Explosions and Unbounded Growth in Nonlinear Delay Differential Equations: Numerical and Asymptotic Analysis

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Mathematics is entirely my own work, that I have exercised reasonable care to ensure the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to extent that such work has been cited and acknowledged within the text of my work.

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Contents

Abstract

This thesis investigates the asymptotic behaviour of a scalar, nonlinear differential equation with a fixed delay, and examines whether the properties of this equation can be replicated by an appropriate discretisation. We begin by considering equations for which the solution explodes in finite–time. Existing work on such explosive equations has dealt with devising numerical schemes for equations with polynomially growing instantaneous feedback, and methods to deal with delayed feedback have not been fully explored. We therefore set out a discretised scheme which replicates all the qualitative features of the continuous–time solution for a more general class of equations. Next, for non-explosive equations which exhibit extremely rapid growth, the rate of growth of the solution depends on the comparative asymptotic nonlinearities of the coefficients of the equation and the magnitude of the delay. Thus we set out conditions on these parameters which characterise the growth rate of the solution, and investigate numerical methods for recovering this rate. Using constructive comparison principles and nonlinear asymptotic analysis, we extend the numerical methods devised for explosive equations for this purpose.

Introduction and Preliminaries

0.1 Overview

This thesis considers the asymptotic behaviour of nonlinear differential equations, as well as the asymptotic behaviour of corresponding numerical approximations. A perusal of the literature on the former topic reveals it to be extensive, even though the topic has only been studied in depth since the 1950s. One of the earliest papers is due to Myskhis [42], which develops characteristic equations. In the last thirty years, the field has been particularly active; much progress has been made in particular on the asymptotic behaviour of linear equations, especially on the asymptotic stability of their equilibria. Important monographs written which are devoted to this topic include [29, 33]. Furthermore, linearisation results comparable to those available for ordinary differential equations are also available [20, 29].

Suppose that it is known that the solution of the equation converges to an equilibrium or grows without bound, but the functional on the right-hand side does not have leading order linear behaviour in the region in which the solution ultimately lies. In the case of *stability*, some results are known; precise information on asymptotic behaviour, including rates of convergence, have been obtained by Haddock and Krisztin [26, 27]. For such problems, the intrinsically nonlinear character of the leading order terms makes exponential convergence of solutions to equilibrium impossible. Therefore, the delicate and precise theory associated with the linear case cannot characterise the asymptotic behaviour. Interesting surveys of this work, and techniques to understand such highly nonlinear systems, is provided in the monograph of Lakshmikantham, Wen and Zhang [34].

It would appear that the case when solutions *grow unboundedly* is less well studied, and one aim of this thesis is to develop techniques to tackle such highly nonlinear equations. Unbounded growth in this case can take two forms, and this is best seen by considering the simple family of scalar ordinary differential equations

$$x'(t) = f(x(t)), \quad t > 0; \quad x(0) = \psi > 0, \tag{0.1.1}$$

where $f(x) = x^\beta$ for $x > 0$. If $0 < \beta < 1$, there is a unique continuous solution of (0.1.1) defined on $[0,\infty)$ which obeys

$$\lim_{t\to\infty} x(t) = +\infty, \tag{0.1.2}$$

and moreover grows at a well–defined rate according to

$$\lim_{t\to\infty} \frac{x(t)}{t^{1/(1-\beta)}} = (1-\beta)^{1/(1-\beta)}. \tag{0.1.3}$$

The case when $\beta = 1$ corresponds to the linear differential equation. However, if $\beta > 1$, the unique continuous solution of (0.1.1) is defined on a finite interval of the form $[0, T)$, where

$$\lim_{t\to T^-} x(t) = +\infty, \tag{0.1.4}$$

and T, often called the *explosion time* or *blow–up time*, depends on the initial data (in fact $T = T_\psi = \psi^{1-\beta}/(\beta-1)$). The asymptotic behaviour of the solution as it approaches the explosion time is also readily determined, and given by

$$\lim_{t\to T_\psi^-} \frac{x(t)}{((\beta-1)(T_\psi - t))^{-1/(\beta-1)}} = 1. \tag{0.1.5}$$

These two distinct types of behaviour are in this thesis termed *unbounded growth* and *explosion*. It is our goal to investigate the asymptotic behaviour of the simplest possible scalar delay differential equation extension to (0.1.1); one suitable candidate is

$$x'(t) = f(x(t)) + g(x(t-\tau)), \quad t > 0; \quad x(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{0.1.6}$$

The fixed delay here is $\tau > 0$. Since our interest is in the unbounded growth or explosion of solutions, it is natural to request that f and g be positive, so that for any initial function ψ which is positive on $[-\tau, 0]$, solutions will tend to infinity as they approach the upper endpoint of their interval of existence. We also wish in general to avoid any complications that might ensue owing to the existence of multiple solution of (0.1.6). So for this reason, we assume f to be locally Lipschitz continuous, and g and ψ for be continuous, though for the purpose of our numerical analysis we often request that g be locally Lipschitz continuous. Since these conditions on f, g and ψ are referred to throughout our work, we list them now:

$$f \in C((0,\infty);(0,\infty)) \text{ is locally Lipschitz continuous,} \tag{0.1.7}$$

$$g \in C((0,\infty);(0,\infty)) \text{ is locally Lipschitz continuous,} \tag{0.1.8}$$

$$\psi \in C([-\tau,0];(0,\infty)). \tag{0.1.9}$$

Our task can be summarised as follows: to find conditions on f, g and τ under which the solutions of (0.1.6) obey (0.1.4) or (0.1.2). In the case that the solution obeys (0.1.2), we attempt to determine the rate at which $x(t) \to \infty$ by finding a deterministic function J and a positive λ (both of which depend explicitly on f, g and τ) such that

$$\lim_{t\to\infty} \frac{J(x(t))}{t} = \lambda, \tag{0.1.10}$$

which is the natural analogue of the asymptotic result (0.1.3) for the ordinary equation. If, on the other hand (0.1.4) occurs, we attempt to determine the rate at which x explodes by finding a deterministic function M (which again can depend on f, g and τ) such that

$$\lim_{t\to T_\psi^-} \frac{M(x(t))}{T_\psi - t} = 1, \tag{0.1.11}$$

which is the natural analogue of the asymptotic result (0.1.5) for the ordinary equation.

It transpires, in the case when x obeys (0.1.4), that the function M is given by $\bar{F}$, defined by

$$\bar{F}(x) = \int_x^\infty \frac{1}{f(u)}\,du, \quad x > 0. \tag{0.1.12}$$

However, in contrast to the ordinary differential equation (0.1.1) we cannot obtain an explicit formula for the explosion time T_ψ in terms of f, g, ψ and τ. Furthermore, there may be circumstances in which we cannot compute a closed form formula for $\bar{F}$. For these reasons, it is desirable to develop reliable, versatile and tractable numerical methods which will reproduce the qualitative asymptotic features of the explosion, approximate the solution adequately on compact intervals, and approximate the explosion time with arbitrary accuracy.

Equally, one may wish to obtain information about the rate of growth of solutions in the non-exploding case characterised by (0.1.3), because the function J and normalising constant λ in (0.1.10) may not be known, or computable in closed form. In contrast to the explosive case, in which the function $M = \bar{F}$ characterises the asymptotic behaviour, it happens that the function J in (0.1.10) which characterises the rate of growth of solutions of (0.1.6) differs according to the relative sizes of $f(x)$ and $g(x)$ as $x \to \infty$, and the magnitude of τ. Therefore it is desirable, amongst other things, that our numerical methods detect the switch between various types of growth function J, provide a computable condition for detecting the switch, and estimate the rate of growth. Naturally, the method should approximate the solution satisfactorily on compact time and space domains.

We have confined our attention to scalar autonomous equations with fixed delay; applications of delay differential equations often require systems of equations or contain non-autonomous features. Our restriction, however, can be justified in a number of ways. First, a complete picture of the asymptotic behaviour of equations of the form (0.1.6) is unavailable, even in the scalar autonomous case, and in fact the rate of growth of solutions in the case when solutions obey (0.1.2) is unexplored and complicated. Second, it is difficult to assess the quality of a general numerical method for a class of differential equations without first identifying some sufficiently rich subclass of problems whose properties are comprehensively understood by purely analytical means. We believe that the analytical work in this thesis generates such a subclass of equations which exemplify the properties of explosion, growth, and dependence on delay, and that the numerical methods presented demonstrate successful strategies as well as potential difficulties for the numerical analysis of more complicated, but related, real–world equations.

0.2 Relevant Literature and Inspiration for the Work

As the outline above indicates, the analysis in this thesis is concerned with the asymptotic behaviour of delay differential equations and reproducing this behaviour for appropriate numerical methods. For a linear equation of the form (0.1.6) (i.e., where f and g are linear functions), a very complete understanding of the asymptotic behaviour exists, with every solution having asymptotic behaviour described by the complex–valued solutions of a characteristic equation [21]. One interesting feature that can be observed is that the fixed delay τ can change the asymptotic behaviour. Even for more general linear differential equations with delay, the picture is very complete: for instance, for most initial functions, the asymptotic behaviour is dominated by the solution of the characteristic equation with largest real part (cf. e.g., [20, 29]). These results cover convergence of solutions to steady states and unbounded growth as well as oscillation of solutions (something we do not investigate here). Of course, linear autonomous equations cannot experience finite time explosions. There is also an extensive theory concerned with perturbations from the linear case; this can include forcing terms independent of the state, or nonlinearities. In the latter case, for autonomous equations, one can show that the exponential rate of growth or convergence is preserved from the underlying linear equation. Recent very sharp results in this direction include [25].

However, when one moves to equations for which the leading order behaviour is not described by linear functions or functionals, the literature is sparse. Indeed, results on finite–time explosion of equations with a fixed delay seem quite limited, and include Ezzinbi and Jazar [22], and Jordan [31]. Other results in which delays can prevent explosions in highly nonlinear equations include Nie and Mei [43], and Redheffer and Redlinger [44].

There is however a nice literature of finite–time explosions in solutions of Volterra integral equations; the point in common with our work is the presence of a superlinear space–dependent term, which in both cases causes the explosion. Some recent and classical works on this topic include Małolepszy and Okrasiński [35], and Bushell and Okrasiński [18]; a work which includes sharp asymptotic estimates on the explosion rate, in a manner related to (0.1.11) is Roberts and Olmstead [45]. Numerical treatment of explosions is given in Kirk [32], with foundational work on techniques appearing in Brunner [16]. A common feature between our work and that for Volterra equations is that the presence of weakly singular convolution kernels, acting on a nonlinear function, in these Volterra equations has an effect comparable to the point concentration of the instantaneous nonlinear term on the right-hand side of (0.1.6). However, in our case, the past and present on the right-hand side may be considered separately owing to the presence of a fixed delay; there is no analogous separation in the Volterra case.

The analysis of finite–time explosions in partial differential equations is another growing field of research which relates to our work. It encompasses both numerical and asymptotic analysis of explosions. Some recent papers include Acosta, Durán and Rossi [1], Brändle, Groisman and Rossi [13], Brändle, Quirós, and Rossi [14]. We also note that spurious blow–up of solutions can arise from misspecifying the discretisation, and this is reported in [24]. The notion of state–dependent discretisation, which we utilise extensively in this thesis to capture rapidly growing solutions, is of major importance in the numerical PDE theory.

Finally, there is a wealth of research on the numerical analysis of functional Volterra and delay differential equations. Important summaries of the state–of–the–art include those of Bellen and Zennaro [10], Brunner and van der Houwen [15], and Hairer and Wanner [28]. An interesting monograph which considers, amongst other things, the asymptotic behaviour of discretisations of differential equations is Stuart and Humphries [47]. We note however, that much of the work concerning the preservation of asymptotic behaviour of the underlying continuous time equation relates to asymptotic stability, and concerns very wide classes of equations in which the coefficients do not depart significantly from linearity. Moreover, the emphasis on preserving the exact asymptotic growth or decay rates of solutions is often of secondary importance to these authors.

Preserving asymptotic features in the discretisation of differential equations which include, but is not limited to boundedness, stability or asymptotic stability, is work on "dynamic consistency", developed by Mickens (see e.g., [37, 38, 39, 40, 41]).

Our work, on the other hand, deals with a narrow class of scalar equations which exhibit a wide variety of nonlinear behaviour, in which solutions grow unboundedly or explode. Indeed, our analysis focuses on recovering exactly the rates at which growth or explosion occur. It can therefore be seen that our work is motivated by the same concerns of dynamic consistency and the properties of A–stability and AN–stability in numerical analysis (cf. e.g., [10, 47]).

Some works which have the same philosophy as this thesis are Appleby, Rodkina and Schurz [5] and Appleby, Berkolaiko, and Rodkina [6], which are concerned with highly nonlinear stochastic differential equations and their continuous and discrete–time analogues. However, these papers are devoted to the study of asymptotic stability, rather than growth or explosion rates.

For this thesis, we have drawn especially on three existing techniques from continuous dynamical systems and their numerical treatment. These are (a) constructive comparison principles; (b) state–dependent meshes for ordinary (stochastic) differential equations; and

(c) continuous extensions of discrete schemes for delay–differential equations to continuous time. The properties of regularly, slowly, and rapidly varying functions are used frequently throughout.

The relevant papers for state dependent meshes for stochastic differential equations include Davila et al. [19] in which solutions explode in finite–time; analytical results about the explosion time occur in Bonder, Groisman, and Rossi [12]. Our simplest state dependent mesh and explosion proofs are inspired by [19]. The importance of such meshes in preventing overshooting of equilibria in stochastic equations is demonstrated in Appleby, Kelly, and Rodkina [8].

For nonlinear delay differential equations which are not linear to a first approximation, results on asymptotic behaviour are sparse. However, a general method which seems successful for determining asymptotic behaviour is a "constructive" comparison principle. The idea here is to construct functions which are upper and lower solutions of the dynamical system. This has been applied to determine exact rates of growth in linear and max–type deterministic equations with unbounded delay (see Appleby and Buckwar [7]) and to polynomial stochastic delay differential equations (see Appleby and Rodkina [2]). In our justification on the assumption of the positivity of the coefficients in our equation, we make use of a result from Burton [17] regarding the existence of a unique, fixed point in delay differential equations with asymptotically constant solutions.

Our proofs of convergence draw heavily on the continuous interpolation methods for stochastic differential equations presented in Mao, Stuart and Higham [30]. Moreover, these continuous–time extensions enable us to recover in a continuous–time (and not only a discrete–time) approximation the growth rates in the underlying delay differential equation (0.1.6).

