592548	0.555908
$p^N(D^+U_1)$	0.207	0.214	0.179	0.134	0.092	0.060

Table 3.4: Exact weighted derivative errors $E^N_{\varepsilon_1}(D^+U_1)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(D^+U_1)$ and orders of convergence $p^N(D^+U_1)$ for the solution u_1 of (3.1a) on the piecewise uniform mesh $\Omega^N_{\varepsilon_1,\varepsilon_2}$ using the transition points (3.6) over the set (3.4).

The orders of convergence from table 3.4 indicate that this choice of mesh will not guarantee that the pointwise weighted derivative errors $E^N(D^+U_1)$ will diminish as N increases. The comparison with table 3.3 clearly demonstrates that the choice of the constant 4 for the factor in σ_2 is appropriate.

3.3 Component wise errors for the constant coefficient problem

Recall that the exact solution of (3.1b) is $u_2(x) = v_2(x) + w_2(x) + z(x)$. A list of the computed errors for the three components of the solution u_2 of example 1 on $\Omega^N_{\varepsilon_1,\varepsilon_2}$ are given in tables 3.5, 3.6 and 3.7. The components $w_2(x)$, $z(x)$ and $v_2(x)$ of $u_2(x)$ the solution of (3.1b) are illustrated in figures 4.5, 4.6 and 4.7 respectively.

	Number of Mesh Points N					
ε_1	64	128	256	512	1024	2048
2^{0}	0.017173	0.008770	0.004435	0.002230	0.001118	0.000560
2^{-2}	0.027958	0.014298	0.007247	0.003650	0.001832	0.000918
2^{-4}	0.054404	0.025059	0.011528	0.005490	0.002748	0.001375
2^{-6}	0.086006	0.042735	0.021124	0.010414	0.005126	0.002521
2^{-8}	0.095002	0.047835	0.023955	0.011964	0.005968	0.002975
2^{-10}	0.097329	0.049154	0.024688	0.012366	0.006186	0.003092
2^{-12}	0.097915	0.049486	0.024873	0.012468	0.006241	0.003122
2^{-14}	0.098062	0.049570	0.024919	0.012493	0.006255	0.003129
2^{-16}	0.098099	0.049591	0.024931	0.012499	0.006258	0.003131
2^{-18}	0.098108	0.049596	0.024934	0.012501	0.006259	0.003132
2^{-20}	**0.098111**	0.049597	**0.024935**	**0.012501**	**0.006259**	**0.003132**
2^{-22}	0.098111	0.049597	0.024935	0.012501	0.006259	0.003132
2^{-24}	0.098111	**0.049598**	0.024935	0.012501	0.006259	0.003132
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-50}	0.098111	0.049598	0.024935	0.012501	0.006259	0.003132
$e^N(v_2)$	0.098111	0.049598	0.024935	0.012501	0.006259	0.003132
$p^N(v_2)$	0.984	0.992	0.996	0.998	0.999	1.000

Table 3.5: Exact errors $E^N_{\varepsilon_1}(v_2)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(v_2)$ and orders of convergence $p^N(v_2)$ for the second regular component $v_2(x)$ of the solution from (3.1b) on the piecewise uniform mesh $\Omega^N_{\varepsilon_1,\varepsilon_2}$ using the transition points (3.3) over the set (3.4).

The $\{\varepsilon_1, \varepsilon_2\}$–uniform convergence of the numerical solution V_2 to v_2, the second regular component of example one is obvious from table 3.5.

Next we illustrate the errors due to the second singular component. The pointwise errors of the singular component with respect to both small parameters decrease as N decreases and this is reflected in table 3.6.

	Number of Mesh Points N					
ε_1	64	128	256	512	1024	2048
2^0	0.048104	0.028230	0.017031	0.009848	0.005571	0.003095
2^{-2}	0.077714	0.042346	0.022789	0.013178	0.007455	0.004141
2^{-4}	0.122207	0.075940	0.045821	0.026497	0.014989	0.008327
2^{-6}	0.122687	0.076239	0.046001	0.026601	0.015048	0.008359
2^{-8}	0.122807	0.076313	0.046046	0.026627	0.015063	0.008367
2^{-10}	0.122837	0.076332	0.046057	0.026634	0.015066	0.008369
2^{-12}	0.122844	0.076336	0.046060	0.026635	0.015140	0.008486
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-18}	**0.122847**	0.076750	0.047790	0.028340	0.016283	0.009130
2^{-22}	0.122847	0.077317	0.048052	0.028463	0.016356	0.009178
2^{-26}	0.122847	0.077454	0.048113	0.028490	0.016370	0.009188
2^{-30}	0.122847	0.077489	0.048127	0.028495	0.016373	**0.009191**
2^{-34}	0.122847	0.077497	0.048131	0.028496	0.016373	0.009191
2^{-38}	0.122847	0.077499	**0.048132**	**0.028497**	**0.016374**	0.009191
2^{-42}	0.122847	**0.077500**	0.048132	0.028497	0.016374	0.009191
2^{-46}	0.122847	0.077500	0.048132	0.028497	0.016374	0.009191
2^{-50}	0.117509	0.077500	0.048132	0.028497	0.016374	0.009191
$e^N(w_2)$	0.122847	0.077500	0.048132	0.028497	0.016374	0.009191
$p^N(w_2)$	0.665	0.687	0.756	0.799	0.833	0.859

Table 3.6: Exact errors $E^N_{\varepsilon_1}(w_2)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(w_2)$ and orders of convergence $p^N(w_2)$ for the second singular component $w_2(x)$ of the solution from (3.1b) on the piecewise uniform mesh $\Omega^N_{\varepsilon_1,\varepsilon_2}$ using the transition points (3.3) over the set (3.4).

Next we illustrate the errors due to the coupling component. In a similar way to the previous two components table 3.7 illustrates the $\{\varepsilon_1, \varepsilon_2\}$–uniform convergence of the numerical approximations Z to z the coupling component of example one. It is also interesting to note that the orders of convergence $p^N(z)$ are similar to the orders $p^N(w_2)$ for the second singular component for the example chosen here.

ε_1	Number of Mesh Points N 64	128	256	512	1024	2048
2^{0}	0.010483	0.005383	0.002726	0.001372	0.000688	0.000345
2^{-2}	0.047912	0.025546	0.013281	0.006766	0.003416	0.001717
2^{-4}	0.168395	0.111065	0.068978	0.039369	0.020453	0.010418
2^{-6}	0.183085	0.120751	0.074994	0.044431	0.025529	0.014322
2^{-8}	0.186759	0.123172	0.076498	0.045322	0.026041	0.014609
2^{-10}	0.187677	0.123778	0.076874	0.045544	0.026169	0.014681
2^{-12}	0.187907	0.123929	0.076968	0.045600	0.026201	0.014699
2^{-14}	0.187965	0.123967	0.076991	0.045614	0.026209	0.014704
2^{-16}	0.187979	0.123976	0.076997	0.045618	0.026211	**0.014705**
2^{-18}	0.187983	0.123979	0.076998	0.045618	0.026211	0.014705
2^{-20}	0.187983	0.123979	0.076999	0.045619	**0.026212**	0.014705
2^{-22}	**0.187984**	**0.123980**	**0.076999**	**0.045619**	0.026212	0.014705
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-28}	0.187984	0.123980	0.076999	0.045619	0.026212	0.014705
2^{-30}	0.187984	0.123979	0.076999	0.045619	0.026212	0.014705
2^{-34}	0.187983	0.123976	0.076999	0.045619	0.026212	0.014705
2^{-38}	0.187973	0.123924	0.076995	0.045619	0.026212	0.014705
2^{-42}	0.187582	0.122901	0.076720	0.045552	0.026155	0.014667
2^{-46}	0.149029	0.094443	0.054514	0.030124	0.015911	0.008188
2^{-50}	0.076018	0.052068	0.032622	0.019199	0.010969	0.006126
$e^N(z)$	0.187984	0.123980	0.076999	0.045619	0.026212	0.014705
$p^N(z)$	0.601	0.687	0.755	0.799	0.834	0.859

Table 3.7: Exact errors $E^N_{\varepsilon_1}(z)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(z)$ and orders of convergence $p^N(z)$ for the second singular component $z(x)$ of the solution from (3.1b) on the piecewise uniform mesh $\Omega^N_{\varepsilon_1,\varepsilon_2}$ using the transition points (3.3) over the set (3.4).

3.4 Counterexamples for one transition point

In this section we outline counterexamples that support the choice of a two transition point mesh. In the first three examples of this section a single transition point is used. Both equations are solved over the same mesh $\bar{\Omega}^N_\sigma$ using a single transition point. We define the mesh using a single transition point σ in the following manner:

$$\bar{\Omega}^N_\sigma = \{x_i\}, \quad x_i = \begin{cases} 2\sigma i/N & if \quad i \leq N/2 \\ x_{i-1} + 2(1-\sigma)/N & if \quad i > N/2 \end{cases}. \tag{3.7}$$

First we examine the single transition point

$$\sigma = \min\{\frac{1}{2}, \frac{4\varepsilon_1 \ln N}{\alpha_1}\}, \tag{3.8}$$

Here we look at $E^N(u_2)$ using the mesh Ω_σ^N over the set (3.4). The lack of $\{\varepsilon_1, \varepsilon_2\}$–uniform convergence of the numerical method over this mesh is clearly demonstrated in table 3.8.