As indicated above, the properties of regular variation enable us to ascertain very precise asymptotic information about both continuous and discrete–time equations. Standard references to the topic of regular variation include Feller [23] and Bingham et al. [11]. Of course, regular variation has proved to be a useful tool for the asymptotic analysis of differential equations (cf. e.g., [36]) and linear non-autonomous differential equations with delay [48] in many other situations.

Some of the work in this thesis has already appeared in abridged form in the scientific literature. [4] forms the basis of Chapter 7 while [3] comprises a significant proportion of Chapter 1.

We make a brief remark that, in addition to the combustion problems alluded to in the PDE and Volterra theory, applications of exploding solutions of differential equations can be found in the study of (random) metal fatigue (see e.g., Sobczyk and Spencer [46]).

0.3 Synopsis of the Thesis

This thesis begins by discussing the limitations of a uniform Euler method. It is shown that when f and g are *sublinear*, the solution to the delay differential equation does not explode in finite–time, and moreover the rate of unbounded growth is determined by the asymptotics of g/f and can be replicated in discrete–time with a uniform discretisation. However when f is *superlinear* (but not sufficiently nonlinear to cause an explosion) and is determining the rate of growth, a uniform method is unsuitable in that it underestimates the growth rate of the differential equation. This motivates the use of more complex methods.

Finite–time explosions of (0.1.6) are introduced in Chapter 2. We begin by stating a necessary and sufficient condition for the presence of explosion, and comment on the inability of explicit and implicit uniform methods to replicate this asymptotic behaviour in discrete–time. We then construct an explicit, state–dependent method, which is essentially the simplest discretisation which allows us to detect the explosion. This enables us to approximate the solution on any compact interval, and thus provides a method for approximating the explosion time with arbitrary accuracy. Under conditions on f being a regularly varying function, we can obtain very precise information on the continuous–time rate of explosion, which is always dependent solely on f. However for equations with coefficients that grow faster than regularly varying, our information on the explosion rate is no longer as precise, and this motivates the use of the more computational intensive method introduced in Chapter 3 which correctly determines the exact rate of explosion of the differential equation.

In Chapter 4, we apply the numerical method introduced in Chapter 2 for the purpose of replicating the growth rate of the superlinear, non-explosive equations for which Chapter 1 demonstrated that a uniform method will not suffice. We see that if g/f tends to a finite limit, this extension is relatively straightforward. The solution to the differential equation grows at a rate dependent solely on f, the numerical solution will indeed pick up this exact asymptotic behaviour, and moreover the approximation error can be controlled on any compact interval. If g/f is no longer tending to a finite limit however, things become a bit more complicated. In Chapter 5, we determine conditions on f, g and τ for the solution to grow at a rate which is identical to that of the ordinary differential equation given by (0.1.1). Examples of such equations are provided along with commentary on their construction.

However when we try to use the method introduced in Chapter 2 to replicate this growth rate in discrete–time, we encounter a problem. Estimating the growth rates in the case where g/f tends to a finite limit made use of a constructive comparison principle (cf. e.g., [2, 7]), but this strategy is no longer effective since g is growing at a more rapid rate. In Chapter 6 we "pretransform" the differential equation to an equation which we know is growing linearly and discretise this equation. However the formula for the transformed equation requires explicit forms for certain functions which are in practice very problematic to compute. It is our strategy to replace them with auxiliary functions which are obtained by applying a state–dependent discretisation to the ordinary differential equation equivalent of (0.1.6). We can then apply the inverse transform to verify that the resulting discrete–time approximation grows a rate identical to that (0.1.6).

In Chapter 7, we look at conditions for which the asymptotic behaviour of the differential is no longer determined solely by f, in that it is now the delayed component of the equation that is responsible for the rate of growth. These conditions rely on the existence of a function that obeys certain asymptotic properties relative to f and g, and while we do not attempt to prove that the existence of such a function is guaranteed, we demonstrate that for a wide range of representative examples, this function can be determined. In Section 8, we consider appropriate numerical methods for replicating these "delay-dominant" rates of growth, and see that a uniform step-size Euler method will in fact recover the essential asymptotic behaviour of the solution to (0.1.6). To this end we are able to consider examples of equations which have identical rates of growth, but for which different components of the equation are responsible for this rate; indicating that the appropriate discretisation method for replicating asymptotic behaviour is in fact independent of the rate of growth of the solution.

0.4 Preliminaries

Notations The following notations are used in this thesis:

$\mathbb{R}$: set of real numbers.

$\mathbb{R}^+$: set of non-negative real numbers.

$C((a,b);(c,d))$: set of continuous functions mapping from (a,b) onto (c,d)

$C^1((a,b);(c,d))$: set of continuously differentiable functions mapping from (a,b) onto (c,d)

$x \vee y$: the maximum value between x and y.

$x \wedge y$: the minimum value between x and y.

$g = O(f)$: $\limsup_{x\to\infty} g(x)/f(x) < \infty$.

$g = o(f)$: $\lim_{x\to\infty} g(x)/f(x) = 0$.

Definitions and Technical Issues The major relevant definitions and theorems on technical issues are given here:

Regularly varying functions (cf. e.g., [11]): A function $m : (0,\infty) \to (0,\infty)$ is *regularly varying at infinity* with index $\alpha \in \mathbb{R}$ if

$$\lim_{x\to\infty} m(\lambda x)/m(x) = \lambda^\alpha \text{ for each } \lambda > 0.$$

We write $m \in \mathrm{RV}_\infty(\alpha)$.

Some properties of functions which are regularly varying at infinity include:

- If $m \in \mathrm{RV}_\infty(\alpha)$, $1/m \in \mathrm{RV}_\infty(-\alpha)$ and if $M(x) := \int_1^x m(u)\,du$, $x > 0$, then $M \in \mathrm{RV}_\infty(\alpha+1)$ for $\alpha > -1$, and $M \in \mathrm{RV}_\infty(0)$ for $\alpha = -1$ if $M(x) \to \infty$ as $x \to \infty$.

- If $m \in \mathrm{RV}_\infty(\alpha)$, $\exists \mu$ such that $\mu(x)/m(x) \to 1$, then $x\mu'(x)/\mu(x) \to \alpha$ as $x \to \infty$.

- If $m \in \mathrm{RV}_\infty(\alpha)$, and x and y are continuous with $x(t)/y(t) \to 1$, $x(t) \to \infty$ and $y(t) \to \infty$ as $t \to \infty$, then $\lim_{t\to\infty} m(x(t))/m(y(t)) = 1$.

Similarly, a function $n : (0,\infty) \to (0,\infty)$ is *regularly varying at zero* with index $\beta \in \mathbb{R}$ if

$$\lim_{x\to 0} n(\lambda x)/n(x) = \lambda^\beta \text{ for each } \lambda > 0.$$

We write $n \in \mathrm{RV}_0(\beta)$.

Some properties of functions which are regularly varying at zero include:

- If $n \in \mathrm{RV}_0(\beta)$, $1/n \in \mathrm{RV}_0(-\beta)$ and if $N(x) := \int_x^1 n(u)\,du$, $x > 0$, then $N \in \mathrm{RV}_0(\beta+1)$ for $\beta > -1$, and $N \in \mathrm{RV}_0(0)$ for $\beta = -1$ if $N(x) \to \infty$ as $x \to 0$.

- If $n \in \mathrm{RV}_0(\beta)$, $\exists \nu$ such that $\nu(x)/n(x) \to 1$, then $x\nu'(x)/\nu(x) \to \beta$ as $x \to 0$.

- If $n \in \mathrm{RV}_0(\beta)$, and x and y are continuous with $x(t)/y(t) \to 1$, $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$, then $\lim_{t\to\infty} n(x(t))/n(y(t)) = 1$.

Banach Fixed Point Theorem (cf. e.g.,[17]): Let (X,d) be a complete metric space and a function $f : X \to X$ be a contracting operation, i.e. there exists a $\lambda \in (0,1)$ such that $d(f(x), f(y)) \leq \lambda d(x,y)$ for any $x, y \in X$. Then there exists a unique $p \in X$ such that $f(p) = p$.

Chapter 1

The Limitations of a Uniform Euler Discretisation

1.1 Introduction

In the study of numerical analysis, one of the simplest methods of constructing an approximation for a solution of a differential equation is the uniform Euler method. This method is very well understood, and a wealth of literature exists on the performance of uniform Euler methods in analysing a broad spectrum of classes of differential equation. It is both easy to construct and straightforward to implement, and moreover in the presence of fixed–time delay it greatly simplifies the analysis of the delayed component of the equation, in that the discretisation parameter can be chosen to avoid any issues arising out of referencing the past values of the numerical method at inputs in between those for which it is defined.

A well-known necessary and sufficient condition for the solution to (0.1.1) to grow unboundedly on the interval $[0,\infty)$ is given by $\int_1^\infty \frac{1}{f(u)}\,du = \infty$. We now demonstrate that the condition for the delay differntial equation (0.1.6) to grow unboundedly is given by

$$\int_1^\infty \frac{1}{f(u)}\,du = \infty, \quad \liminf_{x\to\infty} f(x) > 0. \tag{1.1.1}$$

Theorem 1.1.1. *Let f obey* (0.1.7) *and* (1.1.1)*, g obey* (0.1.8)*, and ψ obey* (0.1.9) *where $\tau > 0$. Then there is $x \in C([-\tau,\infty))$ which is the unique continuous solution of* (0.1.6) *and which moreover obeys* (0.1.2)*.*

Proof. It is evident that there is a unique continuous solution of (0.1.6) on $[-\tau, T)$ where $T \in (0,\infty]$ is such that

$$\lim_{t\to T^-} x(t) = \infty.$$

This limit is ∞ as the positivity of the initial condition together with the positivity of f and g ensure that $x'(t) > 0$ for $t \in (0,T)$. We wish to rule out the possibility that $T < +\infty$. Suppose that $T \in (0,\tau]$. Clearly, if $g_1 = \max_{s\in[-\tau,0]} g(x(s)) \geq 0$, we have

$$x'(t) \leq f(x(t)) + g_1, \quad t \in [0,T).$$

Define $f_1(x) := f(x) + g_1$ for $x \geq 0$. Then, as $x(t) \to \infty$ as $t \to T^-$, we have

$$\int_{x(0)}^{\infty} \frac{1}{f_1(x)}\,dx = \lim_{t \to T^-} \int_0^t \frac{x'(s)}{f_1(x(s))}\,ds \leq T < \infty.$$

Now (1.1.1) implies $\int_{x(0)}^{\infty} 1/f(u)\,du = \infty$ and therefore

$$\int_{x(0)}^{\infty} \left(\frac{1}{f(u)} - \frac{1}{f_1(u)} \right) du = \infty.$$

Thus

$$\int_{x(0)}^{\infty} \frac{1}{f(u)\,(f(u)+g_1)}\,du = \int_{x(0)}^{\infty} \frac{f_1(u) - f(u)}{f(u)f_1(u)}\,du = \int_{x(0)}^{\infty} \left(\frac{1}{f(u)} - \frac{1}{f_1(u)} \right) du = \infty.$$

But $\liminf_{x\to\infty} f(x) > 0$. Since $f(x) > 0$ for $x > 0$ there exists $x^* > 0$ such that $f(x) \geq c_1$ for all $x > x^*$. Now since f is continuous there exists $x_1 \in [0, x^*]$ such that

$$\inf_{x\in[0,x^*]} f(x) = \min_{x\in[0,x^*]} f(x) = f(x_1) =: c_2.$$

Therefore $f(x) \geq c_3 > 0$ for all $x > 0$. Thus $f(u)\,(f(u)+g_1) \geq c_3\,(f(u)+g_1) = c_3 f_1(u)$. So

$$\int_{x(0)}^{\infty} \frac{1}{f(u)\,(f(u)+g_1)}\,du \leq \int_{x(0)}^{\infty} \frac{1}{c_3} \cdot \frac{1}{f_1(u)}\,du < \infty,$$

which gives a contradiction. Hence $T > \tau$

Suppose now that x does not explode in $[0, n\tau]$, but does in $(n\tau, (n+1)\tau]$. This is true for $n = 1$. Clearly, if $g_n = \max_{s\in[(n-1)\tau,n\tau]} g(x(s)) \geq 0$, we have

$$x'(t) \leq f(x(t)) + g_n, \quad t \in [n\tau, T).$$

Define $f_n(x) := f(x) + g_n$ for $x \geq 0$. Then, as $x(t) \to \infty$ as $t \to T^-$, we have

$$\int_{x(n\tau)}^{\infty} \frac{1}{f_n(x)}\,dx = \lim_{t \to T^-} \int_{n\tau}^t \frac{x'(s)}{f_n(x(s))}\,ds \leq T - n\tau < \infty.$$

Now (1.1.1) implies

$$\int_{x(0)}^{\infty} \left(\frac{1}{f(u)} - \frac{1}{f_n(u)} \right) du = \infty.$$

Thus

$$\int_{x(0)}^{\infty} \frac{1}{f(u)\,(f(u)+g_n)}\,du = \int_{x(0)}^{\infty} \frac{f_n(u) - f(u)}{f(u)f_n(u)}\,du = \int_{x(0)}^{\infty} \left(\frac{1}{f(u)} - \frac{1}{f_n(u)} \right) du = \infty.$$

By the same arguments as before, $f(u)\,(f(u)+g_n) \geq c_3\,(f(u)+g_n) = c_3 f_n(u)$. So

$$\int_{x(0)}^{\infty} \frac{1}{f(u)\,(f(u)+g_n)}\,du \leq \int_{x(0)}^{\infty} \frac{1}{c_3} \cdot \frac{1}{f_n(u)}\,du < \infty,$$

which gives a contradiction. Hence $T > (n+1)\tau$. Since this is true for any $n \in \mathbb{N}$, it follows that $T = \infty$.

We have shown that (0.1.6) has interval of existence $[-\tau, \infty)$. Since $\psi(t) > 0$ for $t \in [-\tau, 0]$ and $f(x) > 0$, $g(x) > 0$ for all $x > 0$, we have that $x'(t) > 0$ for all $t > 0$. Therefore $\lim_{t\to\infty} x(t) =: L \in [\psi(0), \infty]$. Suppose that $L > 0$ is finite. Since

$$x(t) = \psi(0) + \int_0^t f(x(s))\,ds + \int_{-\tau}^{t-\tau} g(x(s))\,ds, \quad t \geq \tau,$$

by the continuity of f and g we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(x(s))\,ds = f(L), \quad \lim_{t\to\infty} \frac{1}{t}\int_{-\tau}^{t-\tau} g(x(s))\,ds = g(L).$$

Since $x(t)$ tends to the finite limit L, we get

$$0 = \lim_{t\to\infty} \frac{x(t)}{t} = \lim_{t\to\infty} \frac{\psi(0)}{t} + \frac{1}{t}\int_0^t f(x(s))\,ds + \frac{1}{t}\int_{-\tau}^{t-\tau} g(x(s))\,ds = f(L) + g(L).$$

Since f and g are positive we have $L = 0$, a contradiction. Hence $L = \infty$ and x obeys (0.1.2), as claimed. □

Throughout this thesis, we make of (1.1.1) when we wish to restrict ourselves to solutions which exhibit unbounded growth and do *not* explode in finite–time.

In this chapter we investigate the effectiveness of a uniform Euler method for the purpose of replicating the growth rate of the unique solution x of the delay differential equation given by (0.1.6) when x obeys (0.1.2). That is, we set out conditions for which the linear interpolant $\bar{x}_h$ of the solution x_n to the equation given by

$$x_{n+1}(h) = x_n(h) + hf(x_n(h)) + hg(x_{n-N}(h)), \quad n \in \mathbb{N}, \quad h := \tau/N, \tag{1.1.2a}$$

$$x_n(h) = \psi(nh) > 0, \quad n = -N, -N+1, \ldots, 0. \tag{1.1.2b}$$

exhibits the same rate of growth as the solution to (0.1.6).

For certain classes of functions, these conditions are well understood. For example if f is linear and "dominates" g (in the sense that $g/f \to 0$ as $x \to \infty$), solutions to the delay differential equation grow exponentially. Using a uniform Euler method to estimate this rate of growth leads to a familiar result. Consider the equation given by

$$x'(t) = \alpha x(t) + g(x(t-\tau)), \quad t > 0; \quad x(t) = \psi(t), \quad t \in [-\tau, 0] \tag{1.1.3}$$

where g obeys (0.1.8), ψ obeys (0.1.9) and $\lim_{x\to\infty} g(x)/f(x) = 0$. Discretising this equation gives for $h := \tau/N$ where $N \in \mathbb{N}$

$$x_{n+1}(h) = x_n(h) + h\alpha x_n(h) + hg(x_{n-N}(h)),\ n > 0; \quad x_n(h) = \psi(nh),\ n = -N, \dots, 0. \tag{1.1.4}$$

Since g and ψ are positive, we have that $x_n(h)$ is increasing for $n \geq 0$ and so

$$\lim_{n\to\infty} \frac{x_{n+1}(h)}{x_n(h)} = 1 + h\alpha.$$

Therefore

$$\lim_{t\to\infty} \frac{\log x(t)}{t} = \alpha \text{ whereas } \lim_{n\to\infty} \frac{\log x_n(h)}{nh} = \frac{\log(1+h\alpha)}{h} =: \alpha_h, \tag{1.1.5}$$

that is the growth rate of the difference equation is dependent on the discretisation parameter h. However $\alpha_h \to \alpha$ as $h \to 0$, indicating that the correct growth rate of (1.1.3) can replicated by (1.1.4) with increased computational effort.

This chapter will set out the classes of functions for which the Euler method given by (1.1.2) will replicate the correct growth rate of the solution to (0.1.6), and introduce the classes of functions for which it will not. Throughout this chapter we will assume that condition (1.1.1) is satisfied, which ensures that the solution to (0.1.6) will not explode in finite–time. We begin by setting out a discussion of the main results in Section 1.2, the proofs are deferred to Section 1.3.

The work in this chapter appears mainly in a paper [3], joint with John Appleby and Alexandra Rodkina.

1.2 Main Results on Uniform Euler Methods

Firstly, we show that if f and g are *sublinear* functions (in the sense that $f(x)/x \to 0$, $g(x)/x \to 0$ as $x \to \infty$), the solution to the continuous equation x, the solution to the Euler scheme x_n, and its continuous–time interpolant $\bar{x}_h$ all have the same growth rate. Furthermore in contrast to the linear example given by (1.1.4), this growth rate is independent of the discretisation parameter h. So for sublinear functions an Euler method is an ideal choice to replicate the exact rate of growth of the solution to (0.1.6), as it picks up the correct asymptotics at a very small computational cost.

Let $\tau > 0$ and let f obey (0.1.7) and (1.1.1), g obey (0.1.8) and ψ obey (0.1.9). Then there is a unique continuous and strictly positive x obeying (0.1.6) defined on $t \in [-\tau, \infty)$.

Theorem 1.2.1. *Suppose*

$$\lim_{x\to\infty} f(x)/x = 0, \quad \lim_{x\to\infty} g(x)/x = 0; \quad \textit{and} \tag{1.2.1}$$

$$\textit{there exists } \lambda \in [0,\infty] \textit{ such that } \lambda := \lim_{x\to\infty} g(x)/f(x). \tag{1.2.2}$$

(i) If $\lambda \in [0,\infty)$ *and* $f \in RV_\infty(\alpha)$ *(note* (1.2.1) *implies* $\alpha \leq 1$*), then*

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1 + \lambda, \tag{1.2.3}$$

where F *is defined by*

$$F(x) := \int_\xi^x \frac{1}{f(u)}\,du, \quad x > \xi > 0. \tag{1.2.4}$$

If $\alpha < 1$*, then*

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}((1+\lambda)t)} = 1. \tag{1.2.5}$$

(ii) If $\lambda = \infty$ *and* $g \in RV_\infty(\alpha)$ *(note* (1.2.1) *implies* $\alpha \leq 1$*), then*

$$\lim_{t\to\infty} \frac{G(x(t))}{t} = 1, \tag{1.2.6}$$

where G *is defined by*

$$G(x) := \int_\xi^x \frac{1}{g(u)}\,du, \quad x > \xi > 0. \tag{1.2.7}$$

If $\alpha < 1$*, then*

$$\lim_{t\to\infty} \frac{x(t)}{G^{-1}(t)} = 1. \tag{1.2.8}$$

We can see from the above result that for sublinear equations the growth rate of the solution to (0.1.6) is dependent solely whichever of the feedback functions are asymptotically dominant. Note that when $\lambda \in [0,\infty)$, the solution grows like that of the equivalent ordinary differential equation (ODE) y given by $y'(t) = (1+\lambda)f(y(t))$. Also for $\lambda = \infty$ the growth rate is that of the equivalent ODE z given by $z'(t) = g(z(t))$, independent of the magnitude of the delay τ. Theorem 1.2.1 applies when f and g are asymptotic to e.g., $\phi_1(x) = x^\alpha \log^\beta(x)$ (for $\alpha < 1$ and $\beta \in \mathbb{R}$) or to $\phi_2(x) = x\log^\beta(x)$ or $\phi_3(x) = x(\log\log x)^\beta$

(for $\beta < 0$) as $x \to \infty$. Neither f nor g need be monotone nor tend to infinity as $x \to \infty$. Note that (1.2.1) implies that (1.1.1) is satisfied, namely sublinear equations cannot generate finite–time explosions of the solution to (0.1.6).

Next we show that a uniform Euler method does indeed preserve the growth rates given by Theorem 1.2.1.

Theorem 1.2.2. *Let $N \in \mathbb{N}$, $h := \tau/N$ and $x_n(h)$ given by* (1.1.2) *be the approximation of the solution x of* (0.1.6) *at time $t = nh$.*

(i) If $\lambda \in [0, \infty)$ in (1.2.2) *and $f \in RV_\infty(\alpha)$ (note* (1.2.1) *implies $\alpha \leq 1$), then*

$$\lim_{n\to\infty} \frac{F(x_n(h))}{nh} = 1 + \lambda, \tag{1.2.9}$$

where F is defined by (1.2.4)*. If $\alpha < 1$, then*

$$\lim_{n\to\infty} \frac{x_n(h)}{F^{-1}(nh)} = 1 + \lambda. \tag{1.2.10}$$

(ii) If $\lambda = \infty$ in (1.2.2) *and $g \in RV_\infty(\alpha)$ (note* (1.2.1) *implies $\alpha \leq 1$), then*

$$\lim_{n\to\infty} \frac{G(x_n(h))}{nh} = 1, \tag{1.2.11}$$

where G is defined by (1.2.7)*. If $\alpha < 1$, then*

$$\lim_{n\to\infty} \frac{x_n(h)}{G^{-1}(nh)} = 1. \tag{1.2.12}$$

Now consider the linear interpolant of the discrete–time equation given by (1.1.2). Define $\bar{x}_h \in C([-\tau, \infty), (0, \infty))$ by

$$\bar{x}_h(t) = x_n(h) + (x_{n+1}(h) - x_n(h))(t - nh)/h, \quad t \in [nh, (n+1)h], \quad n \geq 0, \tag{1.2.13a}$$

$$\bar{x}_h(t) = \psi(t),\ t \in [-\tau, 0]. \tag{1.2.13b}$$

So, $\bar{x}_h$ takes the value $x_n(h)$ at time nh for $n \geq 0$ and interpolates linearly between the values of $x_n(h)$ at the times $\{0, h, 2h, \ldots\}$. It is well understood that the error associated with using $\bar{x}_h$ as an approximate for x is controlled on any compact interval by the discretisation parameter h (see e.g [10]) in the sense that for any $T > 0$,

$$\lim_{h\to 0} \sup_{0\leq t\leq T} |x(t) - \bar{x}_h(t)| = 0. \tag{1.2.14}$$

We now summarise that Theorems 1.2.1 and 1.2.2 do indeed show that $\bar{x}_h$ mimics the asymptotic behaviour of x.

Theorem 1.2.3. *Let $N \in \mathbb{N}$, $h := \tau/N$, $x_n(h)$ obey* (1.1.2) *and $\bar{x}_h$ be given by* (1.2.13)*.*

(i) If $\lambda \in [0, \infty)$ in (1.2.2) *and $f \in RV_\infty(\alpha)$ (note* (1.2.1) *implies $\alpha \leq 1$), then*

$$\lim_{t\to\infty} \frac{F(\bar{x}_h(t))}{t} = 1 + \lambda, \tag{1.2.15}$$

where F is defined by (1.2.4)*. If $\alpha < 1$, then*

$$\lim_{t\to\infty} \frac{\bar{x}_h(t)}{x(t)} = 1. \tag{1.2.16}$$

(ii) If $\lambda = \infty$ in (1.2.2) *and $g \in RV_\infty(\alpha)$ (note* (1.2.1) *implies $\alpha \leq 1$), then*

$$\lim_{t\to\infty} \frac{G(\bar{x}_h(t))}{t} = 1, \tag{1.2.17}$$

where G is defined by (1.2.7)*. If $\alpha < 1$, then*

$$\lim_{t\to\infty} \frac{\bar{x}_h(t)}{x(t)} = 1. \tag{1.2.18}$$

Next for the main result of the chapter, in which we consider superlinear equations, that is equations for which f obeys $f(x)/x \to \infty$ as $x \to \infty$. For the purpose of this chapter we add that f dominates g, or that g/f is bounded. For such equations, the solution to (0.1.6) behaves asymptotically as the solution to the equivalent ODE y given by $y'(t) = f(y(t))$.

Theorem 1.2.4. *Suppose*

$$\lim_{x\to\infty} f(x)/x = \infty, f \in RV_\infty(1), \int_1^\infty 1/f(u)\,du = \infty; \quad \textit{and} \tag{1.2.19a}$$

$$\textit{there exists } \Lambda \in [0,\infty) \textit{ such that } \Lambda := \limsup_{x\to\infty} \frac{g(x)}{f(x)}, \tag{1.2.19b}$$

then

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1. \tag{1.2.20}$$

Functions obeying (1.2.19) include those asymptotic to $\phi_5(x) = x\log^\beta x$ for $\beta \in (0,1]$ (but not $\beta > 1$) or to $\phi_6(x) = x(\log\log x)^\beta$ for $\beta > 0$ (but not $\beta \leq 0$).

Under the conditions of Theorem 1.2.4, the solution $x_n(h)$ of (1.1.2) and the interpolant $\bar{x}_h$ given by (1.2.13) have different growth rates from the solution x of (0.1.6), irrespective of the mesh size $h > 0$. This is in contrast to the linear equation given by (1.1.3), in which the exact growth rate could be obtained by letting $h \to 0$. For superlinear equations where

g/f tends to a finite limit, a uniform Euler method will underestimate the growth rate of the solution to (0.1.6), suggesting that we may need special meshes to deal with such equations. This problem is revisited in Chapter 4.

Theorem 1.2.5. *Suppose the conditions of Theorem 1.2.4 hold and let $N \in \mathbb{N}$, $h := \tau/N$, $x_n(h)$ obey* (1.1.2) *and $\bar{x}_h$ be given by* (1.2.13). *Also suppose there is a function f_1 such that*

$$x \mapsto f_0(x) = \frac{f_1(x)}{x} \text{ is positive, non-decreasing on } (X_1, \infty), \ \lim_{x\to\infty} \frac{f_1(x)}{f(x)} = 1. \quad (1.2.21)$$

If H is defined by

$$H(x) = \int_{\psi(0)}^{x} 1/\{u \log(1 + f(u)/u)\}\, du, \quad x > \psi(0), \quad (1.2.22)$$

then

$$\lim_{n\to\infty} \frac{H(x_n(h))}{n} = 1, \quad \lim_{t\to\infty} \frac{H(\bar{x}_h(t))}{t} = \frac{1}{h}. \quad (1.2.23)$$

The proof of Theorem 1.2.5 is facilitated by the following Lemma

Lemma 1.2.1. *Let $k > 0$. Suppose $\phi \in C((0,\infty);(0,\infty))$, $\phi \in RV_\infty(1)$, $\phi(y)/y \to \infty$ as $y \to \infty$, $\int_1^\infty 1/\phi(u)\, du = +\infty$, and there is a function ϕ_1 with $\phi_1(y)/\phi(y) \to 1$ as $y \to \infty$ such that $\phi_0 : (y_1, \infty) \to (0,\infty) : y \mapsto \phi_0(y) := \phi_1(y)/y$ is non-decreasing. If*

$$y_{n+1}(k) = y_n(k) + k\phi(y_n(k)),\ n \geq 0; \quad y_0(k) = \xi > 0, \quad (1.2.24)$$

then

$$\lim_{n\to\infty} \frac{K(y_n(k))}{n} = 1, \quad (1.2.25)$$

where

$$K(y) = \int_{\xi}^{y} \frac{1}{u \log(1 + \phi(u)/u)}\, du, \quad y > \xi. \quad (1.2.26)$$

Note that (1.1.1) implies $H(x) \to \infty$ as $x \to \infty$. To see this, put $y = f(x)/x > 0$ in the inequality $\log(1 + y) < y$, $y > 0$. Thus $1/\{x \log(1 + f(x)/x)\} > 1/f(x)$, and integration gives $H(x) \geq F(x)$. Condition (1.1.1) implies $F(\infty-) = \infty$, proving the claim. Indeed as $H(x)/F(x) \to \infty$ as $x \to \infty$, the Euler scheme always underestimates the growth rate of the solution of (0.1.6). The second limit in (1.2.23) implies

$$\lim_{t\to\infty} \frac{F(\bar{x}_h(t))}{t} = 0.$$

irrespective of the discretisation parameter h, but x obeys

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1.$$

The second limit in (1.2.23) is derived using the method of proof of Theorem 1.2.3.