ε_1	Number of Mesh Points N 64	128	256	512	1024	2048
2^0	0.110146	0.106153	0.104121	0.103096	0.102581	0.102323
2^{-2}	0.115683	0.111908	0.107893	0.105227	0.103711	0.102905
2^{-4}	0.134639	0.105250	0.110845	0.110407	0.107329	0.104996
2^{-6}	0.164083	0.100627	0.109305	0.110615	0.109491	0.107786
2^{-8}	0.172233	0.102562	0.108898	0.110521	0.109486	0.107800
2^{-10}	0.193500	0.143140	0.108794	0.110497	0.109486	0.107804
2^{-12}	0.210177	0.180567	0.151304	0.110491	0.109486	0.107805
2^{-14}	0.217904	0.188866	0.174471	0.158576	0.131130	0.107805
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-18}	0.220299	0.191233	0.180885	0.174879	0.170742	0.166302
2^{-22}	0.220449	0.191377	0.181265	0.175807	0.172916	0.171306
2^{-26}	0.220458	0.191386	0.181289	0.175864	0.173050	0.171609
2^{-30}	0.220459	0.191387	0.181290	0.175868	0.173058	0.171628
2^{-34}	**0.220459**	**0.191387**	**0.181290**	**0.175868**	**0.173059**	**0.171629**
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-50}	0.220459	0.191387	0.181290	0.175868	0.173059	0.171629
$E^N(u_2)$	0.220459	0.191387	0.181290	0.175868	0.173059	0.171629
$p^N(u_2)$	0.204	0.078	0.044	0.023	0.012	0.001

Table 3.8: Exact errors $E_{\varepsilon_1}^N(u_2)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(u_2)$ and orders of convergence $p^N(u_2)$ for the solution u_2 of (3.1b) on the piecewise uniform mesh Ω_σ^N using the transition point (3.8) over the set (3.4).

Next we construct a counterexample in a similar way to that done in table 3.8 where we look at $E^N(u_1)$ over the set S . We now take the single transition point

$$\sigma = \min\{\frac{1}{2}, \frac{4\varepsilon_2 \ln N}{\alpha_2}\}. \tag{3.9}$$

We see from table 3.9 that the numerical method is not $\{\varepsilon_1, \varepsilon_2\}$–uniformly convergent for u_1 using Ω_σ^N, (3.9) and (3.4).

The next counterexample is constructed where we take the single transition point

$$\sigma = \min\{\frac{1}{2}, \frac{4\varepsilon_1 \ln N}{\alpha_1}, \frac{4\varepsilon_2 \ln N}{\alpha_2}\} \tag{3.10}$$

over the corresponding mesh Ω_σ^N. The computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(u_1)$ and $E^N(u_2)$ are both shown in table 3.10 where it is clear that this single transition point does not guarantee the $\{\varepsilon_1, \varepsilon_2\}$–uniform convergence.

ε_1	Number of Mesh Points N 64	128	256	512	1024	2048
2^0	0.012657	0.006278	0.003126	0.001560	0.000779	0.000389
2^{-2}	0.069221	0.036181	0.018573	0.009413	0.004739	0.002378
2^{-4}	0.198993	0.115603	0.064410	0.034211	0.017666	0.008982
2^{-6}	0.222933	0.222933	0.183545	0.107057	0.061368	0.032652
2^{-8}	0.247303	0.212291	0.206884	0.206146	0.178620	0.104357
2^{-10}	0.254192	0.216951	0.211367	0.208215	0.205151	0.201713
2^{-12}	0.255922	0.218943	0.212482	0.209474	0.206562	0.203097
2^{-14}	0.256356	0.219441	0.212761	0.209788	0.206911	0.203485
2^{-16}	0.256464	0.219566	0.212831	0.209867	0.206999	0.203582
2^{-18}	0.256491	0.219597	0.212848	0.209886	0.207021	0.203606
2^{-20}	0.256498	0.219605	0.212853	0.209891	0.207026	0.203612
2^{-22}	0.256499	0.219607	0.212854	0.209892	0.207027	0.203613
2^{-24}	**0.256500**	0.219607	**0.212854**	**0.209893**	**0.207028**	**0.203614**
2^{-26}	0.256500	0.219607	0.212854	0.209893	0.207028	0.203614
2^{-28}	0.256500	**0.219608**	0.212854	0.209893	0.207028	0.203614
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-50}	0.256500	0.219608	0.212854	0.209893	0.207028	0.203614
$e^N(u_1)$	0.256500	0.219608	0.212854	0.209893	0.207028	0.203614
$p^N(u_1)$	0.224	0.045	0.020	0.020	0.024	0.029

Table 3.9: Exact errors $E^N_{\varepsilon_1}(u_1)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(u_1)$ and orders of convergence $p^N(u_1)$ for the solution u_1 of (3.1a) on the piecewise uniform mesh Ω^N_σ using the transition point (3.9) over the set (3.4).

	Number of Mesh Points N 64	128	256	512	1024	2048
$E^N(u_1)$	0.202358	0.222933	0.183545	0.206146	0.178620	0.201713
$E^N(u_2)$	0.279982	0.302553	0.255950	0.282447	0.245100	0.277103

Table 3.10: The computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(u_1)$ and $E^N(u_2)$ for the solutions u_1 and u_2 of (3.1) on the piecewise uniform mesh Ω^N_σ using the transition point (3.10) over the set (3.4).

Interpolation

Here we attempt to solve problem (3.1) by using a different mesh for each equation. The first equation (3.1a) has only one dependent variable it is then appropriate to obtain the numerical approximations U_1^N using the piecewise uniform mesh $\Omega^N_{\sigma_1}$ using the single transition point

$$\sigma_1 = \min\{\frac{4\varepsilon_1 \ln N}{\alpha_1}, \frac{1}{2}\}. \tag{3.11}$$

The solution u_1 of the first equation (3.1a) is independent of ε_2. The $\{\varepsilon_1, \varepsilon_2\}$–uniform convergence validated in table 3.11 can be viewed as an ε_1–uniform convergence and the choice of transition point (3.11) provides this.

	Number of Mesh Points N					
ε_1	64	128	256	512	1024	2048
2^0	0.006278	0.003126	0.001560	0.000779	0.000389	0.000195
2^{-2}	0.036181	0.018573	0.009413	0.004739	0.002378	0.001191
2^{-4}	0.097887	0.057332	0.032539	0.017666	0.008982	0.004530
2^{-6}	0.120774	0.069897	0.039771	0.022112	0.012112	0.006559
2^{-8}	0.127133	0.073534	0.041799	0.023257	0.012741	0.006900
2^{-10}	0.128768	0.074503	0.042321	0.023555	0.012903	0.006988
2^{-12}	0.129180	0.074746	0.042452	0.023631	0.012943	0.007010
2^{-14}	0.129283	0.074808	0.042485	0.023649	0.012954	0.007016
2^{-16}	0.129309	0.074823	0.042493	0.023654	0.012956	0.007017
2^{-18}	0.129315	0.074827	0.042495	0.023655	0.012957	0.007017
2^{-20}	0.129317	**0.074828**	**0.042496**	**0.023656**	**0.012957**	**0.007017**
2^{-22}	0.129317	0.074828	0.042496	0.023656	0.012957	0.007017
2^{-24}	**0.129318**	0.074828	0.042496	0.023656	0.012957	0.007017
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-50}	0.129318	0.074828	0.042496	0.023656	0.012957	0.007017
$e^N(u_1)$	0.129318	0.074828	0.042496	0.023656	0.012957	0.007017
$p^N(u_1)$	0.790	0.816	0.845	0.868	0.885	0.896

Table 3.11: Exact errors $E_{\varepsilon_1}^N(u_1)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(u_1)$ and orders of convergence $p^N(u_1)$ for the approximations U_1^N of (3.13a) to the solution u_1 of (3.1a) on the piecewise uniform mesh $\Omega_{\sigma_1}^N$ using the transition point (3.11) over the set (3.4).

Since the second equation (3.1b) depends on the small parameter ε_2 we attempt to obtain the numerical approximations U_2^N using the piecewise uniform mesh $\Omega_{\sigma_2}^N$ and the single transition point

$$\sigma_2 = \min\{\frac{4\varepsilon_2 \ln N}{\alpha_2}, \frac{1}{2}\}. \tag{3.12}$$

Before we can find the approximations U_2^N, we need numerical approximations to the solution u_1 of (3.1a) at each of the mesh points of $\Omega_{\sigma_2}^N$. We interpolate the numerical approximations U_1^N obtained over $\Omega_{\sigma_1}^N$ using the transition point (3.11). The interpolated values over the mesh $\Omega_{\sigma_2}^N$ are represented by $\bar{U}_1^N$. The second equation is then solved over $\Omega_{\sigma_2}^N$ using the interpolated values $\bar{U}_1^N$.

We summarise the corresponding numerical problem to (3.1) where the first equation is solved on the mesh $\Omega_{\sigma_1}^N$ and the second equation is solved on the mesh

$\Omega^N_{\sigma_1}$ using the interpolated values $\bar{U}^N_1$.

$$\begin{aligned} \varepsilon_1\delta^2 U_{1,i} + 3D^+U_{1,i} &= 15x_i^4, \quad x_i \in \Omega^N_{\sigma_1}, & (3.13a)\\ \varepsilon_2 U_{2,i} + 2D^+U_{2,i} + 2.75D^+\bar{U}_{1,i} &= 0.6e^{x_i}, \quad x_i \in \Omega^N_{\sigma_2}, & (3.13b) \end{aligned}$$

where $U_{1,0} = u_1(0)$, $U_{1,N} = u_1(1)$, $U_{2,0} = u_2(0)$ and $U_{2,N} = u_2(1)$.

It would appear that glancing at the exact errors $E^N_{\varepsilon_1}(u_2)$ and at the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(u_2)$ in table 3.12 that the approximations U^N_2 are $\{\varepsilon_1, \varepsilon_2\}$–uniformly convergent but looking at the orders of convergence we see that the orders are decreasing which suggests the rate of convergence is slowing down thus suggesting the interpolating method is not parameter–uniform.

	Number of Mesh Points N					
ε_1	64	128	256	512	1024	2048
2^0	0.050392	0.030124	0.017880	0.010259	0.005761	0.003183
2^{-2}	0.061996	0.035249	0.021374	0.012483	0.007087	0.003942
2^{-4}	0.149181	0.083880	0.047104	0.025014	0.012625	0.006319
2^{-6}	0.192375	0.111707	0.063630	0.035674	0.019192	0.010168
2^{-8}	0.206435	0.123700	0.074944	0.045319	0.025807	0.013956
2^{-10}	0.210092	0.127038	0.078971	0.050929	0.032649	0.020332
2^{-12}	0.211014	0.127885	0.080001	0.052479	0.035155	0.024266
2^{-14}	0.211246	0.128098	0.080260	0.052870	0.035794	0.025348
2^{-16}	0.211303	0.128151	0.080325	0.052968	0.035954	0.025621
2^{-18}	0.211318	0.128164	0.080341	0.052993	0.035995	0.025689
2^{-20}	0.211321	0.128168	0.080345	0.052999	0.036005	0.025706
2^{-22}	0.211322	0.128169	0.080346	0.053000	0.036007	0.025710
2^{-24}	**0.211323**	**0.128169**	**0.080346**	**0.053001**	**0.036008**	**0.025711**
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-46}	0.211323	0.128169	0.076990	0.046474	0.028462	0.017820
2^{-50}	0.134607	0.075994	0.046770	0.033300	0.021997	0.013837
$e^N(u_2)$	0.211323	0.128169	0.080346	0.053001	0.036008	0.025711
$p^N(u_2)$	0.721	0.674	0.600	0.558	0.486	0.376

Table 3.12: Exact errors $E^N_{\varepsilon_1}(u_2)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $E^N(u_2)$ and orders of convergence $p^N(u_2)$ for the approximations U^N_2 of (3.13b) to the solution u_2 of (3.1b) on the piecewise uniform mesh $\Omega^N_{\sigma_2}$ using the transition point (3.12) over the set (3.4) and using the interpolated values of $\bar{U}^N_1$ obtained on $\Omega^N_{\sigma_1}$ which used the transition point (3.11) .