1.3 Proofs

Proof of Theorem 1.2.1 Since $x(t) > 0$ for $t \geq -\tau$ and f and g are positive, it follows that $x'(t) > 0$ for $t > 0$. This implies $x(t) \to L \in (0, \infty]$ as $t \to \infty$. Now if $L < \infty$, integrate (0.1.6) over $[0, t]$, divide by t and let $t \to \infty$ to get the contradiction $f(L) + g(L) = 0$, so

$$\lim_{t\to\infty} x(t) = \infty$$

Now since $x'(t) > 0$ for $t > 0$ we have that $x(t - \tau) < x(t)$ for $t \geq \tau$. Therefore for $t \geq \tau$

$$0 < \frac{x'(t)}{x(t)} = \frac{f(x(t))}{x(t)} + \frac{g(x(t-\tau))}{x(t)} < \frac{f(x(t))}{x(t)} + \frac{g(x(t-\tau))}{x(t-\tau)}.$$

Since f and g obey (1.2.1) and $x(t) \to \infty$ as $t \to \infty$ this implies that $x'(t)/x(t) \to 0$ as $t \to \infty$ and so

$$\lim_{t\to\infty} \frac{x(t-\tau)}{x(t)} = 1. \tag{1.3.1}$$

In case (i), since $f \in \mathrm{RV}_\infty(\alpha)$, (1.3.1) implies $f(x(t-\tau))/f(x(t)) \to 1$ as $t \to \infty$. Hence using this, (1.2.2) and diving both sides of (0.1.6) by $f(x(t))$ we get

$$\lim_{t\to\infty} \frac{x'(t)}{f(x(t))} = 1 + \lambda. \tag{1.3.2}$$

Thus for every $\varepsilon \in (0, 1)$ there is a $T_\varepsilon > 0$ such that for all $t > T_\varepsilon$ we have

$$(1 + \lambda)(t - T_\varepsilon) \leq \int_{T_\varepsilon}^{t} \frac{x'(s)}{f(x(s))}\, ds \leq (1 + \varepsilon)(1 + \lambda)(t - T_\varepsilon).$$

Using the definiton of F, dividing both sides of the equation by t, and then letting $t \to \infty$ and $\varepsilon \to 0^+$ yields (1.2.3). (1.2.5) follows since $f \in \mathrm{RV}_\infty(\alpha)$ implies $F \in \mathrm{RV}_\infty(1 - \alpha)$. As $\alpha < 1$, $F^{-1} \in \mathrm{RV}_\infty(1/(1 - \alpha))$. By (1.2.3),

$$\lim_{t\to\infty} \frac{F^{-1}(F(x(t)))}{F^{-1}((1 + \lambda)t)} = \lim_{t\to\infty} \frac{x(t)}{F^{-1}((1 + \lambda)t)} = 1.$$

In case (ii), since $g \in \mathrm{RV}_\infty(\alpha)$, (1.3.1) implies $g(x(t-\tau))/g(x(t)) \to 1$ as $t \to \infty$. Hence using this, (1.2.2) and diving both sides of (0.1.6) by $g(x(t))$ we get

$$\lim_{t\to\infty} \frac{x'(t)}{g(x(t))} = 1. \tag{1.3.3}$$

Proceeding as in (i) yields (1.2.6) and (1.2.8) follows.

Proof of Theorem 1.2.2 Firstly, define the function $a \in C((0,\infty);(0,\infty))$ by $a(x) = f(x) + g(x)$. In case (i), $a(x)/f(x) \to 1+\lambda$ as $x \to \infty$, and since $f \in \mathrm{RV}_\infty(\alpha)$, this implies $a \in \mathrm{RV}_\infty(\alpha)$. Similarly in case (ii), $a(x)/g(x) \to 1$ as $x \to \infty$, and since $g \in \mathrm{RV}_\infty(\alpha)$, this implies $a \in \mathrm{RV}_\infty(\alpha)$. So since a is a regularly varying function, there exists $b \in C^1((0,\infty);(0,\infty))$ such that

$$\lim_{x\to\infty} \frac{b(x)}{a(x)} = 1; \quad \lim_{x\to\infty} \frac{xb'(x)}{b(x)} = \alpha. \tag{1.3.4}$$

Let $B(x) := \int_{\psi(0)}^{x} 1/b(u)\,du$ for $x > \psi(0)$, so $B \in C^2((0,\infty);(0,\infty))$.

Now consider the uniform Euler scheme give by (1.1.2). Since $\psi(nh) > 0$ and f and g are positive, we have that $x_n(h) > 0$ for $n \geq -N$ and also is also increasing for $n \geq 0$. Therefore there exists $L \in (0,\infty]$ such that

$$\lim_{n\to\infty} x_n(h) = L.$$

If $L < \infty$, by taking limits across (1.1.2) we get $f(L) + g(L) = 0$. But since f and g are positive, this is a contradiction, so we must have $L = \infty$. Since $x_n(h)$ is increasing, we have that $x_n(h) > x_{n-N}(h)$ and so

$$1 < \frac{x_{n+1}(h)}{x_n(h)} \leq 1 + \frac{f(x_n(h))}{x_n(h)} + \frac{g(x_{n-N}(h)}{x_{n-N}(h)}$$

and since $x_n(h) \to \infty$ as $n \to \infty$, using (1.2.1) we obtain

$$\lim_{n\to\infty} \frac{x_{n+1}(h)}{x_n(h)} = 1. \tag{1.3.5}$$

Next define $h_n := h(f(x_n(h))/a(x_n(h)) + g(x_{n-N}(h))/a(x_n(h)))$ so that $x_{n+1}(h) = x_n(h) + h_n a(x_n(h))$ for $n \geq 0$. Then in case (i) we have

$$\lim_{n\to\infty} \frac{f(x_n(h))}{a(x_n(h))} = \frac{1}{1+\lambda}$$

since $x_n(h) \to \infty$ as $n \to \infty$. Now (1.3.5) implies $x_{n-N}(h)/x_n(h) \to 1$ as $n \to \infty$, and since $a \in \mathrm{RV}_\infty(\alpha)$ we have $a(x_{n-N}(h))/a(x_n(h)) \to 1$ as $n \to \infty$. Therefore since $x_n(h) \to \infty$ as $n \to \infty$

$$\lim_{n\to\infty} \frac{g(x_{n-N}(h))}{a(x_n(h))} = \frac{\lambda}{1+\lambda}$$

Combining these two limits we see that $h_n \to h$ as $n \to \infty$.

By Taylor's Theorem, there is an $\xi_n(h) \in [x_n(h), x_{n+1}(h)]$ such that

$$B(x_{n+1}(h)) - B(x_n(h)) = B'(x_n(h))h_n a(x_n(h)) + \frac{1}{2}B''(\xi_n(h))h_n^2 a^2(x_n(h)). \tag{1.3.6}$$

Now by (1.2.1), $a(x)/x \to 0$ as $x \to \infty$. Since

$$1 \le \frac{\xi_n(h)}{x_n(h)} \le 1 + h_n \frac{a(x_n(h))}{x_n(h)}$$

it follows that $\xi_n(h)/x_n(h) \to 1$ as $n \to \infty$. Consider the right-hand side of (1.3.6). Firstly, $B'(x_n(h))h_n a(x_n(h)) = h_n a(x_n(h))/b(x_n(h)) \to h$ as $n \to \infty$. Rewrite the second term to get

$$-\frac{1}{2}h_n^2 \frac{b'(\xi_n(h))}{b^2(\xi_n(h))} a^2(x_n(h)) = -\frac{1}{2}h_n^2 \frac{\xi_n(h)b'(\xi_n(h))}{b(\xi_n(h))} \cdot \frac{b(x_n(h))}{b(\xi_n(h))} \cdot \frac{x_n(h)}{\xi_n(h)} \cdot \frac{a(x_n(h))}{x_n(h)} \cdot \frac{a(x_n(h))}{b(x_n(h))}.$$

Since $\xi_n(h) \to \infty$ as $n \to \infty$, we have that

$$\lim_{n\to\infty} \frac{\xi_n(h)b'(\xi_n(h))}{b(\xi_n(h))} = \alpha$$

by (1.3.4). Also since $\xi_n(h)/x_n(h) \to 1$ as $n \to \infty$, this implies that $b(x_n(h))/b(\xi_n(h)) \to 1$ as $n \to \infty$. The fourth factor tends to zero by (1.2.1), and since $x_n(h) \to \infty$ the fifth factor tends to unity by (1.3.4). Since $h_n \to h$, we have

$$\lim_{n\to\infty} B''(\xi_n(h))h_n^2 a^2(x_n(h)) = 0.$$

Therefore by (1.3.6), $B(x_{n+1}(h)) - B(x_n(h)) \to h$ as $n \to \infty$ and so

$$\lim_{n\to\infty} \frac{B(x_n(h))}{nh} = 1. \tag{1.3.7}$$

Now in case (i), by L'Hôpital's rule and the first part of (1.3.4)

$$\lim_{x\to\infty} F(x)/B(x) = \lim_{x\to\infty} b(x)/f(x) = 1 + \lambda,$$

so (1.2.9) is proved. (1.2.10) follows since $f \in \mathrm{RV}_\infty(\alpha)$ implies $F \in \mathrm{RV}_\infty(1-\alpha)$. As $\alpha < 1$, $F^{-1} \in \mathrm{RV}_\infty(1/(1-\alpha))$. By (1.2.9),

$$\lim_{n\to\infty} \frac{F^{-1}(F(x_n(h)))}{F^{-1}(nh)} = \lim_{n\to\infty} \frac{x_n(h)}{F^{-1}(nh)} = 1.$$

In case (ii), (1.2.11) and (1.2.12) follow similarly as $\lim_{x\to\infty} G(x)/B(x) = 1$.

Proof of Theorem 1.2.3 This follows from the proof of Theorem 1.2.2. Note that for every $t > 0$ there is an $n(t) \in \mathbb{N}$ such that $t \in [n(t)h, (n(t)+1)h)$, so $x_{n(t)}(h) \le \bar{x}_h(t) < x_{n(t)+1}(h)$. In case (i) as F is increasing,

$$\frac{n(t)h}{t}\frac{1}{n(t)h}F(x_{n(t)}(h)) \le \frac{1}{t}F(\bar{x}_h(t)) \le \frac{(n(t)+1)h}{t}\frac{1}{(n(t)+1)h}F(x_{n(t)+1}(h)).$$

As $n(t)h/t \to 1$ as $t \to \infty$, (1.2.9) implies

$$\lim_{t\to\infty}\frac{F(\bar{x}_h(t))}{t} = 1+\lambda. \tag{1.3.8}$$

To prove that $\bar{x}_h(t)/x(t) \to 1$ as $t \to \infty$, note that $f \in \mathrm{RV}_\infty(\alpha)$ implies $F \in \mathrm{RV}_\infty(1-\alpha)$. As $\alpha < 1$, $F^{-1} \in \mathrm{RV}_\infty(1/(1-\alpha))$. By (1.3.8)

$$\lim_{t\to\infty}\frac{F^{-1}(F(\bar{x}_h(t)))}{F^{-1}(t)} = 1, \tag{1.3.9}$$

and so by (1.2.5), $\bar{x}_h(t)/x(t) \to 1$ as $t \to \infty$ follows. The proof is similar in case (ii) where $\lambda = \infty$.

Proof of Theorem 1.2.4 Firstly, condition (1.2.19) ensures that (0.1.6) has a unique, continuous solution defined on all of $[-\tau, \infty)$. Since $x(t) > 0$ for $t \ge -\tau$ and f and g are positive, it follows that $x'(t) > 0$ for $t > 0$ and so $x(t) \to \infty$ as $t \to \infty$ using the same reasoning as in the proof of Theorem 1.2.1. Thus

$$\lim_{t\to\infty}\frac{x'(t)}{x(t)} \ge \lim_{t\to\infty}\frac{f(x(t))}{x(t)} = \infty$$

by (1.2.19). Hence for every $M > 0$ there is a $T_M > 0$ such that $x'(t)/x(t) > M$ for $t > T_M$. Thus for $t > \tau + T_M$,

$$\log\left(\frac{x(t)}{x(t-\tau)}\right) = \int_{t-\tau}^{t}\frac{x'(s)}{x(s)}\,ds \ge M\tau,$$

so

$$\lim_{t\to\infty}\frac{x(t)}{x(t-\tau)} = \infty. \tag{1.3.10}$$

Hence for every $\varepsilon > 0$ there is a $T_1(\varepsilon) > 0$ such that $x(t-\tau) < \varepsilon x(t)$ for $t > T_1(\varepsilon)$. Since $f \in \mathrm{RV}_\infty(1)$, there exists a function c such that

$$\lim_{x\to\infty}\frac{c(x)}{f(x)} = 1; \quad \lim_{x\to\infty}\frac{xc'(x)}{c(x)} = 1, \tag{1.3.11}$$

with c increasing on $[x_2,\infty)$ for some $x_2>1$. Since $x(t)\to\infty$ as $t\to\infty$ there is a $T_2>0$ such that $x(t-\tau)>x_2$ for $t>T_2$. Let $T_3(\varepsilon)=\max(T_2,T_1(\varepsilon))$. Then for $t>T_3(\varepsilon)$, $c(x_2)<c(x(t-\tau))<c(\varepsilon x(t))$. So as c satisfies (1.3.11) and $f\in \mathrm{RV}_\infty(1)$, this implies $c(\varepsilon x)/c(x)\to\varepsilon$ as $x\to\infty$ and so

$$\limsup_{t\to\infty}\frac{f(x(t-\tau))}{f(x(t))}=\limsup_{t\to\infty}\frac{f(x(t-\tau))}{c(x(t-\tau))}\cdot\frac{c(x(t-\tau))}{c(\varepsilon x(t))}\cdot\frac{c(\varepsilon x(t))}{c(x(t))}\cdot\frac{c(x(t))}{f(x(t))}\le\varepsilon.$$

Thus $f(x(t-\tau))/f(x(t))\to 0$ as $t\to\infty$. Using this, (1.2.19b) and dividing both sides of (0.1.6) by $f(x(t))$ yields

$$\lim_{t\to\infty}\frac{x'(t)}{f(x(t))}=1.$$

Integration gives (1.2.20).