3.5 A variable coefficient problem

Even though we obtained the exact solution for the constant coefficient problem of example one, the exact solution is generally not easily obtained. We need to address how we measure the errors when the exact solution is not available. The errors for the numerical solutions U_k^N $(k = 1, 2)$ are obtained by comparing the numerical solution at each N with the linear interpolant of the numerical solution on the finest mesh available, here that is the mesh when $N = 8192$ [2]. We then define the approximate errors $e^N_{\varepsilon_1,\varepsilon_2}(u_k)$, $e^N_{\varepsilon_1}(u_k)$ and the computed parameter–uniform maximum pointwise error $e^N(u_k)$ as follows:

$$
\begin{aligned}
e^N_{\varepsilon_1,\varepsilon_2}(u_k) &= \| U_k^N - \bar{U}_k^{8192} \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}}, \quad k = 1, 2,\\
e^N_{\varepsilon_1}(u_k) &= \max_{\varepsilon_2} e^N_{\varepsilon_1,\varepsilon_2}(u_k) \quad k = 1, 2,\\
e^N(u_k) &= \max_{\varepsilon_1} e^N_{\varepsilon_1}(u_k) \quad k = 1, 2,
\end{aligned}
$$

where U_k^N are the numerical approximations for u_k at a particular N, ε_1 and ε_2. The $\bar{U}_k^{8192}$ are the interpolated values at the N mesh points using the numerical solution U_k^{8192}.

An approximation to p the $\{\varepsilon_1, \varepsilon_2\}$–uniform rate of convergence, is determined using the double mesh method [2]. This involves the double–mesh differences where we define $D^N_{\varepsilon_1,\varepsilon_2}(u_k)$, $D^N_{\varepsilon_1}(u_k)$ and the computed parameter–uniform maximum double mesh difference $D^N(u_k)$ as follows:

$$
\begin{aligned}
D^N_{\varepsilon_1,\varepsilon_2}(u_k) &= \| U_k^N - \bar{U}_k^{2N} \|_{\Omega^N_{\varepsilon_1,\varepsilon_2}}, \quad k = 1, 2,\\
D^N_{\varepsilon_1}(u_k) &= \max_{\varepsilon_2} D^N_{\varepsilon_1,\varepsilon_2}(u_k) \quad k = 1, 2,\\
D^N(u_k) &= \max_{\varepsilon_1} D^N_{\varepsilon_1}(u_k) \quad k = 1, 2,
\end{aligned}
$$

where U_k^N are the numerical approximations to u_k at a particular N, ε_1 and ε_2 and $\bar{U}_k^{2N}$ are the interpolated values at the N mesh points using the numerical solution U_k^{2N}. Here the double mesh orders of convergence are defined by $p_d^N = log_2(\frac{D^N}{D^{2N}})$.

Consider the following variable coefficient problem:

Example Two

$$
\begin{aligned}
\varepsilon_1 u_1''(x) + 3e^{3x} u_1'(x) &= 8.0e^{x}, &\text{(3.14a)}\\
\varepsilon_2 u_2''(x) + 2e^{0.5x} u_2' + 2.8\cos(x) u_1'(x) &= 0.6e^{0.8x}, &\text{(3.14b)}
\end{aligned}
$$

where $x \in \Omega$, $u_1(0) = 0$, $u_1(1) = 0$, $u_2(0) = 0$ and $u_2(1) = 0$.

The corresponding discrete problem for $x_i \in \Omega^N_{\varepsilon_1,\varepsilon_2}$ is:

$$\begin{aligned} \varepsilon_1\delta^2 U_{1,i} + 3e^{3x_i}D^+U_{1,i} &= 8e^{x_i}, && (3.15a)\\ \varepsilon_2 U_{2,i} + 2e^{0.5x_i}D^+U_{2,i} + 2.8\cos(x_i)D^+U_{1,i} &= 0.6e^{0.8x_i}, && (3.15b)\end{aligned}$$

where $U_{1,0} = u_1(0)$, $U_{1,N} = u_1(1)$, $U_{2,0} = u_2(0)$ and $U_{2,N} = u_2(1)$.

The coefficients in (3.14) are chosen so that $\| U_1^{8192} \|_{\Omega_{\varepsilon_1,\varepsilon_1}}$ and $\| U_2^{8192} \|_{\Omega_{\varepsilon_1,\varepsilon_1}}$ are approximately one. In table 3.13 the approximate errors $e^N(u_1)$, the double mesh differences $D^N(u_1)$ and the double mesh orders of convergence for the solution u_1 of example two are illustrated. It is evident that the $\{\varepsilon_1, \varepsilon_2\}$–uniform convergence holds.

	Number of Mesh Points N					
ε_1	64	128	256	512	1024	2048
2^{0}	0.019025	0.009915	0.004989	0.002442	0.001146	0.000493
2^{-2}	0.064629	0.035028	0.018087	0.008976	0.004243	0.001831
2^{-4}	0.186232	0.111873	0.064734	0.033942	0.016469	0.007203
2^{-6}	**0.187954**	**0.129728**	**0.083546**	**0.048474**	**0.026287**	**0.012533**
2^{-8}	0.183362	0.126820	0.081815	0.047559	0.025793	0.012309
2^{-10}	0.181731	0.125846	0.081200	0.047210	0.025609	0.012222
2^{-12}	0.181288	0.125584	0.081032	0.047113	0.025558	0.012198
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
2^{-18}	0.181139	0.125497	0.080976	0.047081	0.025540	0.012189
2^{-22}	0.181137	0.125496	0.080975	0.047080	0.025540	0.012189
2^{-26}	0.181137	0.125495	0.080975	0.047080	0.025540	0.012189
2^{-30}	0.181136	0.125495	0.080975	0.047080	0.025540	0.012189
2^{-34}	0.181117	0.125493	0.080972	0.047076	0.025539	0.012189
2^{-38}	0.180799	0.125463	0.080919	0.047024	0.025523	0.012189
2^{-42}	0.175602	0.124469	0.079625	0.046562	0.025170	0.012062
2^{-46}	0.106052	0.071673	0.043772	0.024827	0.013105	0.006172
2^{-50}	0.106052	0.071673	0.043772	0.024827	0.013105	0.006172
$e^N(u_1)$	0.187954	0.129728	0.083546	0.048474	0.026287	0.012533
$D^N(u_1)$	0.036445	0.032676	0.025081	0.017474	0.009266	0.005335
$p_d^N(u_1)$	0.157	0.382	0.521	0.915	0.796	0.830

Table 3.13: Approximate errors $e^N_{\varepsilon_1}(u_1)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $e^N(u_1)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–double mesh differences $D^N(u_1)$ and orders of convergence $p_d^N(u_1)$ for the solutions u_1 of (3.14b) on the piecewise uniform mesh $\Omega^N_{\varepsilon_1,\varepsilon_2}$ using the transition points (3.3) over the set (3.4).

In table 3.14 the approximate errors $e^N(u_2)$, the double mesh differences $D^N(u_2)$ and the orders of convergence for the solution u_2 of example two are illustrated and we see that the $\{\varepsilon_1, \varepsilon_2\}$–uniform convergence holds.

	Number of Mesh Points N					
ε_1	64	128	256	512	1024	2048
2^0	0.066428	0.042822	0.025678	0.014103	0.007014	0.003285
2^{-2}	0.080571	0.054359	0.033212	0.018275	0.008787	0.004043
2^{-4}	0.247149	0.151528	0.086999	0.045782	0.022228	0.009725
2^{-6}	**0.257211**	**0.178906**	**0.115435**	**0.067050**	**0.036414**	**0.017393**
2^{-8}	0.250239	0.174543	0.113018	0.065843	0.035770	0.017111
2^{-10}	0.247526	0.172907	0.112038	0.065302	0.035485	0.016978
2^{-12}	0.246778	0.172458	0.111765	0.065148	0.035404	0.016940
2^{-14}	0.246586	0.172343	0.111695	0.065108	0.035383	0.016930
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-20}	0.246523	0.172305	0.111672	0.065095	0.035376	0.016927
2^{-24}	0.246522	0.172305	0.111672	0.065095	0.035376	0.016927
2^{-28}	0.246521	0.172305	0.111672	0.065095	0.035376	0.016927
2^{-32}	0.246516	0.172305	0.111671	0.065094	0.035375	0.016927
2^{-34}	0.246500	0.172305	0.111670	0.065092	0.035375	0.016927
2^{-38}	0.246160	0.172305	0.111639	0.065043	0.035366	0.016927
2^{-42}	0.240519	0.171967	0.110510	0.064789	0.035081	0.016830
2^{-46}	0.138907	0.095631	0.060018	0.034229	0.018403	0.008768
2^{-50}	0.138907	0.092288	0.060018	0.034229	0.018403	0.008768
$e^N(u_2)$	0.257211	0.178906	0.115435	0.067050	0.036414	0.017393
$D^N(u_2)$	0.047570	0.043434	0.033629	0.023554	0.012614	0.007367
$p_d^N(u_2)$	0.131	0.369	0.514	0.901	0.776	0.827

Table 3.14: Approximate errors $e_{\varepsilon_1}^N(u_2)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–uniform errors $e^N(u_2)$, the computed $\{\varepsilon_1, \varepsilon_2\}$–double mesh differences $D^N(u_2)$ and orders of convergence $p_d^N(u_2)$ for the solutions u_2 of (3.14b) on the piecewise uniform mesh $\Omega_{\varepsilon_1,\varepsilon_2}^N$ using the transition points (3.3) over the set (3.4).