Proof of Lemma 1.2.1 Set $d(u):=\log(1+k\phi(e^u)/e^u)$, $u\in\mathbb{R}$. Since $y_n(k)$ given by (1.2.24) satisfies $y_n(k)>0$ for $n\ge 0$, let $u_n(k)=\log y_n(k)$ so that

$$u_{n+1}(k)=u_n(k)+d(u_n(k)),\ n\ge 0. \tag{1.3.12}$$

Since $\phi\in\mathrm{RV}_\infty(1)$, there is $\phi_2\in C^1$ such that $\phi_2(y)/\phi(y)\to 1$, $y\phi_2'(y)/\phi_2(y)\to 1$ as $y\to\infty$ and ϕ_2 is positive on (y_2,∞) for some $y_2:=e^{u_2}$. Define

$$d_2(u):=\log(1+\phi_2(e^u)/e^u),\ u>u_2$$

so $d_2\in C^1$ is positive. Also $\phi(y)/y\to\infty$, $\phi_2(y)/\phi(y)\to 1$ as $y\to\infty$ imply

$$\lim_{u\to\infty}\frac{d_2(u)}{d(u)}=1. \tag{1.3.13}$$

Since $y\phi_2'(y)/\phi_2(y)\to 1$ and $\phi_2(y)/y\to\infty$ as $y\to\infty$

$$\lim_{u\to\infty}d_2'(u)=\lim_{u\to\infty}\frac{\phi_2(e^u)/e^u}{1+\phi_2(e^u)/e^u}\cdot\left(\frac{e^u\phi_2'(e^u)}{\phi_2(e^u)}-1\right)=0.$$

Now let $d_1(u)=\log(1+\phi_1(e^u)/e^u)$, $u>u_1:=\log(y_1)$. d_1 is positive and increasing, as $y\mapsto\phi_1(y)/y$ is increasing on (y_1,∞), and as $\phi(y)/y\to\infty$ and $\phi_1(y)/\phi(y)\to 1$ as $y\to\infty$, we get $d_1(u)/d(u)\to 1$ as $u\to\infty$.

Set $K_2(u)=\int_{u_2}^u 1/d_2(v)\,dv$, $u>u_2$, so $K_2\in C^2(u_2,\infty)$, $K_2'(u)=1/d_2(u)$ and $K_2''(u)=-d_2'(u)/d_2^2(u)$. Since $u_n(k)\to\infty$ as $n\to\infty$, there is an $N_1>1$ such that $u_n(k)>\max(u_1,u_2)$, $n\ge N_1$. By Taylor's theorem, there is a $\xi_n(k)\in[u_n(k),u_{n+1}(k)]$ such that

$$K_2(u_{n+1}(k))-K_2(u_n(k))=\frac{d(u_n(k))}{d_2(u_n(k))}-\frac{1}{2}d_2'(\xi_n(k))\frac{d^2(u_n(k))}{d_1^2(u_n(k))}\cdot\frac{d_1^2(u_n(k))}{d_1^2(\xi_n(k))}\cdot\frac{d_1^2(\xi_n(k))}{d_2^2(\xi_n(k))}. \tag{1.3.14}$$

Consider the right-hand side of (1.3.14). As $d(u)/d_2(u) \to 1$ as $u \to \infty$, the first term tends to unity as $n \to \infty$. For the second term, as $\xi_n(k) \to \infty$ as $n \to \infty$ and $d_1(u)/d_2(u) \to 1$ as $u \to \infty$ we have $d_1^2(u_n(k))/d_1^2(\xi_n(k)) \to 1$ as $n \to \infty$. Also as d_1^2 is increasing on (u_1, ∞) and $\xi_n(k) \geq u_n(k) > u_1$, $d_1^2(u_n(k))/d_1^2(\xi_n(k)) \leq 1$. Next $d^2(u_n(k))/d_1^2(u_n(k)) \to 1$ as $n \to \infty$ since $d(u)/d_1(u) \to 1$ as $u \to \infty$. Finally as $\xi_n(k) \to \infty$ as $n \to \infty$, $d_2'(\xi_n(k)) \to 0$ as $n \to \infty$. Thus $K_2(u_{n+1}(k)) - K_2(u_n(k)) \to 1$ as $n \to \infty$ and so

$$\lim_{n\to\infty} \frac{K_2(u_n(k))}{n} = 1. \tag{1.3.15}$$

Now define

$$k_2(y) := \int_{y_2}^{y} 1/(w d_2(\log w))\, dw,$$

so $K_2(u_n(k)) = k_2(y_n(k))$ and define

$$d_0(u) := \log(1 + \phi(e^u)/e^u),\ u \in \mathbb{R}$$

so that $\lim_{u\to\infty} d_0(u)/d(u) = 1$. Note that $K(y)$ given by (1.2.26) satisfies $K(y) := \int_1^y 1/(w d_0(\log w))\, dw$. By L'Hôpital's rule and using $d_2(u)/d(u) \to 1$ as $u \to \infty$ we get $k_2(y)/K(y) \to 1$ as $y \to \infty$, which gives (1.2.25).

Proof of Theorem 1.2.5 Using the same reasoning as in Theorem 1.2.2, we have that $x_n(h) \to \infty$ as $n \to \infty$. Now since $x_{n+1}(h) > x_n(h) + hf(x_n(h))$ and the first part of (1.2.19) holds,

$$\lim_{n\to\infty} \frac{x_{n+1}(h)}{x_n(h)} = \infty \text{ and therefore } \lim_{n\to\infty} \frac{x_{n-N}(h)}{x_n(h)} = 0. \tag{1.3.16}$$

Again since $f \in \mathrm{RV}_\infty(1)$, there exists a function c obeying (1.3.11) with c increasing on $[x_2, \infty)$ for some $x_2 > 1$. So there is a $N_2 := N_2(h)$ such that $x_{n-N}(h) > x_2$ for $n > N_2$ and by (1.3.16) for every $\varepsilon > 0$ an $N_3 := N_3(\varepsilon; h)$ such that $x_{n-N}(h) < \varepsilon x_n(h)$ for $n \geq N_3$. Thus for $n \geq N_4 := \max(N_2, N_3)$ we have $c(x_{n-N}(h)) < c(\varepsilon x_n(h))$, and by (1.3.11) and since $f \in \mathrm{RV}_\infty(1)$

$$\limsup_{n\to\infty} \frac{f(x_{n-N}(h))}{f(x_n(h))} = \limsup_{n\to\infty} \frac{f(x_{n-N}(h))}{c(x_{n-N}(h))} \cdot \frac{c(x_{n-N}(h))}{c(\varepsilon x_n(h))} \cdot \frac{c(\varepsilon x_n(h)}{c(x_n(h))} \cdot \frac{c(x_n(h))}{f(x_n(h))} \leq \varepsilon. \tag{1.3.17}$$

Thus $f(x_{n-N}(h))/f(x_n(h)) \to 0$ as $n \to \infty$, and by (1.2.19b), $g(x_{n-N}(h))/f(x_n(h)) \to 0$ as $n \to \infty$. Set

$$h_n := \frac{hf(x_n(h)) + hg(x_{n-N}(h))}{c(x_n(h))}.$$

Then $x_{n+1}(h) = x_n(h) + h_n c(x_n(h))$ and $h_n \to h$ as $n \to \infty$. Hence there is a $N_5 > 0$ such that for $n \geq N_5$, $h/2 < h_n < 2h$ and $x_n(h) > 4x_2$. So for $n \geq N_5$,

$$x_{n+1}(h) > x_n(h) + \frac{h}{2}c(x_n(h)) \text{ and } x_{n+1}(h) < x_n(h) + 2hc(x_n(h)).$$

Now define two sequences $y_n^-(h)$ and $y_n^+(h)$ by

$$y_{n+1}^-(h) = y_n^-(h) + \frac{h}{2}c(y_n^-(h));\ n > N_5; \quad y_{N_5}^-(h) = \frac{x_{N_5}(h)}{2}$$
$$y_{n+1}^+(h) = y_n^+(h) + 2hc(y_n^+(h));\ n > N_5; \quad y_{N_5}^+(h) = 2x_{N_5}(h).$$

As c is increasing on $[x_2, \infty)$, $x_2 < y_n^-(h) < x_n(h) < y_n^+(h)$ for $n > N_5$ and as K given by (1.2.26) is increasing on $(1, \infty)$, $K(y_n^-(h)) < K(x_n(h)) < K(y_n^+(h))$ for $n > N_5$.

Now since $f(x)/c(x) \to 1$ as $x \to \infty$ we have by (1.2.19) and (1.2.21) that $c \in \mathrm{RV}_\infty(1)$, $c(y)/y \to \infty$ as $y \to \infty$, $c_1 := f_1$ is such that $c_1(y)/c(y) \to 1$ as $y \to \infty$ and $y \mapsto c_0(y) := c_1(y)/y$ is non-decreasing on (X_1, ∞). Hence by applying Lemma 1.2.1 to the sequences $y_n^-(h)$ and $y_n^+(h)$ we get

$$\lim_{n\to\infty} \frac{K(y_n^-(h))}{n} = 1 \text{ and } \lim_{n\to\infty} \frac{K(y_n^+(h))}{n} = 1$$

and so

$$\lim_{n\to\infty} \frac{K(x_n(h))}{n} = 1. \tag{1.3.18}$$

Finally as $f(x)/c(x) \to 1$ and $c(x)/x \to \infty$ as $x \to \infty$, L'Hôpital's rule gives $H(x)/K(x) \to 1$ as $x \to \infty$, so the first part of (1.2.23) holds.

To prove the second part of (1.2.23), note that for every $t > 0$ there is an $n(t) \in \mathbb{N}$ such that $t \in [n(t)h, (n(t)+1)h)$, so $x_{n(t)}(h) \leq \bar{x}_h(t) < x_{n(t)+1}(h)$. As H is increasing,

$$\frac{n(t)h}{t}\frac{1}{n(t)h}H(x_{n(t)}(h)) \leq \frac{1}{t}H(\bar{x}_h(t)) \leq \frac{(n(t)+1)h}{t}\frac{1}{(n(t)+1)h}H(x_{n(t)+1}(h)).$$

As $n(t)h/t \to 1$ as $t \to \infty$, the first part of (1.2.23) implies

$$\lim_{t\to\infty} \frac{H(\bar{x}_h(t))}{t} = \frac{1}{h}. \tag{1.3.19}$$

Chapter 2

Explosions

2.1 Introduction

As we have seen in Theorem 1.2.4, nonlinear differential equations can grow at very rapid rates. For example solutions to equations which obey the criteria of Theorem 1.2.4 include those which grow at rates asymptotic to iterated exponentials, that is

$$\lim_{t\to\infty} \frac{x(t)}{\exp_n(t)} = 1$$

where $\exp_n(x) := \exp(\exp(\exp \ldots \exp(x)))$ is the n-th fold composition of exponential functions. Moreover, it was demonstrated that under the conditions of Theorem 1.2.5, a uniform Euler discretisation would not replicate these rapid rates of growth.

However, for certain classes of equations the solution may "explode" in finite–time, that is the feedback is sufficiently nonlinear to guarantee that there exists some $T > 0$ such that x is continuous on $[-\tau, T)$,

$$\lim_{t\to T^-} x(t) = \infty,$$

and T is referred to as the "explosion time" or "blow-up time" of the solution. Equations which exhibit explosive rates of growth include hyperbolic equations like $x(t) = 1/(1-t)$ for $t \in [-\tau, 1)$. Such equations are frequently encountered in epidemiology, population dynamics and in the study of fracture mechanics. The characteristics of these explosions have been studied for equations which do not involve delay [1, 13, 14]. It is therefore natural to examine the impact of nonlinear, fixed–time delayed feedback on the presence of these explosions, the time at which they occur and the rate at which the solution explodes. In this chapter, we study the limiting behaviour of explosive equations which satisfy (0.1.6), analysing the role of the feedback functions f and g, the initial function ψ and the delay τ on the asymptotics of the finite–time explosion.

The uniform Euler method described by (1.1.2) will give incorrect information about the presence of a finite–time explosion in the solution to the delay differential equation. Indeed it will suggest that the solution $t \mapsto x(t)$ is defined for all $t \geq 0$, in contrast to (0.1.4) which illustrated that the solution will have finite interval of existence. Furthermore, an implicit method will not work for explosive equations. Consider the implicit equation

$$x_{n+1} = x_n + hf(x_{n+1}) + hg(x_{n-N}), \quad n \geq 0, \text{ for } h := \tau/N \text{ where } N \in \mathbb{N}, \tag{2.1.1}$$

where the underlying continuous–time equation obeys (0.1.4). If a solution exists, it must be increasing. So it either tends to a limit or tends to infinity as $n \to \infty$. Suppose it has a limit $L \in (0, \infty)$. This implies

$$L = L + hf(L) + hg(L) > L$$

since f is positive. Thus we must have $x_n \to \infty$ as $n \to \infty$. Next, it follows that since f obeys (2.2.1), $f(x)/x \to \infty$ as $x \to \infty$, and there is an $x^* = x^*(h) > 0$ such that for all $x > x^*$ we have $f(x)/x > 2/h$. Now since $x_n \to \infty$ as $n \to \infty$ there exists an $N_1(h) > 0$ such that $x_n > x^*(h)$ for all $n \geq N_1(h)$. By hypothesis there is an $x_{N_1(h)+1} > 0$ which obeys

$$x_{N_1(h)+1} = x_{N_1(h)} + hf(x_{N_1(h)+1}) + hg(x_{N_1(h)-N}).$$

However

$$\begin{aligned} x_{N_1(h)+1} &= x_{N_1(h)} + hf(x_{N_1(h)+1}) + hg(x_{N_1(h)-N}) \\ &> x_{N_1(h)} + hf(x_{N_1(h)+1}) \\ &> x_{N_1(h)} + 2x_{N_1(h)+1} > x_{N_1(h)+1}, \end{aligned}$$

a contradiction. Thus if f obeys (2.2.1) we cannot construct a solution to (2.1.1) and an implicit method is an unreliable guide to the presence of explosions. This suggests that alternative numerical methods must be used. Furthermore, the unsuitability of the implicit method described above also applies to non-explosive equations covered by Theorem 1.2.4 for which an explicit uniform Euler method underestimates the rate of growth of the solution in accordance with Theorem 1.2.5.