Chapter 4

Graphical output

In this chapter some graphical outputs are provided which illustrate the need for two transition points. This is done by highlighting the "double–layer effect" of the solution to the second differential equation. Each of the components of the second solution are also illustrated. When $\frac{\varepsilon_1}{\alpha_1} \leq \frac{\varepsilon_2}{\alpha_2}$ the dependence of the coupling component z on the ratio $\frac{\varepsilon_1}{\varepsilon_2}$ is also shown. The graphical outputs of the exact solutions from (3.1) the constant coefficient problem are also provided. Finally a counterexample is given which verifies that a standard comparison principle does not hold in general for a convection–diffusion system.

4.1 The "double–layer" effect

The constant coefficient problem

First the constant coefficient problem (3.1) is examined. The exact solution $u_1(x)$

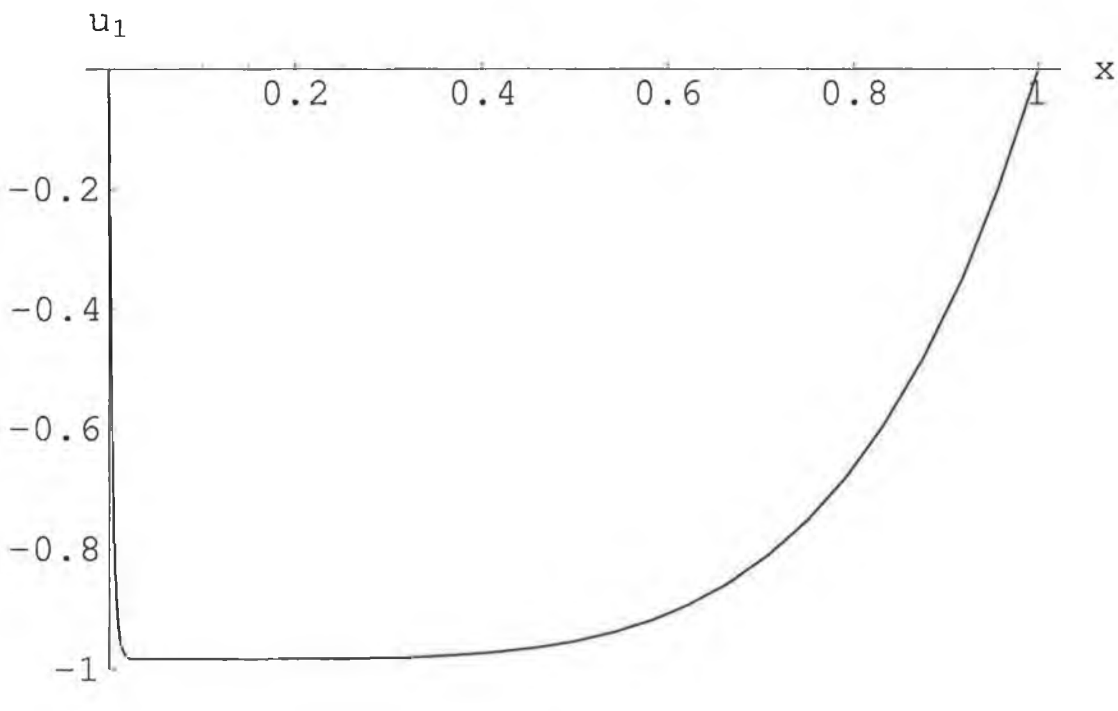

Figure 4.1: The solution $u_1(x)$, of (3.1a) when $\varepsilon_1 = 0.01$.

of (3.1) is illustrated over $\bar{\Omega}$ in figure 4.1 where $\varepsilon_1 = 0.01$. The exact solution $u_2(x)$ is illustrated over $\bar{\Omega}$ in 4.2 where $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.0001$.

Figure 4.1 shows that the solution $u_1(x)$ has only one "layer", that is there is only one steep gradient. This is due to the first singular component $w_1(x)$ which is discussed in [2]. The view of $u_2(x)$ over the domain $[0, 1]$ in figure 4.2 hides the "double–layer" effect. There are two steep gradients, one due to the second singular component $w_2(x)$ and the other due to the coupling component $z(x)$. This "double–layer" further demonstrates the need for two transition points. Two further "zooms" illustrate the two layers where $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.0001$. The first layer which is of order ε_2 is shown in figure 4.3 where the solution u_2 is illustrated over the domain $[0, 0.002)$. The second layer which is of order ε_1 is highlighted in figure 4.4 where the solution u_2 is illustrated over the interval $[0, 0.1)$.

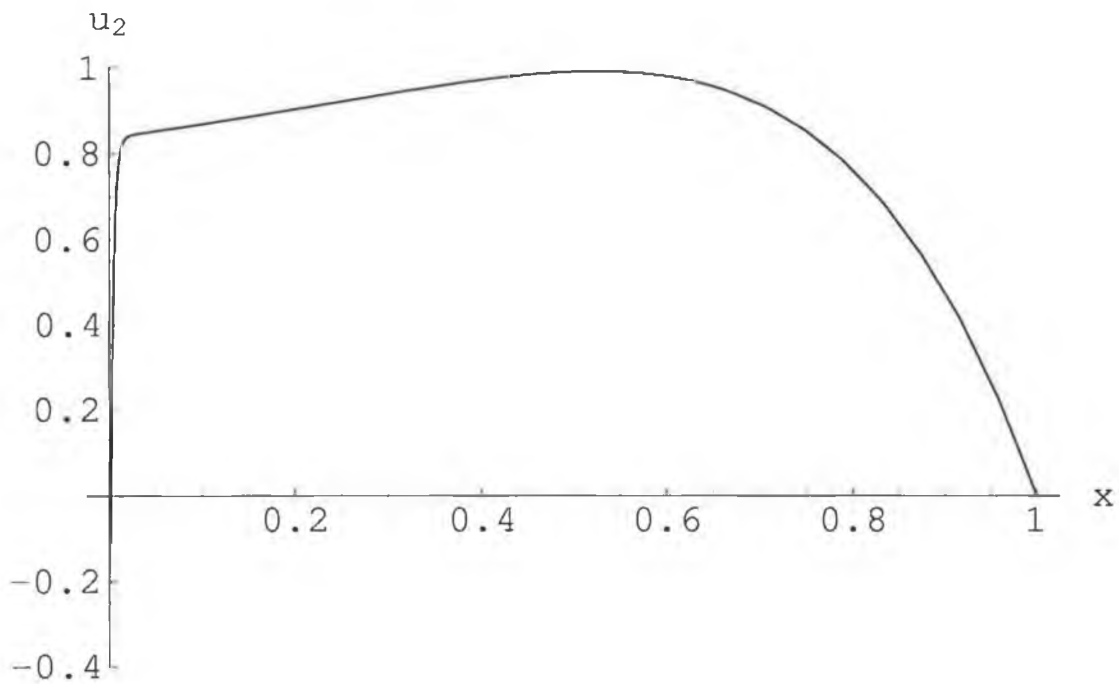

Figure 4.2: The solution $u_2(x)$ of (3.1b) when $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.0001$.

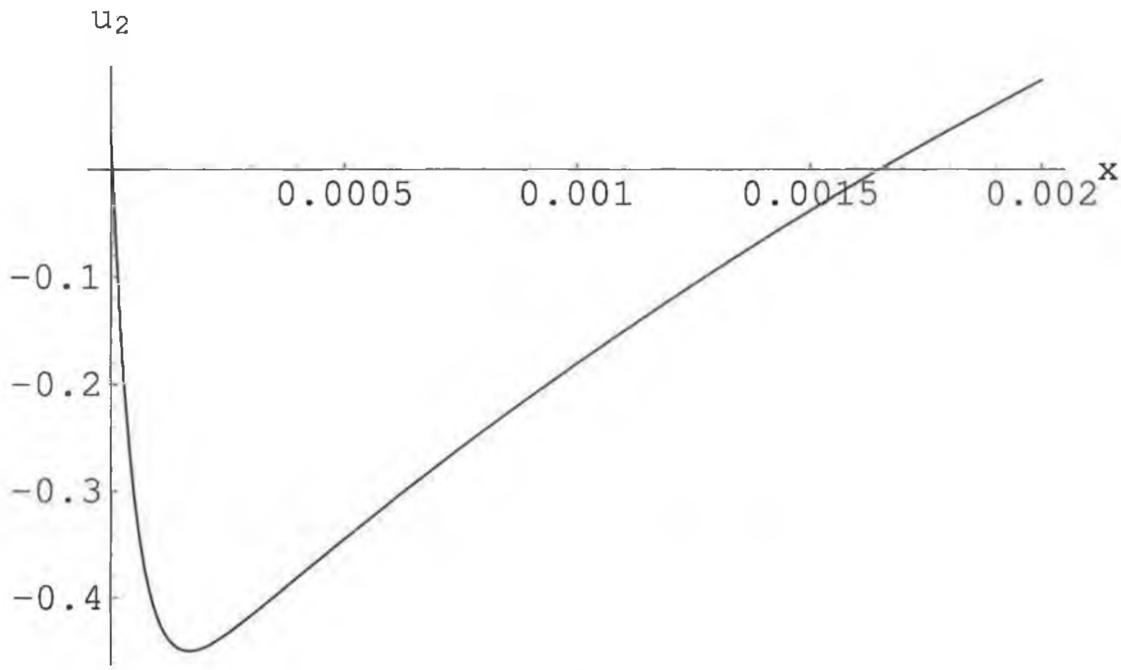

Figure 4.3: The "double–layer" effect, the solution $u_2(x)$ of (3.1b) when $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.0001$ over the interval [0, 0.002).

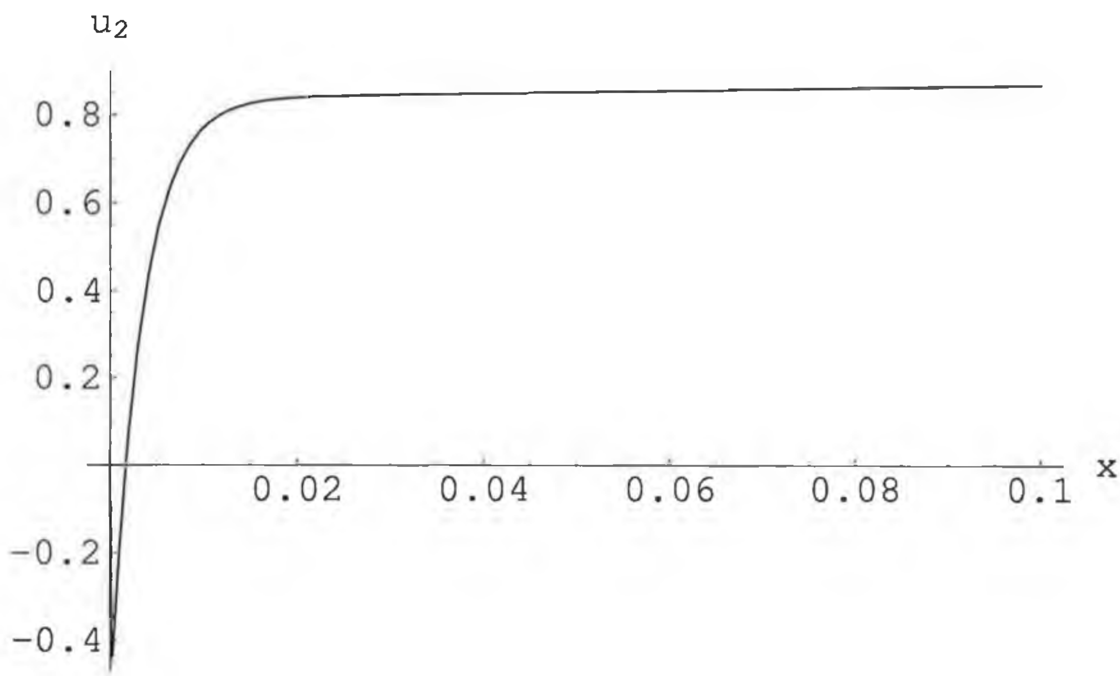

Figure 4.4: The "double–layer" effect, the solution $u_2(x)$ of (3.1b) when $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.0001$ over the interval [0, 0.1).

4.2 The components of u_2

Recall the decomposition of u_2 into the regular component v_2, the singular component w_2 and the coupling component z. The regular component and the singular component resemble those of u_1 as discussed in [2]. The case when $\frac{\varepsilon_1}{\alpha_1} \leq \frac{\varepsilon_2}{\alpha_2}$ is considered separately from that when $\frac{\varepsilon_2}{\alpha_2} \leq \frac{\varepsilon_1}{\alpha_1}$.

The components of u_2 when $\frac{\varepsilon_2}{\alpha_2} \leq \frac{\varepsilon_1}{\alpha_1}$

The solution u_2 of (3.1) has already been illustrated in Figures 4.2–4.4 where $\varepsilon_1 = 0.01$ and $\varepsilon_2 = 0.0001$. The decomposition of u_2 into its three components w_2, z and v_2 for (3.1) are illustrated in Figures 4.5, 4.6 and 4.7 respectively over the above values of ε_1 and ε_2.

The "double layer" effect is due to both the second singular component and the coupling component. The "first layer" or steep gradient is due to the singular component w_2 which is of order ε_2 and this is illustrated in figure 4.5.

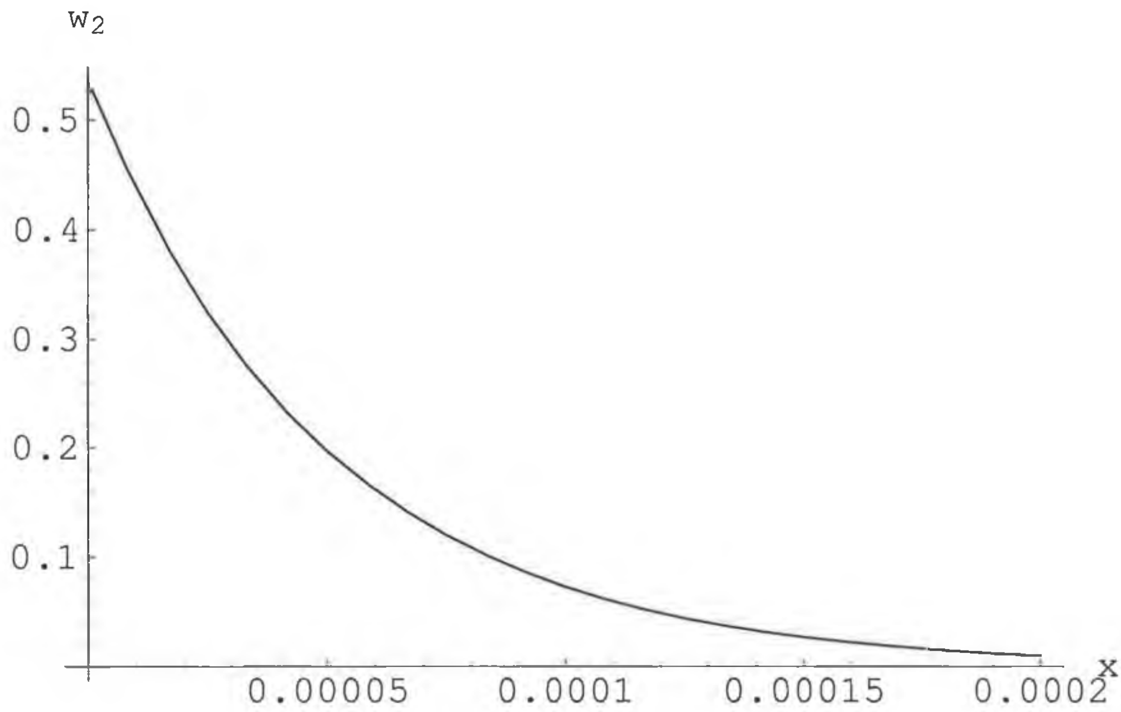

Figure 4.5: $w_2(x)$, the singular component of $u_2(x)$,the solution from (3.1b) when $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.0001$ over the interval [0, 0.0002).

The "second layer" or steep gradient is due to the coupling component z which is of order ε_1 and this is illustrated in figure 4.6. In the constant coefficient problem (3.1) the coupling component resembles the first singular component. This is borne out by the fact that for problem (3.1) $z(x) = Cw_1(x) + \varepsilon_2^2 z_2(x)$ where C is independent of ε_1 and ε_2 and the second term is small when ε_2 is small. In problem (3.1) the coupling component $z(x)$ also depends on the coefficient of $u_1'(x)$ in (3.1b). The regular component $v_2(x)$ is shown in figure 4.7.

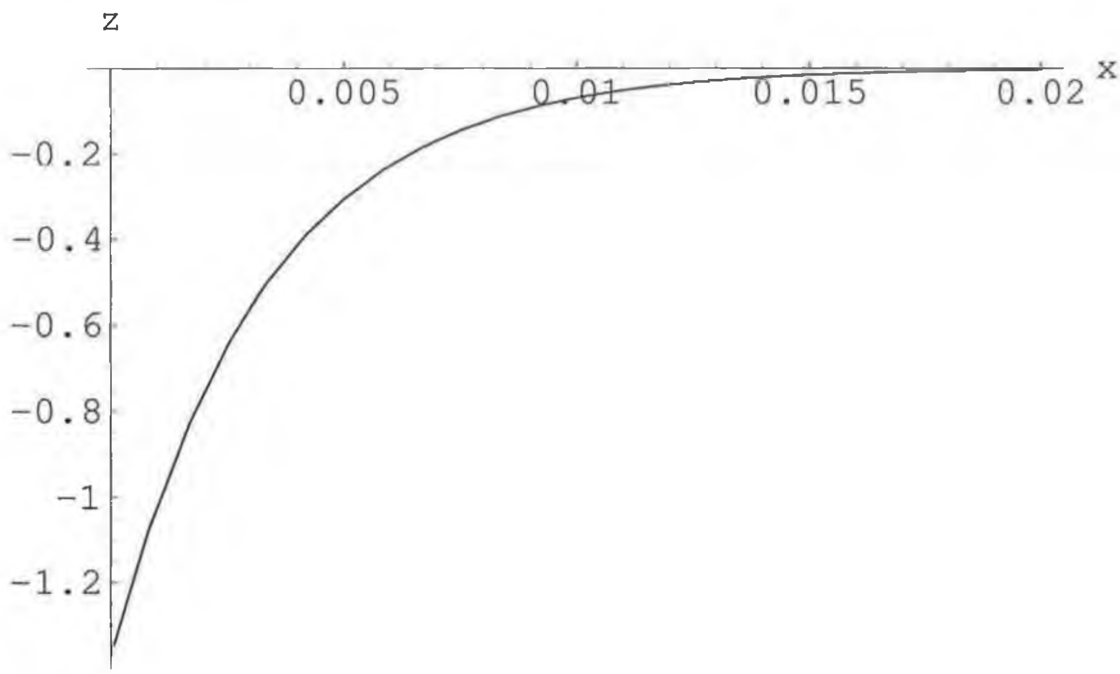

Figure 4.6: $z(x)$, the coupling component of $u_2(x)$, the solution from (3.1b), $\varepsilon_1 = 0.01$ when $\varepsilon_2 = 0.0001$ over the interval [0, 0.02).