Existing work has been done on devising numerical schemes for explosive equations, specifically equations with polynomially growing instantaneous feedback (cf. e.g., [1, 13, 14]), but equations involving delay have not been fully explored. Here we describe a state–dependent discretisation which inherits the appropriate explosion asymptotics for a wider range of possible feedback functions and involves the presence of delay. Much consideration has been given to make the scheme as versatile and extendable as possible, as we will use the techniques developed in this chapter to replicate the growth rates of the non-explosive, superlinear delay differential equations introduced in Theorem 1.2.4.

In Section 2.2, we state the condition for a finite–time explosion of the solution. Comments on hypotheses are included, namely the positivity of g and it's contribution to the presence of an explosion. The state–dependent numerical scheme is described in Section 2.3 along with some remarks on its construction, and the existence of an explosion of this approximation is demonstrated in Section 2.4. Section 2.5 shows that the error associated with using the approximation can be controlled and that the continuous–time explosion can be approximated with arbitrary precision by the by discrete–time explosion. In section 2.6, we determine the rates of explosion of the solutions to both the continuous and discrete–time equations, and includes some representative examples. Section 2.7 considers what happens in the absence of any monotonicity assumptions on f, where the limitations of the numerical method described in Section 2.3 are demonstrated through constructing pathological examples and considering alternative discretisations. We use the arguments of Section 2.7 to justify that we must assume some sort of monotonicity assumption on f. Finally, certain proofs are deferred to Section 2.8.

2.2 Existence of Explosion

A well-known necessary and sufficient condition for the presence of a finite–time explosion of the solution to (0.1.1) is given by

$$\int_1^\infty \frac{1}{f(u)}\,du < \infty. \tag{2.2.1}$$

It is straightforward to show that this condition also holds for the delay differential equation given by (0.1.6).

Theorem 2.2.1. *Let f obey* (0.1.7) *and* (2.2.1)*, g obey* (0.1.8) *and ψ obey* (0.1.9)*. Then there exists a finite $T > 0$ and $x \in C([-\tau, T); (0, \infty))$ such that x is increasing on $[0, T)$,*

x is the unique continuous solution of (0.1.6)*, and*

$$\lim_{t\to T^-} x(t) = \infty. \tag{2.2.2}$$

Proof. It is evident that there is a unique continuous solution of (0.1.6) on $[-\tau, T)$ where $T \in (0,\infty]$ is such that

$$\lim_{t\to T^-} x(t) = \infty.$$

This limit is ∞ as the positivity of the initial condition together with the positivity of f and g ensure that $x'(t) > 0$ for $t \in (0,T)$. We wish to rule out the possibility that $T = +\infty$. Suppose that it is the case that $T = +\infty$. Clearly since g is positive,

$$x'(t) > f(x(t)), \quad t > 0.$$

Therefore as $\lim_{t\to T^-} x(t) = +\infty$,

$$\int_{\psi(0)}^{\infty} \frac{1}{f(u)}\,du \geq T = \infty, \quad t > 0.$$

But f obeys (2.2.1), which gives a contradiction. Therefore T is finite and moreover the positivity of f and g ensure that $x(t) \to \infty$ as $t \to T^-$. □

The number T is called the *explosion time* of x. If condition (2.2.1) does not hold, but (0.1.8) and (0.1.9) hold, then x cannot explode in finite–time (cf. Theorem 1.1.1). Therefore (2.2.1) is a necessary and sufficient condition for an explosion of (0.1.6). Throughout this thesis, we make of (2.2.1) when we wish to restrict ourselves to solutions which explode and therefore and do *not* exhibit unbounded growth.

Suppose also that

$$\exists \phi \in C((0,\infty);(0,\infty)) \text{ such that } \phi \text{ is nondecreasing and } \lim_{x\to\infty} \frac{f(x)}{\phi(x)} = 1. \tag{2.2.3}$$

This assumption, which is essential to the reliability of our numerical method, is discussed in the following section.

2.2.1 Discussion on hypotheses

Since (2.2.1) implies $\limsup_{x\to\infty} f(x) \to \infty$, it is perhaps natural to assume the monotonicity of f. However we need only assume the existence of a positive, monotone function

ϕ which obeys (2.2.3) (i.e. the *asymptotic monotonicity* of f) for our method to correctly predict the existence or nonexistence of explosions of the unique solution of (0.1.6). In Section 2.6, where we determine the growth rate of our state–dependent numerical scheme, we make use of the property of regular variation of a function. Note that if f is indeed regularly varying, such a positive, monotone, asymptotically equivalent function ϕ is guaranteed to exist. However in the absence of any sort of monotonicity our method will be unreliable, as we will demonstrate in Section 2.7. Without monotonicity, we can construct a family of pathological examples for which the method either fails to detect an explosion of the continuous equation or incorrectly diagnoses the presence of an explosion when the solution does not explode.

Throughout the remainder of this section, where we will justify the positivity assumption on g, we assume f to be monotone to make the analysis more convenient. It is worth noting that g need only be continuous, but assuming local Lipschitz continuity is useful for control of the error estimates. The positivity of g cannot be relaxed if a finite–time explosion of (0.1.6) is to be guaranteed. If the positivity does not hold, we can obtain different asymptotic results for x depending on the initial condition ψ. To see this, we will first consider the equation

$$x'(t) = f(x(t)) - f(x(t-\tau)), \quad t > 0, \tag{2.2.4a}$$

$$x(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{2.2.4b}$$

We show that we can have an explosion or solutions tending to a finite limit as $t \to \infty$ of (2.2.4) according as to whether the initial function ψ is small or large. This shows that an explosion can be suppressed if there is a *negative* nonlinear delayed feedback term, the initial condition is sufficiently small, and the nonlinear function f is superlinear local to zero.

Theorem 2.2.2. *Suppose that ψ in $C([-\tau,0];(0,\infty))$ is increasing and f is monotonically increasing and suppose that f obeys*

$$f \in C^1([0,\infty),(0,\infty)), \quad f'(0) = 0, \quad \int_1^\infty 1/f(u)\,du < +\infty. \tag{2.2.5}$$

(i) Suppose that $\delta > 0$ is such that $0 < \psi(t) \le \delta/2$ for $t \in [-\tau, -\tau/2]$. If $\psi(0) > \delta/2$

is sufficiently large, then there exists $T_\psi \in (0, \tau/2)$ *such that the unique solution of* (2.2.4) *obeys* $\lim_{t\to T_\psi^-} x(t) = \infty$.

(ii) There exists $\delta > 0$ *sufficiently small such that if* $0 < \psi(t) \leq \delta/2$ *for all* $t \in [-\tau, 0]$ *then there is a finite* $L \in (\psi(0), \delta)$ *such that*

$$L = \psi(0) - \int_{-\tau}^{0} f(\psi(s))\, ds + \tau f(L), \tag{2.2.6}$$

and the unique solution of (2.2.4) *obeys* $\lim_{t\to\infty} x(t) = L$.

The next result shows that we can have *any* negative nonlinear delayed feedback and any delay (however short) and still consider initial functions which guarantee that the solution of the delayed equation explodes in finite time.

Theorem 2.2.3. *Suppose that* ψ *in* $C([-\tau, 0]; (0, \infty))$, $f \in C^1((0, \infty), (0, \infty))$ *is increasing and suppose that* f *obeys* (2.2.1). *Suppose* $g \in C((0, \infty), (0, \infty))$. *Let* x *be the unique solution of*

$$x'(t) = f(x(t)) - g(x(t - \tau)), \quad t > 0, \tag{2.2.7a}$$

$$x(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{2.2.7b}$$

If $\psi(0)$ *is sufficiently large, then there exists* $T_\psi > 0$ *such that the unique solution of* (2.2.7) *obeys*

$$\lim_{t\to T_\psi^-} x(t) = \infty \tag{2.2.8}$$

Next we show that the explosion can always be contained, provided the initial condition ψ is decreasing and $f(0) = 0$. In this case, we choose a controlling nonlinear function g which always exceeds f. We extend f and g to be zero on $(-\infty, 0]$.

Theorem 2.2.4. *Suppose that* f *is increasing and is in* $C^1((0, \infty), (0, \infty))$ *and suppose that* f *obeys* (2.2.1). *Suppose that* $g \in C((0, \infty), (0, \infty))$ *is increasing and*

$$g(x) > f(x), \quad x > 0. \tag{2.2.9}$$

Extend f *and* g *to* $(-\infty, 0]$ *so that* $g(x) = f(x) = 0$ *for* $x \leq 0$. *Let* $\psi \in C([-\tau, 0]; (0, \infty))$ *be decreasing. Then there exists a finite* $L \leq 0$ *such that the unique solution of* (2.2.7) *obeys*

$$\lim_{t\to\infty} x(t) = L. \tag{2.2.10}$$

So we can see that the presence of explosions in equations with *negative* delayed feedback depends on the relative sizes of the functions f and g, the magnitude of the delay τ and the nature of the initial function ψ. We do not investigate these equations in this thesis and accordingly consider equations with *positive* delayed feedback which guarantee the presence of a finite–time explosion.

2.3 Construction of State–Dependent Discretisation

We now construct a parameterised sequence $x_n(\Delta)$ and an associated continuous interpolating function $\bar{X}_\Delta$ which will approximate the solution of (0.1.6) and mimic its asymptotic behaviour. Let $\Delta \in (0, \tau f(\psi(0)))$, and define $N_\Delta \in \mathbb{N}$ so that

$$N_\Delta \frac{\Delta}{f(\psi(0))} \leq \tau, \quad \frac{(N_\Delta + 1)\Delta}{f(\psi(0))} > \tau.$$

Now define $t_{-N_\Delta}(\Delta) = -\tau$ and

$$t_n(\Delta) = \frac{n\Delta}{f(\psi(0))}, \quad n = -N_\Delta + 1, \ldots, 0.$$

Note that $\Delta < \tau f(\psi(0))$ ensures that $N_\Delta \geq 1$, that is we have at least one mesh point on the initial interval $[-\tau, 0]$. Then

$$t_{-N_\Delta+1}(\Delta) - t_{-N_\Delta}(\Delta) = -\frac{(N_\Delta - 1)\Delta}{f(\psi(0))} + \tau < \frac{(N_\Delta + 1)\Delta}{f(\psi(0))} - \frac{(N_\Delta - 1)\Delta}{f(\psi(0))} = \frac{2\Delta}{f(\psi(0))},$$

and

$$t_{-N_\Delta+1}(\Delta) - t_{-N_\Delta}(\Delta) = -\frac{(N_\Delta - 1)\Delta}{f(\psi(0))} + \tau \geq -\frac{(N_\Delta - 1)\Delta}{f(\psi(0))} + N_\Delta \frac{\Delta}{f(\psi(0))} = \frac{\Delta}{f(\psi(0))}.$$

Also define

$$x_n(\Delta) = \psi(t_n(\Delta)), \quad n = -N_\Delta, \ldots, 0, \tag{2.3.1}$$

$$X_\Delta(t) = \psi(t_n(\Delta)), \quad t \in [t_n(\Delta), t_{n+1}(\Delta)), \quad n = -N_\Delta, \ldots, -1 \tag{2.3.2}$$

and

$$\bar{X}_\Delta(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{2.3.3}$$

Next we extend $(t_n(\Delta))$ for $n \geq 0$ by

$$t_{n+1}(\Delta) = t_n(\Delta) + \frac{\Delta}{f(x_n(\Delta))}, \tag{2.3.4}$$

where $(x_n(\Delta))_{n\geq 0}$ and X_Δ are defined by

$$x_{n+1}(\Delta) = x_n(\Delta) + f(x_n(\Delta))(t_{n+1}(\Delta) - t_n(\Delta)) + \int_{t_n(\Delta)}^{t_{n+1}(\Delta)} g(X_\Delta(s-\tau))\, ds, \quad n = 0, 1, 2, \dots \tag{2.3.5}$$

and

$$X_\Delta(t) = x_n(\Delta), \quad t \in [t_n(\Delta), t_{n+1}(\Delta)). \tag{2.3.6}$$

Furthermore $\bar{X}_\Delta$ is defined for $n \geq 0$ by

$$\bar{X}_\Delta(t) = x_n(\Delta) + f(x_n(\Delta))(t - t_n(\Delta)) + \int_{t_n(\Delta)}^{t} g(X_\Delta(s-\tau))\, ds, \quad t \in [t_n(\Delta), t_{n+1}(\Delta)). \tag{2.3.7}$$

Remark 2.3.1. We note that these constructions do indeed yield a well-defined, positive and increasing sequence $(x_n(\Delta))_{n\geq 0}$, a well-defined, nonnegative and increasing sequence $(t_n(\Delta))_{n\geq 0}$ and a well-defined continuous function $\bar{X}_\Delta$ on the interval $[-\tau, T_\Delta)$ where $T_\Delta > 0$ could be finite or infinite. Let

$$(x_j(\Delta))_{0\leq j\leq n} \text{ is well-defined, positive and increasing,} \tag{2.3.8a}$$

$$(t_j(\Delta))_{0\leq j\leq n} \text{ is well-defined, nonnegative and increasing,} \tag{2.3.8b}$$

$$\bar{X}_\Delta \in C([-\tau, t_n(\Delta)) \text{ is well-defined and positive.} \tag{2.3.8c}$$

These statements are true for $n = 0$. If these statements are true at level n, we see that $f(x_n(\Delta)) > 0$, and so $t_{n+1}(\Delta) > t_n(\Delta) \geq 0$ is well-defined, nonnegative and increasing. Since $(x_j(\Delta))_{0\leq j\leq n}$ is well-defined, X_Δ is well-defined on $[0, t_{n+1}(\Delta))$. It is also positive, by hypothesis. Since $t_{n+1}(\Delta) - \tau < t_{n+1}(\Delta)$ we have $g(X_\Delta(s-\tau)) > 0$ for $s \in [t_n(\Delta), t_{n+1}(\Delta))$ and so $x_{n+1}(\Delta) > x_n(\Delta) > 0$ is well-defined, positive and increasing. Thus $\bar{X}_\Delta$ is well-defined and positive on $[t_n(\Delta), t_{n+1}(\Delta))$. Therefore (2.3.8a) and (2.3.8b) have been proven at level $n+1$, and so are true for all $n \geq 0$. Finally, we prove that $\bar{X}_\Delta \in C([-\tau, t_n(\Delta)); (0,\infty))$ for all $n \in \mathbb{N}$ and is increasing on $[0, t_n(\Delta))$ for all $n \in \mathbb{N}$. First, we deal with the continuity. $\bar{X}_\Delta = \psi$ is continuous on $[-\tau, 0) = [-\tau, t_0(\Delta))$. For each $n \geq 0$, $\bar{X}_\Delta$ is continuous on $[t_n(\Delta), t_{n+1}(\Delta))$. Since

$$\begin{aligned}
&\lim_{t \to t_{n+1}(\Delta)^-} \bar{X}_\Delta(t) \\
&= x_n(\Delta) + f(x_n(\Delta))(t_{n+1}(\Delta) - t_n(\Delta)) + g(\bar{X}_\Delta(t_n(\Delta) - \tau))\,(t_{n+1}(\Delta) - t_n(\Delta)) \\
&= x_{n+1}(\Delta) = \bar{X}_\Delta(t_{n+1}(\Delta)),
\end{aligned}$$

we see that $\bar{X}_\Delta$ is continuous at $t_{n+1}(\Delta)$ for all $n \geq 0$. Since it is also continuous at $t_0(\Delta) = 0$, we see that $\bar{X}_\Delta$ is continuous on $[-\tau, t_n(\Delta))$ for $n \geq 0$. $\bar{X}_\Delta$ is increasing on $[t_n(\Delta), t_{n+1}(\Delta))$ for each $n \geq 0$ since $f(x_n(\Delta)) > 0$ and $g(X_\Delta(s-\tau)) > 0$ for $s \in [t_n(\Delta), t_{n+1}(\Delta))$, and so increasing on $[0, t_n(\Delta))$ because $\bar{X}_\Delta$ is continuous on $[0, t_n(\Delta))$.