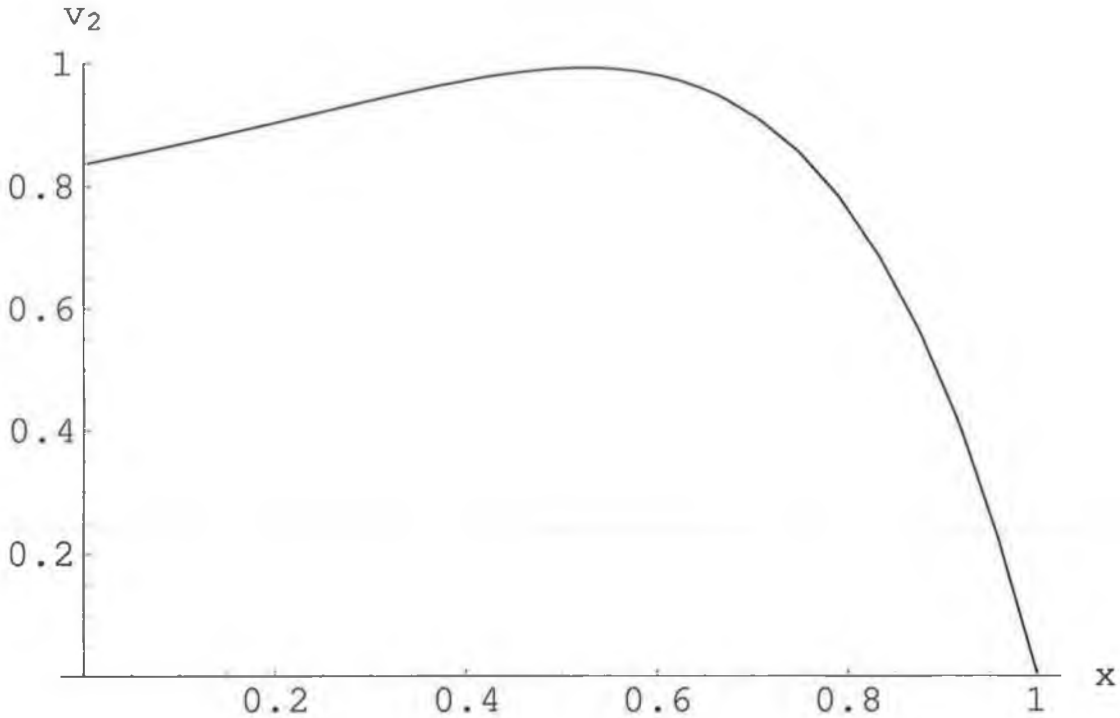

Figure 4.7: $v_2(x)$, the smooth component of u_2, the solution from (3.1b) when $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.0001$.

The components of u_2 when $\frac{\varepsilon_2}{\alpha_2} \leq \frac{\varepsilon_1}{\alpha_1}$

When $\frac{\varepsilon_1}{\alpha_1} \leq \frac{\varepsilon_2}{\alpha_2}$ the decomposition of $u_2(x)$ is carried out in a similar manner to the previous case except for the coupling component. The second singular component $w_2(x)$ and the second regular component $v_2(x)$ are similar to those illustrated for the previous case in figures 4.5 and 4.7 respectively.

Here coupling component $z(x)$ depends on $\frac{\varepsilon_1}{\varepsilon_2}$ and $z(0)$ and $z(1)$ are both zero. This dependence is illustrated in figures 4.8 and 4.9 from example (3.1b) where $\varepsilon_1 = 0.001$ and $\varepsilon_1 = 0.0005$ respectively and $\varepsilon_2 = 0.01$ in both cases. Note that in figures 4.8 and 4.9 the value of ε_2 is fixed and ε_1 is halved between figure 4.8 and figure 4.9 with a corresponding halving of the maximum value of z.

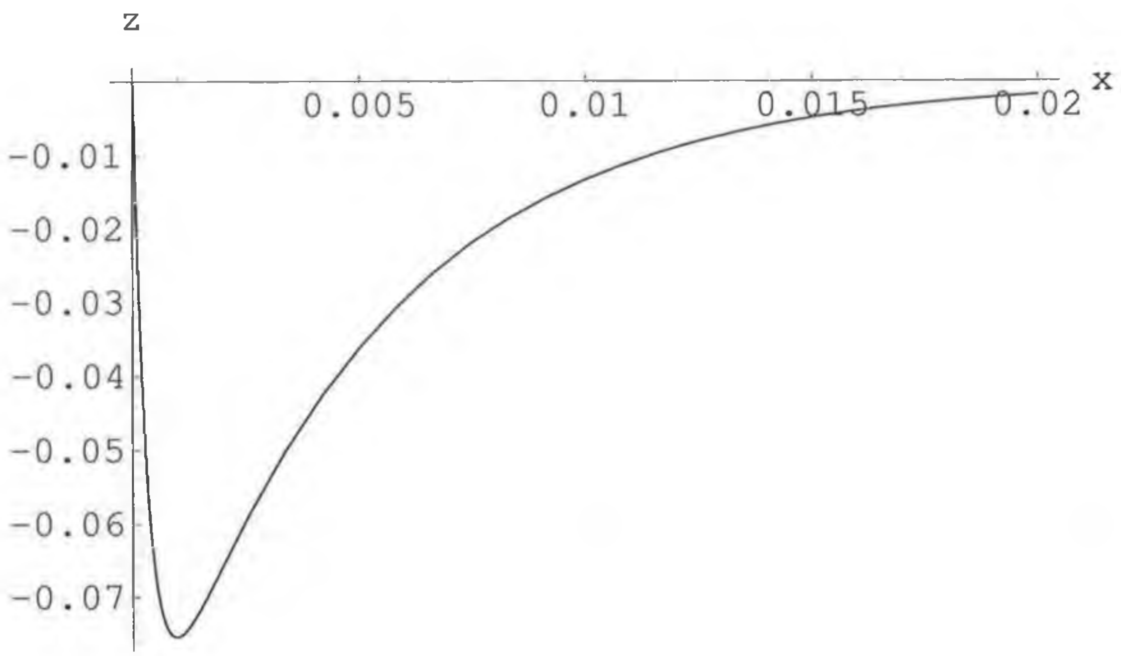

Figure 4.8: $z(x)$, the coupling component of $u_2(x)$, the solution from (3.1b) when $\varepsilon_1 = 0.001$, $\varepsilon_2 = 0.01$ over the interval $[0,\ 0.02)$.

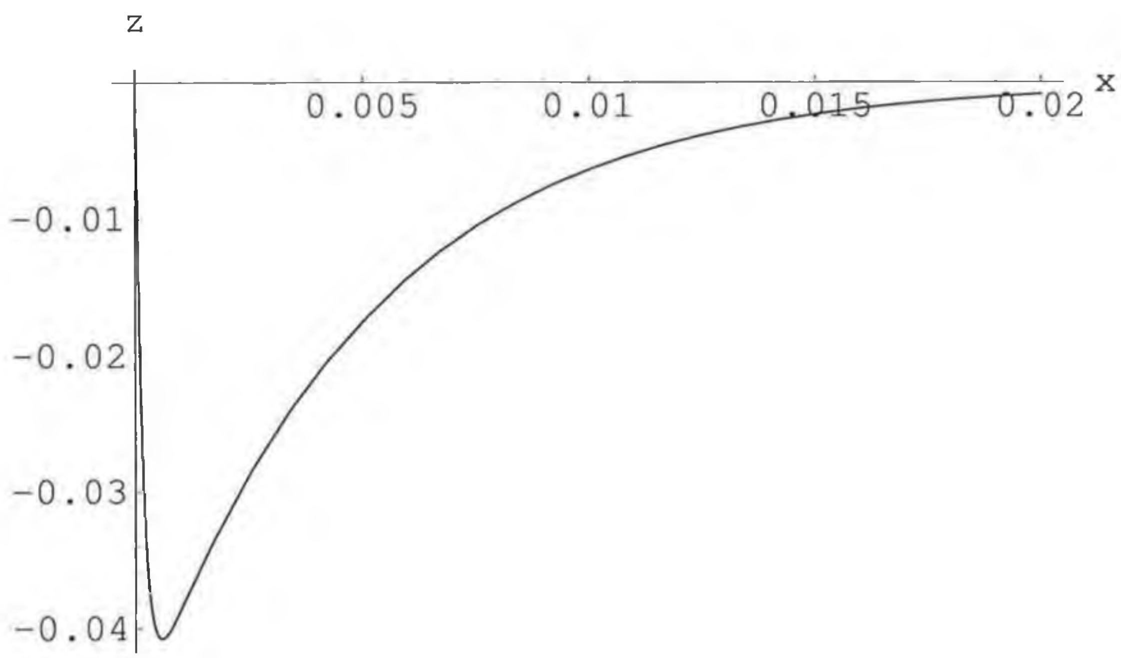

Figure 4.9: $z(x)$, the coupling component of $u_2(x)$, the solution from (3.1b) when $\varepsilon_1 = 0.0005$, $\varepsilon_2 = 0.01$ over the interval $[0,\ 0.02)$.

4.3 A counterexample for a standard comparison principle

Finally in this section we illustrate a counterexample to show that a standard comparison principle [2] does not hold in general for our system of convection–diffusion differential equations (1.1). Consider the following problem where $\tilde{L}_1$ and $\tilde{L}_2$ are defined when $x \in \bar{\Omega}$ as follows:

$$\tilde{L}_1\vec{u} \equiv 0.01u_1''(x) + 0.1u_1'(x) = -1000, \tag{4.1a}$$

$$\tilde{L}_2\vec{u} \equiv 0.01u_2''(x) + 0.1u_2'(x) - 0.08u_1'(x) = -1, \tag{4.1b}$$

where $\vec{u}$ is the vector $\{u_1(x), u_2(x)\}$ and we define $\vec{u} \geq 0$ to be $u_1(x) \geq 0$ and $u_2(x) \geq 0$. We also take the boundary conditions $\vec{u} \geq 0$ on $\{0, 1\}$ and define $\mathbf{L}\vec{u} = \{\tilde{L}_1\vec{u}, \tilde{L}_2\vec{u}\}$.