Remark 2.3.2. Note that for $s \in [t_n(\Delta), t_{n+1}(\Delta)]$, there can only be finitely many values of $X_\Delta(s-\tau)$. To see this, let $m(n) \in \mathbb{Z}$ such that $m(n) \geq -N_\Delta$ and $X_\Delta(t_n(\Delta) - \tau) = x_{m(n)}(\Delta)$. Clearly $m(n) < n$ and $t_{m(n)} \leq t_n(\Delta) - \tau$.

Now $t_{n+1}(\Delta) - \tau < t_{n+1}(\Delta)$, it follows that $X_\Delta(t_{n+1}(\Delta) - \tau)$ can only assume the values $x_{m(n)}(\Delta), x_{m(n)+1}(\Delta), \ldots, x_n(\Delta)$, and therefore for $s \in [t_n(\Delta), t_{n+1}(\Delta)]$ that $X_\Delta(s-\tau)$ can only assume the values $x_{m(n)}(\Delta), x_{m(n)+1}(\Delta), \ldots, x_n(\Delta)$.

Moreover is f is non-decreasing, it can be shown that $X_\Delta(s-\tau)$ can only assume the values $x_{m(n)}(\Delta)$ or $x_{m(n)+1}(\Delta)$ on $s \in [t_n(\Delta), t_{n+1}(\Delta)]$.

2.4 Explosions in the Numerical Method

We now show that the function $\bar{X}_\Delta$ explodes in finite time and mimics other properties of the solution x of (0.1.6) (cf. Theorem 2.2.1). We will make use of the following Lemma:

Lemma 2.4.1. *Let f obey* (0.1.7), (2.2.1) *and* (2.2.3)*. Let $\Delta \in (0, \tau f(\psi(0)))$ and $x_n(\Delta)$ be defined by* (2.3.5)*. Then*

(i)

$$\int_1^\infty \frac{1}{f(u)}\,du < \infty \text{ if and only if } \int_1^\infty \frac{1}{\phi(u)}\,du < \infty \tag{2.4.1}$$

(ii)

$$\sum_{j=0}^\infty \frac{\Delta}{f(x_j(\Delta))} < \infty \text{ if and only if } \sum_{j=0}^\infty \frac{\Delta}{\phi(x_j(\Delta))} < \infty. \tag{2.4.2}$$

Theorem 2.4.1. *Let f obey* (0.1.7)*,* (2.2.1) *and* (2.2.3)*, g obey* (0.1.8)*, and ψ obey* (0.1.9) *where $\tau > 0$. Let $\Delta \in (0, \tau f(\psi(0)))$ and $\bar{X}_\Delta$ be defined by* (2.3.7)*. Then there exists a finite $T_\Delta > 0$ such that $\bar{X}_\Delta \in C([-\tau, T_\Delta); (0, \infty))$, is increasing on $[0, T_\Delta)$,*

$$T_\Delta := \lim_{n\to\infty} t_n(\Delta) \tag{2.4.3}$$

and

$$\lim_{t\to T_\Delta^-} \bar{X}_\Delta(t) = \infty. \tag{2.4.4}$$

Moreover we have that

$$T_\Delta \le \frac{\Delta}{\underline{R}\phi(\psi(0))} + \frac{1}{\underline{R}} \int_{\psi(0)}^{\infty} \frac{1}{\phi(u)}\, du \tag{2.4.5}$$

where $\underline{R} := \inf_{x>0} \frac{f(x)}{\phi(x)} \in (0, \infty)$, and so there exist $C > 0$, $\Delta^* > 0$ such that

$$T_\Delta < C \text{ for } \Delta < \Delta^*. \tag{2.4.6}$$

However we note that if condition (2.2.1) is modified to condition (1.1.1), X_Δ does not explode in finite time. Therefore the numerical scheme does not produce a spurious explosion in the case where the continuous solution does not explode.

Theorem 2.4.2. *Let f obey* (0.1.7)*,* (2.2.1) *and* (2.2.3)*, g obey* (0.1.8)*, and ψ obey* (0.1.9) *where $\tau > 0$. Then there is $x \in C([-\tau, \infty))$ which is the unique continuous solution of* (0.1.6) *and which moreover obeys $\lim_{t\to\infty} x(t) = \infty$ (see Theorem 1.1.1). Let $\Delta \in (0, \tau f(\psi(0)))$, $\bar{X}_\Delta$ be defined by* (2.3.7)*, Then $\bar{X}_\Delta \in C([-\tau, \infty); (0, \infty))$ is increasing on $[0, \infty)$ and*

$$\lim_{t\to\infty} \bar{X}_\Delta(t) = \infty.$$

These results are investigated in later chapters.

2.5 Convergence of the Scheme

We now state the first main results in the chapter which deal not only with qualitative results (i.e., the existence of an explosion in $\bar{X}_\Delta$ given by (2.3.7)) but also establish quantitative properties of the approximate solution $\bar{X}_\Delta$. In this section, we concentrate on two such properties, namely: (i) the supnorm error between the true solution x and the continuous approximation $\bar{X}_\Delta$ and (ii) the error between the true explosion time T and the approximate explosion time T_Δ.

In Theorem 2.5.1, we show that given that the functions x and $\bar{X}_\Delta$ are compared when they do not exceed some arbitrary ceiling M, $\bar{X}_\Delta$ can be made arbitrarily close to x by choosing the parameter $\Delta < \Delta(M)$ sufficiently small.

Theorem 2.5.1. *Let f obey* (0.1.7)*, g obey* (0.1.8)*, and $\psi \in C([-\tau, 0]; (0, \infty))$ where $\tau > 0$. Then there exists a unique continuous solution x of* (0.1.6) *and there exists a finite $T > 0$ such that x is increasing on $[0, T)$ and x obeys* (2.2.2)*.*

Let $\Delta \in (0, \tau f(\psi(0)))$, let $\bar{X}_\Delta$ be defined by (2.3.7) *and let $L(f, \psi) := \inf_{x \geq \psi(0)} f(x) > 0$. If ρ_M and $\bar{\rho}_M(\Delta)$ are defined by*

$$\rho_M = \inf\{t \geq 0 : x(t) \geq M\}, \tag{2.5.1}$$

$$\bar{\rho}_M(\Delta) = \inf\{t \geq 0 : \bar{X}_\Delta(t) \geq M\}, \tag{2.5.2}$$

where $M \in (\psi^*, \infty)$ with $\psi^* := \max_{s \in [-\tau, 0]} \psi(s)$, we have

$$\lim_{\Delta \to 0} \sup_{t \in [0, \rho_M \wedge \bar{\rho}_{2M}(\Delta)]} \left| x(t) - \bar{X}_\Delta(t) \right| = 0. \tag{2.5.3}$$

Theorem 2.5.1 deals with the error between x and $\bar{X}_\Delta$ on a compact interval, on which both functions are well-defined. Therefore Theorem 2.5.1 shows that the time ρ_M at which x hits the threshold M (for arbitrary M) can be approximated with arbitrary precision by the time $\bar{\rho}_M(\Delta)$ at which $\bar{X}_\Delta$ hits M. In fact this theorem holds whether f obeys (2.2.1) or (1.1.1), as both $x(t)$ and $\bar{X}_\Delta(t)$ are finite on $t \in [0, \rho_M \wedge \bar{\rho}_{2M}(\Delta)]$ regardless of which condition is satisfied. Furthermore, f need not assume any degree of monotonicity.

We now show that under condition (2.2.1) the explosion time T of x can be approximated with arbitrary precision by $\bar{\rho}_M(\Delta)$.

Theorem 2.5.2. *Let f obey* (0.1.7)*,* (2.2.1) *and* (2.2.3)*, g obey* (0.1.8)*, and ψ obey* (0.1.9) *where $\tau > 0$. Then there exists a unique continuous solution x of* (0.1.6) *and there exists a finite $T > 0$ such that x is increasing on $[0, T)$ and x obeys* (2.2.2)*.*

Let $\Delta \in (0, \tau f(\psi(0)))$, let $\bar{X}_\Delta$ be defined by (2.3.7) *and let $L(f, \psi) := \inf_{x \geq \psi(0)} f(x) > 0$. If ρ_M and $\bar{\rho}_M(\Delta)$ are defined by* (2.5.1) *and* (2.5.2) *where $M \in \left(\max_{t \in [-\tau, 0]} \psi(t), \infty\right)$, we* *have*

$$\lim_{M\to\infty}\lim_{\Delta\to 0}|\bar{\rho}_M(\Delta)-T|=0.$$

Note that $\bar{\rho}_M(\Delta)$ can be obtained explicitly from the numerical scheme for any $M > 0, \Delta \in (0, \tau f(\psi(0)))$, in contrast to T and even T_Δ, which cannot be determined in finite time by the algorithm. Theorem 2.5.2 can be useful if it is not know in advance that condition (2.2.1) is satisfied and x obeys (2.2.2). Since we can approximate ρ_M with arbitrary precision by $\bar{\rho}_M(\Delta)$, it is possible to investigate the convergence of these approximations of ρ_M as $M \to \infty$. The observation that these values are approaching a finite limit, which is very apparent given the nature of explosive equations, indicates the presence of an explosion of the solution of (0.1.6). However if f does not satisfy condition (2.2.3), such a conclusion would be unwise as in the absence of monotonicity the numerical method may give false information about the presence of an explosion, as we will demonstrate in Section 2.7. To prove Theorem 2.5.2 we make use of the following Lemma.

Lemma 2.5.1. *Let f obey* (0.1.7)*,* (2.2.1) *and* (2.2.3)*, g obey* (0.1.8)*, ψ obey* (0.1.9) *where $\tau > 0$, $M \in \left(\max_{t\in[-\tau,0]}\psi(t), \infty\right)$ and let ρ_M and $\bar{\rho}_M(\Delta)$ be defined by* (2.5.1) *and* (2.5.2)*. Then there exists $\Delta(M) \in (0, \tau f(\psi(0)))$ such that*

$$\bar{\rho}_{2M}(\Delta) \geq \rho_M, \rho_{2M} \geq \bar{\rho}_M(\Delta) \textit{ for } \Delta < \Delta(M). \tag{2.5.4}$$

Proof of Theorem 2.5.2 Now, note that for $\Delta < \Delta(M)$

$$\bar{\rho}_M(\Delta) - T = \bar{\rho}_M(\Delta) - \rho_{2M} + \rho_{2M} - T < \bar{\rho}_M(\Delta) - \rho_{2M} \leq 0$$

Note also that for $\Delta < \Delta(M/2)$

$$\bar{\rho}_M(\Delta) - T = \bar{\rho}_M(\Delta) - \rho_{M/2} + \rho_{M/2} - T \geq \rho_{M/2} - T$$

Thus for all $\Delta < \min(\Delta(M), \Delta(M/2))$,

$$\rho_{M/2} - T \leq \bar{\rho}_M(\Delta) - T < 0.$$

Therefore

$$\limsup_{\Delta\to 0}|\bar{\rho}_M(\Delta) - T| \leq T - \rho_{M/2}.$$

Hence

$$\lim_{M\to\infty}\lim_{\Delta\to 0}|\bar{\rho}_M(\Delta)-T|=0.$$

and the proof is complete.

2.6 Explosion Rate of Continuous and Discrete Equations

In this section we determine the rate of explosion of the solution of (0.1.6).

Theorem 2.6.1. *Let f obey* (0.1.7), (2.2.1) *and* (2.2.3), *g obey* (0.1.8), *and let ψ obey* (0.1.9) *where $\tau > 0$. Then there is a unique continuous solution x of* (0.1.6), *and there exists $T > 0$ such that x obeys* (2.2.2). *Moreover, with $\bar{F}$ defined by* (0.1.12), *we have*

$$\lim_{t\to T^-}\frac{\bar{F}(x(t))}{T-t}=1. \tag{2.6.1}$$

Proof. We first notice that (0.1.7), (2.2.1) and (2.2.3) imply that $f(x)\to\infty$ as $x\to\infty$. Therefore by (0.1.7) and (2.2.2) we have

$$\lim_{t\to T^-} f(x(t))=\infty.$$

By (0.1.8) and the continuity of x on $[-\tau, T-\tau+\tau/2]$, we have

$$\lim_{t\to T^-} g(x(t-\tau))=g(x(T-\tau)),$$

which is finite. By these observations and (0.1.6), we have

$$\lim_{t\to T^-}\frac{x'(t)}{f(x(t))}=1.$$

Notice that $x'(t)\geq f(x(t))$, $t\in(0,T)$. Therefore, for every $\varepsilon>0$, there exists $T_\varepsilon\in(0,T)$ such that

$$1\leq\frac{x'(t)}{f(x(t))}<1+\varepsilon,\quad t\in(T_\varepsilon,T).$$

Hence, by integrating across this inequality and using the definition of $\bar{F}$ in (0.1.12), we have

$$T-t\leq\bar{F}(x(t))\leq(1+\varepsilon)(T-t),\quad t\in(T_\varepsilon,T),$$

from which (2.6.1) immediately follows. $\square$

As a result of Theorem 2.6.1, we see that regularly varying equations with weaker nonlinear feedback produce faster explosion rates. To illustrate this, consider the following example:

Example 2.6.1. Suppose that

$$x_1'(t) = f_1(x_1(t)) + g(x_1(t-1)),\ t > 0; \quad x_1(t) = 1/(1-t),\ t \in [-1, 0], \tag{2.6.2a}$$

$$x_2'(t) = f_2(x_2(t)) + g(x_2(t-1)),\ t > 0; \quad x_2(t) = 1/(1-t)^2,\ t \in [-1, 0], \tag{2.6.2b}$$

where $f_1(x) = x^2 - \frac{x}{(x+1)}$, $f_2(x) = 2x^{3/2} - \frac{x}{(x+2\sqrt{x}+1)}$ and $g(x) = x$. We can easily show that

$$x_1(t) = 1/(1-t),\ t \in [-1, 1), \tag{2.6.3a}$$

$$x_2(t) = 1/(1-t)^2,\ t \in [-1, 1). \tag{2.6.3b}$$

Clearly $\lim_{x\to\infty} \frac{f_1(x)}{f_2(x)} = \infty$ but $\lim_{t\to 1^-} \frac{x_1(t)}{x_2(t)} = 0$.