Consider the following standard minimum principle for one equation [2]:

$$\begin{aligned} \text{If } u &\geq 0 \quad \text{on} \quad \{0,1\}, \\ \text{and } Lu &\leq 0 \quad \text{on} \quad (0,1), \\ \text{then } u &\geq 0 \quad \text{on} \quad [0,1]. \end{aligned} \tag{4.2}$$

It is of interest to note that the natural extension of the standard comparison principle holds for an analogous system reaction–diffusion differential equations [4], [5], that is,

$$\begin{aligned} \text{If } \vec{u} &\geq 0 \quad \text{on} \quad \{0,1\}, \\ \text{and } \mathbf{L}\vec{u} &\leq 0 \quad \text{on} \quad (0,1), \\ \text{then } \vec{u} &\geq 0 \quad \text{on} \quad [0,1]. \end{aligned} \tag{4.3}$$

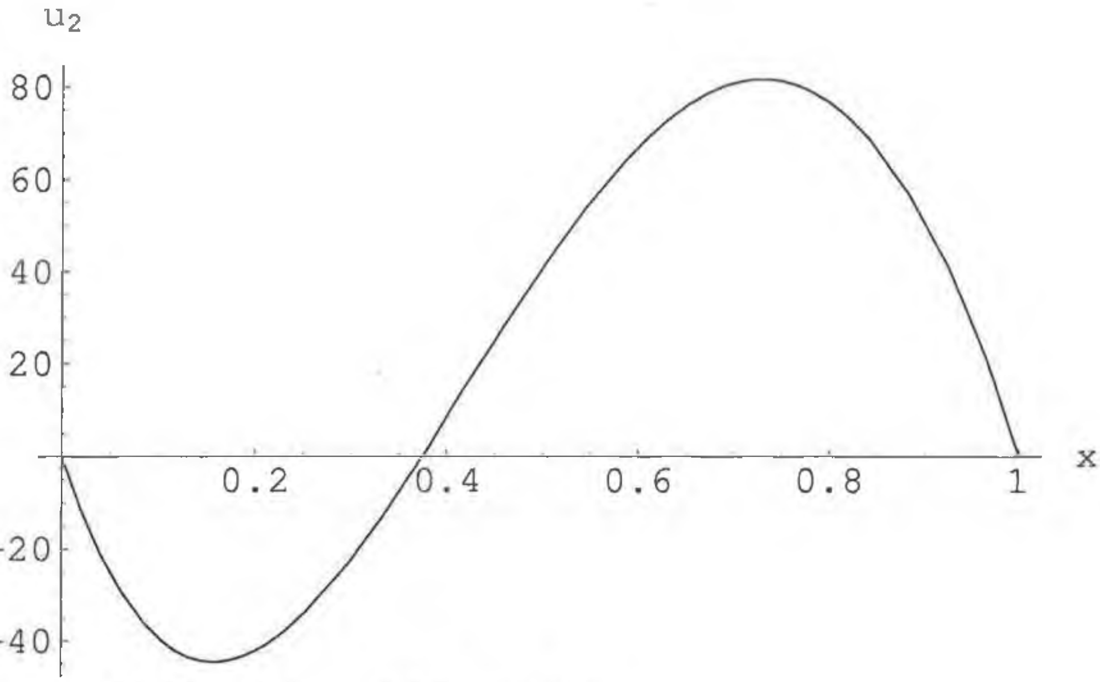

Figure 4.10: The solution $u_2(x)$ from (4.1b).

Clearly in (4.1) $\vec{u}(x) \geq 0$ when $x \in \{0, 1\}$ and $\mathbf{L}\vec{u} \leq 0$ on $(0, 1)$. The solution $u_1(x) \geq 0$ over the domain $[0, 1]$ by using the maximum principle (4.2) but the solution $u_2(x) \geq 0$ does not hold for all $x \in [0, 1]$ as illustrated in figure 4.10. Then the standard comparison principle (4.3) does not apply to the system (4.1).

Conclusion

This thesis considered a Dirichlet problem for a system of two singularly perturbed convection–diffusion ordinary differential equations. A finite difference numerical method whose solutions converged pointwise to the continuous solutions independent of the small parameters was considered.

The system where the first differential equation had only one dependent variable was considered while the second equation had two dependent variables. The solution from the first equation was decomposed into regular and singular components using the "Shishkin" decomposition [2]. The solution from the second equation was decomposed into three components. The first two components, the regular and singular components mirrored those of the first equation. The coupling component was the new feature. The solution to the second equation had a "double–layer", one layer was due to the second singular component while the other was due to the coupling component.

The finite difference numerical method was constructed over a piecewise Shishkin type mesh involving two transition points. The numerical approximations were decomposed in an analogous manner to the continuous solutions. The convergence of the numerical method was analysed separately over each component. The main aspects of the analysis involved the use of the weighted derivative results from the first components which were used in the analysis for the second regular component and for the coupling component. The use of the Andreyev–Savin [1] stability technique enabled the completion of the analysis for these components.

Appendix A

Lemma 16

Given any $k \geq 1$ and $N \geq 2$,

$$(1 + \frac{2k \ln N}{N})^{-\frac{N}{2k}} \leq \frac{2k}{N} \quad k, N > 0.$$

Proof

We start with a result from [6],

$$(1 + \frac{2 \ln N}{N})^{-\frac{N}{2}} \leq \frac{2}{N}.$$

Let $M = \frac{N}{k}$, $1 \leq k$ then

$$(1 + \frac{2k \ln N}{N})^{-\frac{N}{2k}} \leq (1 + \frac{2k \ln(\frac{N}{k})}{N})^{-\frac{N}{2k}} = (1 + \frac{2 \ln M}{M})^{-\frac{M}{2}} \leq \frac{2}{M} = \frac{2k}{N}. \qquad \blacksquare$$

Here we prove two technical lemmas which are used for the sharper weighted derivative estimates.

Lemma 17

Let $\{Y_i\}$ be the solution of the difference equation

$$Y_j - Y_{j-1} + a_j Y_j = a_j b_j, \ \forall j \tag{A1}$$

where $a_j > 0$, $b_j > 0$. Then

$$(P_i) \quad |Y_i| \leq \frac{|Y_k| \, (1 + \alpha)^{-(i-k-1)}}{1 + a_i} + \| \, b \, \|, \quad i \geq k + 1$$

where

$$a_j \geq \alpha, \quad k < j \leq i - 1; \quad \| \, b \, \| = \max_{k+1 \leq j \leq i} \{b_j\}.$$

Proof

Using (A1) we get

$$|Y_j| \leq \frac{1}{1+a_j}\,|Y_{j-1}|+\frac{a_j}{1+a_j}\,\|\,b\,\| \tag{A2}$$

which yields that (P_i) is true for $i = k+1$. We proceed by induction. Assume that (P_i) true for $i = m > k+1$. For $i = m+1$ we use (A2) and get

$$\begin{aligned}|Y_{m+1}| &\leq \frac{1}{1+a_{m+1}}\,|Y_m| + \frac{a_{m+1}}{1+a_{m+1}}\,\|\,b\,\| \\ &\leq \frac{|Y_k|\,(1+\alpha)^{-(m-k)}}{1+a_{m+1}} + \|\,b\,\|\,.\end{aligned}$$

This completes the proof. ■

Lemma 18

Let $\{Y_i\}$ be the solution of the difference equation

$$D^+Y_j - D^+Y_{j-1} + a_j(Y_{j+1} - Y_j) = b_j,\ \ \forall j \tag{A3}$$

Then (P_i)

$$D^+Y_{k-i} = D^+Y_k + a_kY_{k+1} - a_{k-i}Y_{k-i+1} - \sum_{j=k-i+1}^{k}(a_j - a_{j-1})Y_j - \sum_{j=k-i+1}^{k} b_j,$$

where $1 \leq i < k$.

Proof

Using (A3) it is easy to check that (P_i) is true for $i = 1$. We proceed by induction where we assume (P_i) is true for $i = m$. Then for $i = m+1$ we use (A3) to obtain

$$D^+Y_{k-(m+1)} = D^+Y_{k-m} + a_{k-m}Y_{k-m+1} - a_{k-m}Y_{k-m} - b_{k-m}$$

Then substituting in for D^+Y_{k-m} using (P_m) gives us the desired result (P_{m+1}). ■

Appendix B

The stability technique of Andreyev and Savin

Andreyev and Savin [1] have considered a singularly perturbed boundary value problem $Lw(x) = \varepsilon w''(x) + a(x)w'(x) = f(x)$ where $w(0) = w(1) = 0$ on $\bar{\Omega}$. They then considered its discretisation over an arbitrary mesh. They studied the properties of the discrete Green's function for the discrete operator using an upwind finite difference scheme. The uniform boundedness of the discrete Green's function yielded a stability result for the discrete problem with a stability constant independent of ε. Their upwind operator differed from the one used in this thesis thus we simply check that analogous arguments hold for the upwind operator used here.

We define an arbitrary non–uniform mesh $\bar{\Omega}^N = \{x_i : 0 = x_0 < ... < x_N = 1\}$ and $\Omega^N = \{x_i : i = 1, ..., N-1\}$. For two arbitrary mesh functions $u = \{u_i\}$ and $v = \{v_i\}$ defined on Ω^N but zero at the end points, we start by defining the discrete scalar product

$$(u, v)_1 = \sum_{i=1}^{N-1} \bar{h}_i u_i v_i, \quad u_0 = u_N = v_0 = v_N = 0.$$

The norm $\| u \|_1$ is defined to be $|(1, u)_1|$.

Construction of the adjoint problem

Given the problem

$$L^N u_i = \varepsilon \delta^2 u_i + a(x_i) D^+ u_i = f_i, \;\; 1 \le i \le N-1, \;\; u_0 = u_N = 0, \tag{B1}$$

we construct the adjoint problem $(L^N)^* v_i = f_i$ where $v_0 = v_N = 0$. Then $(v, L^N u)_1 = (u, (L^N)^* v)_1$. This is done for the upwind operator used in this thesis.

Start by defining $\zeta_i = \frac{1}{h_i} + \frac{1}{h_{i+1}}$ and $\eta_i = \frac{\bar{h}_{i-1}}{h_i \bar{h}_i} - \frac{1}{h_{i+1}}$. Then $(v, L^N u)_1 =$

$$\begin{aligned}
\sum_{i=1}^{N-1} \bar{h}_i v_i L^N u_i &= \sum_{i=1}^{N-1} [\varepsilon v_i (D^+ u_i - D^- u_i) + \frac{a(x_i)\bar{h}_i (u_{i+1} - u_i) v_i}{h_{i+1}}] \\
&= \sum_{i=1}^{N-1} [\varepsilon v_i (\frac{u_{i+1}}{h_{i+1}} - \zeta_i u_i + \frac{u_{i-1}}{h_i}) + \frac{a(x_i)\bar{h}_i (u_{i+1} - u_i) v_i}{h_{i+1}}] \\
&= \sum_{i=1}^{N-1} v_i [\varepsilon (\frac{u_{i+1}}{h_{i+1}} - \zeta_i u_i + \frac{u_{i-1}}{h_i} + \frac{\bar{h}_i a_i u_{i+1}}{h_{i+1}} - \frac{\bar{h}_i a_i u_i}{h_{i+1}})] =
\end{aligned}$$

$$\varepsilon (\sum_{i=2}^{N} \frac{v_{i-1} u_i}{h_i} - \sum_{i=1}^{N-1} \zeta_i v_i u_i + \sum_{i=0}^{N-2} \frac{v_{i+1} u_i}{h_{i+1}}) + \sum_{i=2}^{N} \frac{\bar{h}_{i-1} a_{i-1} u_i v_{i-1}}{h_i} - \sum_{i=1}^{N-1} \frac{\bar{h}_i a_i u_i v_i}{h_{i+1}}.$$

Using the initial conditions $u_0 = u_N = 0$ and $v_0 = v_N = 0$ then $(v, L^N u)_1$

$$\begin{aligned}
&= \sum_{i=1}^{N-1} [\varepsilon u_i (\frac{v_{i+1}}{h_{i+1}} - \zeta_i v_i + \frac{v_{i-1}}{h_i}) + \frac{\bar{h}_{i-1} u_i}{h_i} (a_{i-1} v_{i-1} - a_i v_i) + \sum_{i=1}^{N-1} \bar{h}_i \eta_i a_i u_i v_i] \\
&= \sum_{i=1}^{N-1} [\varepsilon u_i (D^+ v_i - D^- v_i) - \bar{h}_{i-1} u_i D^- (a_i v_i) + \bar{h}_i \eta_i a_i u_i v_i] \\
&= \sum_{i=1}^{N-1} \bar{h}_i u_i (\varepsilon \delta^2 v_i - \frac{\bar{h}_{i-1} D^- (a_i v_i)}{\bar{h}_i} + \eta_i a_i v_i).
\end{aligned}$$

Thus $(L^N)^*$ is the adjoint operator and the adjoint problem is given by

$$(L^N)^* v_i = \varepsilon \delta^2 v_i - \frac{\bar{h}_{i-1}}{\bar{h}_i} D^- (a(x_i) v_i) + \eta_i a(x_i) v_i = f_i$$

$i = 1, ..., N-1$ with $v_0 = v_N = 0$.

Definition of the discrete Green's functions

We now consider the discrete Green's function $G(x_i, \xi_k)$ of problem (B1). As a function of x_i, for a fixed ξ_k it is defined by the relations

$$\begin{aligned}
L^N G(x_i, \xi_k) &= \delta(x_i, \xi_k), \qquad x_i \in \Omega^N, \quad \xi_k \in \Omega^N, \\
G(0, \xi_k) &= G(1, \xi_k) = 0, \qquad \xi_k \in \Omega^N,
\end{aligned} \tag{B2}$$

where $\delta(x_i, \xi_k)$ is the discrete analog of the delta function

$$\delta(x_i, \xi_k) = \begin{cases} \bar{h}_i^{-1} & \text{if} \quad i = k \\ 0 & \text{otherwise} \end{cases}.$$

For the variable ξ_k with a fixed value of x_i, for a fixed ξ_k, Green's function $G(x_i, \xi_k)$ then satisfies the conditions

$$\begin{aligned} (L^N)^* G(x_i, \xi_k) &= \delta(x_i, \xi_k), \qquad x_i \in \Omega^N, \quad \xi_k \in \Omega^N, \\ G(x_i, 0) &= G(x_i, 1) = 0, \qquad x_i \in \Omega^N. \end{aligned} \tag{B3}$$

Recall that the norm $\| v \|_{\Omega^N} = \max_{1 \le j \le N-1} \{|v_j|\}$, over Ω^N an arbitrary mesh.

Theorem

Consider the problem (B1) where $a_i \geq \alpha > 0$ then $G(x_i, \xi_k)$, the Green's function as defined in (B2) is non-positive and uniformly bounded with respect to ε and

$$-\frac{2}{\alpha} \leq G(x_i, \xi_k) \leq 0.$$

The solution of (B1) satisfies

$$\| u \|_{\Omega^N} \leq \frac{2 \| L^N u \|_1}{\alpha} = \frac{2}{\alpha} \sum_{i=1}^{N-1} \bar{h}_i \left| L^N u_i \right|. \tag{B4}$$

Note if Green's function is of one sign then L^N is an M-matrix.

Proof

We start by noting that the solution of (B1) can be represented with the help of the discrete Green's function in the form

$$u_i = (G(x_i, \xi_k), L^N u_k)_1 = \sum_{k=1}^{N-1} \bar{h}_k G(x_i, \xi_k) L^N u_k. \tag{B5}$$

The proof is carried out in the following stages:
(i) Using (B2) and the discrete minimum principle then

$$G(x_i, \xi_k) \leq 0. \tag{B6}$$

(ii) Next a lower bound is established for $G(x_i, \xi_k)$ using (B3).
(iii) Once a bound for Green's function that is independent of the small parameter is established, it is easy to establish the stability result (B4) using (B5).

(ii) We show $-\frac{2}{\alpha} \leq G(x_i, \xi_k)$. Let the point $\xi_m \in \Omega^N$ be such that

$$\min_{\xi_k \in \Omega^N} G(x_i, \xi_k) = G(x_i, \xi_m), \quad x_i \in \Omega^N.$$

The lower bound of $-\frac{2}{\alpha}$ for $G(x_i, \xi_k)$ is achieved by using (B3) where

$$
\begin{aligned}
1 &\geq \sum_{k=1}^{m} \delta(x_i, \xi_k)\bar{h}_k = \sum_{k=1}^{m} (L^N)^* G(x_i, \xi_k)\bar{h}_k \\
&= \sum_{k=1}^{m} \bar{h}_k[\varepsilon\delta^2 G(x_i, \xi_k) - \frac{\bar{h}_{k-1}}{\bar{h}_k} D^-(a_k G(x_i, \xi_k)) + \eta_k a_k G(x_i, \xi_k)] \\
&= \sum_{k=1}^{m} [\varepsilon(D^+G(x_i, \xi_k) - D^-G(x_i, \xi_k)) + \frac{\bar{h}_{k-1}a_{k-1}G(x_i, \xi_{k-1})}{h_k} - \frac{\bar{h}_k a_k G(x_i, \xi_k)}{h_{k+1}}].
\end{aligned}
$$

Summing, we get a telescoping of the first two terms and also a telescoping of the third and fourth terms thus the last sum

$$
\begin{aligned}
&= \varepsilon(D^+G(x_i, \xi_m) - D^-G(x_i, \xi_1)) + \frac{\bar{h}_0 a_0 G(x_i, \xi_0)}{h_1} - \frac{\bar{h}_m a_m G(x_i, \xi_m)}{h_{m+1}} \\
&= \varepsilon(\frac{G(x_i, \xi_{m+1}) - G(x_i, \xi_m)}{h_{m+1}} - \frac{G(x_i, \xi_1)}{h_1}) - \frac{\bar{h}_m a_m G(x_i, \xi_m)}{h_{m+1}}
\end{aligned}
$$

where we take account the initial terms. By the choice of m and using (B6) then

$$
\begin{aligned}
\frac{G(x_i, \xi_{m+1}) - G(x_i, \xi_m)}{h_{m+1}} &\geq 0, \\
-\frac{G(x_i, \xi_1)}{h_1}) &\geq 0, \\
-\frac{\bar{h}_m a_m G(x_i, \xi_m)}{h_{m+1}} &\geq 0,
\end{aligned}
$$

thus

$$
0 \leq -\frac{\bar{h}_m a_m G(x_i, \xi_m)}{h_{m+1}} \leq 1.
$$

Since $\alpha < a_m$ and $\frac{1}{2} < \frac{\bar{h}_m}{h_{m+1}}$ then from $\frac{\alpha}{2} \leq \frac{\bar{h}_m a_m}{h_{m+1}}$ we obtain

$$
0 \leq -\frac{\alpha G(x_i, \xi_m)}{2} \leq -\frac{\bar{h}_m a_m G(x_i, \xi_m)}{h_{m+1}} \leq 1.
$$

Thus $-\frac{2}{\alpha} \leq G(x_i, \xi_m) \leq 0$. Combine this with (B5) to obtain $\| u \|_{\Omega^N} \leq \frac{2\|L^N u\|_1}{\alpha}$.■

The following lemma is a corollary of the theorem.

Lemma 19

Consider the operator L^N for the problem:

$$L^N y_i = \varepsilon\delta^2 y_i + a_i D^+ y_i, \quad i = 1, ..., N-1, \quad y_0 = 0,$$

where $|y_j| \le B$ some bounded value for all $1 < k \le j \le N-1$ then

$$\| y \|_{\Omega^N} \le \sum_{i=1}^{k-1} |L^N y_i| + CB.$$

Proof

Let $\bar{y}(x) = x_k y(x) - y(x_k)x$, then $\bar{y}(0) = \bar{y}(x_k) = 0$. Then we can use the stability technique [1] on $\bar{y}$ over $[0, x_k)$ where $\bar{y}_0 = \bar{y}_k = 0$ and note the following where $L^N(x_i) = -a(x_i)$:

$$\begin{aligned} y(x_i) &= \frac{\bar{y}(x_i)}{x_k} + \frac{y(x_k)x_i}{x_k}, && \text{(B7)} \\ |L^N \bar{y}| &\le x_k |L^N y| + C|y(x_k)|. && \text{(B8)} \end{aligned}$$

Then

$$\begin{aligned} \| y \|_{\Omega^N} &\le \frac{\| \bar{y} \|_{\Omega^N}}{x_k} + |y(x_k)| \\ &\le C\sum_{i=1}^{k-1} \frac{\bar{h}_i |L^N \bar{y}_i|}{x_k} + |y(x_k)| \\ &\le C\sum_{i=1}^{k-1} \bar{h}_i |L^N y_i| + Cy(x_k)\sum_{i=1}^{k-1} \frac{\bar{h}_i}{x_k} + C|y(x_k)| \\ &\le C\sum_{i=1}^{k-1} \bar{h}_i |L^N y_i| + CB, \end{aligned}$$

where we used (B7) and the fact that $\frac{x_i}{x_k} \le 1$, $(i < k)$, the theorem, (B8) and the fact that $\sum \frac{\bar{h}_i}{x_k} = 1$.

■

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