In general, amongst equations with regularly varying feedback, $RV_\infty(1)$ equations which are just sufficient to satisfy condition (2.2.1) produce the quickest explosion rates. This example is particularly interesting, as one might naturally expect equations with more extreme nonlinearities as $x \to \infty$ to give rise to the quickest explosion rates. However equations with feedback that is more rapidly growing than regularly varying, for example $f(x) \sim e^x$, will produce even quicker explosion rates.

Our task now is to show, under appropriate hypotheses on $\bar{F}$, that $x_n(\Delta)$ and $\bar{X}_\Delta$ obey appropriate asymptotic analogues to (2.6.1). To do this, we need the following preparatory result.

Lemma 2.6.1. *Let f obey* (0.1.7), (2.2.1) *and* (2.2.3), *g obey* (0.1.8), *and ψ obey* (0.1.9) *where $\tau > 0$. Let $\Delta \in (0, \tau f(\psi(0)))$ and let $(x_n(\Delta))_{n\geq 0}$ be defined by* (2.3.5). *Then*

$$\lim_{n\to\infty} \frac{x_n(\Delta)}{n} = \Delta. \tag{2.6.4}$$

Proof. In Theorem 2.4.1 it was already established that $T_\Delta > 0$ defined by (2.4.3) is finite. We establish (2.6.4) in the cases when $T_\Delta \leq \tau$ or $T_\Delta > \tau$.

If $T_\Delta \leq \tau$, then $t_n(\Delta) < \tau$ for all $n \geq 0$. Thus $t_n(\Delta) - \tau < 0$, so $\bar{X}_\Delta(t_n(\Delta) - \tau) = \psi(t_n(\Delta) - \tau)$. Therefore by (0.1.8), (0.1.9) and (2.4.3) we have

$$\lim_{n\to\infty} g(\bar{X}_\Delta(t_n(\Delta) - \tau)) = g(\psi(T_\Delta - \tau)). \tag{2.6.5}$$

Now, inserting (2.3.4) in (2.3.5) gives

$$x_{n+1}(\Delta) = x_n(\Delta) + \Delta + \frac{\Delta}{f(x_n(\Delta))} g(\psi(t_n(\Delta) - \tau)). \tag{2.6.6}$$

Since $x_n(\Delta) \to \infty$ as $n \to \infty$ and since (2.2.1) and (2.2.3) imply $f(x) \to \infty$ as $x \to \infty$, we have by (2.6.5) that

$$\varepsilon_n(\Delta) := \frac{\Delta}{f(x_n(\Delta))} g(\psi(t_n(\Delta) - \tau)) \to 0, \quad \text{as } n \to \infty.$$

Using this and (2.6.6) we have (2.6.4).

If $T_\Delta > \tau$, the fact that $0 \le t_n(\Delta) \uparrow T_\Delta$ as $n \to \infty$ implies that there is $n' \ge 0$ such that

$$0 \le t_{n'}(\Delta) < T_\Delta - \tau \le t_{n'+1}(\Delta) < T_\Delta. \tag{2.6.7}$$

To check the strictness of the inner two inequalities, consider the possibility that $t_n(\Delta) \neq T_\Delta - \tau$ for all $n \ge 0$ and the possibility that there is an $n \ge 0$ such that $t_n(\Delta) = T_\Delta - \tau$. We now claim that there exists $\bar{n}_\Delta \ge 0$ such that

$$t_{n'}(\Delta) < t_n(\Delta) - \tau \le t_{n'+1}(\Delta), \quad n \ge \bar{n}_\Delta. \tag{2.6.8}$$

To prove (2.6.8), note that $t_n(\Delta) < T_\Delta$ for all $n \ge 0$. Thus for all $n \ge 0$, by (2.6.7)

$$t_n(\Delta) - \tau < T_\Delta - \tau \le t_{n'+1}(\Delta),$$

so the second member of (2.6.8) holds. To prove the first member of (2.6.8), first note that $t_{n'}(\Delta) + \tau < T_\Delta$. Therefore $\varepsilon_0 := \frac{1}{2}(T_\Delta - t_{n'}(\Delta) - \tau) > 0$. Since $t_n(\Delta) \uparrow T_\Delta$ as $n \to \infty$, there exists $n(\Delta) > 0$ such that

$$0 < T_\Delta - t_n(\Delta) < \varepsilon_0, \quad n > n(\Delta).$$

Thus $T_\Delta < t_n(\Delta) + \varepsilon_0$, and because $t_{n'}(\Delta) + \tau < T_\Delta$, we have

$$t_{n'}(\Delta) < T_\Delta - \tau < t_n(\Delta) - \tau + \varepsilon_0, \quad n > n(\Delta).$$

Thus

$$\begin{aligned} t_n(\Delta) - \tau > T_\Delta - \tau - \varepsilon_0 &= T_\Delta - \tau - \frac{1}{2}(T_\Delta - t_{n'}(\Delta) - \tau) \\ &> T_\Delta - \tau - (T_\Delta - t_{n'}(\Delta) - \tau) = t_{n'}(\Delta), \end{aligned}$$

so $t_{n'}(\Delta) < t_n(\Delta) - \tau$ for all $n > n(\Delta)$. Choosing $\bar{n}_\Delta = n(\Delta) + 1$ now gives the first member of (2.6.8).

(2.6.8) and the monotonicity of $\bar{X}_\Delta$ on $[t_{n'}(\Delta), t_{n'+1}(\Delta)]$ imply that

$$\bar{X}_\Delta(t_n(\Delta) - \tau) \in [x_{n'}(\Delta), x_{n'+1}(\Delta)], \quad n \geq \bar{n}_\Delta.$$

Therefore for $n \geq \bar{n}_\Delta$,

$$\underline{g}_\Delta := \min_{x \in [x_{n'}(\Delta), x_{n'+1}(\Delta)]} g(x) \leq g(\bar{X}_\Delta(t_n(\Delta) - \tau)) \leq \max_{x \in [x_{n'}(\Delta), x_{n'+1}(\Delta)]} g(x) =: \overline{g}_\Delta.$$

Defining $\varepsilon_n(\Delta) = \Delta g(\bar{X}_\Delta(t_n(\Delta) - \tau))/f(x_n(\Delta))$, we have

$$\frac{\Delta}{f(x_n(\Delta))} \underline{g}_\Delta \leq \varepsilon_n(\Delta) \leq \frac{\Delta}{f(x_n(\Delta))} \overline{g}_\Delta, \quad n \geq \bar{n}_\Delta,$$

and so $\varepsilon_n(\Delta) \to 0$ as $n \to \infty$. Moreover by (2.3.4) and (2.3.5), we have

$$x_{n+1}(\Delta) = x_n(\Delta) + \Delta + \varepsilon_n(\Delta), \quad n \geq 0,$$

which therefore implies (2.6.4). □

We are now in a position to determine the asymptotic behaviour of $(x_n(\Delta))_{n \geq 0}$ and $\bar{X}_\Delta(t)$ as $n \to \infty$ and as $t \to T_\Delta^-$ respectively.

We now state two results which enable us to determine the asymptotic behaviour for classes of f which grow at increasingly rapid rates. Commentary and examples will be supplied after the statements.

Theorem 2.6.2. *Let f obey* (0.1.7), (2.2.1) *and* (2.2.3), *g obey* (0.1.8), *and ψ obey* (0.1.9) *where $\tau > 0$. Let $\Delta \in (0, \tau f(\psi(0)))$ and let $(t_n(\Delta))_{n \geq 0}$, $(x_n(\Delta))_{n \geq 0}$ and $\bar{X}_\Delta$ be as defined by* (2.3.4), (2.3.5) *and* (2.3.7). *Then by Theorem 2.4.1 there exists $T_\Delta \in (0, \infty)$ such that*

$$\lim_{t \to T_\Delta^-} \bar{X}_\Delta(t) = \infty, \quad T_\Delta = \lim_{n \to \infty} t_n(\Delta).$$

If $\bar{F}$ is defined by (0.1.12) *and $\bar{F} \in RV_\infty(\beta)$ for some $\beta \leq 0$, then*

$$\lim_{n \to \infty} \frac{\bar{F}(x_n(\Delta))}{T_\Delta - t_n(\Delta)} = 1, \tag{2.6.9}$$

and

$$\lim_{t \to T_\Delta^-} \frac{\bar{F}(\bar{X}_\Delta(t))}{T_\Delta - t} = 1. \tag{2.6.10}$$

Theorem 2.6.2 applies to for example polynomially growing f. We now give a couple of representative examples.

Example 2.6.2. Suppose $f(x) \sim cx^\alpha$ as $x \to \infty$, where $c > 0$, $\alpha > 1$. Then

$$\bar{F}(x) \sim \frac{1}{c}\int_x^\infty u^{-\alpha}\,du = \frac{1}{c}\frac{x^{-(\alpha-1)}}{\alpha-1} \text{ as } x \to \infty,$$

and so $\bar{F} \in RV_\infty(1-\alpha)$. Thus the hypotheses of Theorem 2.6.2 are satisfied with $\beta = 1-\alpha < 0$. Then

$$\lim_{t\to T_\Delta^-}(T_\Delta - t)^{1/(\alpha-1)}\bar{X}_\Delta(t) = \left(\frac{1}{c(\alpha-1)}\right)^{1/(\alpha-1)}.$$

By Theorem 2.6.1, the solution of (0.1.6) obeys

$$\lim_{t\to T^-}(T-t)^{1/(\alpha-1)}x(t) = \left(\frac{1}{c(\alpha-1)}\right)^{1/(\alpha-1)},$$

and moreover $\lim_{\Delta\to 0^+} T_\Delta = T$.

Among equations which satisfy the conditions of Theorem 2.6.2, those for which the rate of growth of f is just sufficient to cause an explosion produce the strongest rates of explosion in accordance with Example 2.6.1. This is demonstrated in the following examples:

Example 2.6.3. Suppose $f(x) \sim cx(\log x)^\alpha$ as $x \to \infty$, where $c > 0$ $\alpha > 1$. Then

$$\bar{F}(x) \sim \frac{1}{c}\int_x^\infty \frac{1}{u(\log u)^\alpha}\,du = \frac{1}{c}\frac{(\log x)^{-(\alpha-1)}}{\alpha-1} \text{ as } x \to \infty,$$

and so $\bar{F} \in RV_\infty(0)$, which satisfies the conditions of Theorem 2.6.2 with $\beta = 0$. So we have

$$\lim_{t\to T_\Delta^-}(T_\Delta - t)^{1/(\alpha-1)}\log\bar{X}_\Delta(t) = \left(\frac{1}{c(\alpha-1)}\right)^{1/(\alpha-1)}.$$

By Theorem 2.6.1, the solution of (0.1.6) obeys

$$\lim_{t\to T^-}(T-t)^{1/(\alpha-1)}\log x(t) = \left(\frac{1}{c(\alpha-1)}\right)^{1/(\alpha-1)},$$

and moreover $\lim_{\Delta\to 0^+} T_\Delta = T$.

Example 2.6.4. Suppose $f(x) \sim cx\log x(\log\log x)^\alpha$ as $x \to \infty$, where $c > 0$, $\alpha > 1$. Then

$$\bar{F}(x) \sim \frac{1}{c}\int_x^\infty \frac{1}{u\log u(\log\log u)^\alpha}\,du = \frac{1}{c}\frac{(\log\log x)^{-(\alpha-1)}}{\alpha-1} \text{ as } x \to \infty,$$

and so $\bar{F} \in RV_\infty(0)$, which satisfies the conditions of Theorem 2.6.2 with $\beta = 0$. So we have

$$\lim_{t\to T_\Delta^-} (T_\Delta - t)^{1/(\alpha-1)} \log\log \bar{X}_\Delta(t) = \left(\frac{1}{c(\alpha-1)}\right)^{1/(\alpha-1)}.$$

By Theorem 2.6.1, the solution of (0.1.6) obeys

$$\lim_{t\to T^-} (T - t)^{1/(\alpha-1)} \log\log x(t) = \left(\frac{1}{c(\alpha-1)}\right)^{1/(\alpha-1)},$$

and moreover $\lim_{\Delta\to 0^+} T_\Delta = T$.

Theorem 2.6.3. *Let f obey* (0.1.7), (2.2.1) *and* (2.2.3), *g obey* (0.1.8), *and ψ obey* (0.1.9) *where $\tau > 0$. Let $\Delta \in (0, \tau f(\psi(0)))$ and let $(x_n(\Delta))_{n\ge 0}$, $(t_n(\Delta))_{n\ge 0}$ and $\bar{X}_\Delta$ be as defined by* (2.3.7). *Then by Theorem 2.4.1 there exists $T_\Delta \in (0,\infty)$ such that*

$$\lim_{t\to T_\Delta^-} \bar{X}_\Delta(t) = \infty, \quad T_\Delta = \lim_{n\to\infty} t_n(\Delta).$$

If $\bar{F}$ is defined by (0.1.12) *and $\bar{F}^{-1} \in RV_0(0)$, then*

$$\lim_{n\to\infty} \frac{x_n(\Delta)}{\bar{F}^{-1}(T_\Delta - t_n(\Delta))} = 1, \tag{2.6.11}$$

and

$$\lim_{t\to T_\Delta^-} \frac{\bar{X}_\Delta(t)}{\bar{F}^{-1}(T_\Delta - t)} = 1. \tag{2.6.12}$$

Theorem 2.6.3 includes exponentially growing f. We notice that in Theorem 2.6.2 we made an assumption on the asymptotic behaviour of $\bar{F}$ at infinity, while in Theorem 2.6.3 we made an asymptotic assumption on $\bar{F}^{-1}$ at zero. It transpires however that the latter hypothesis proves to be a natural way to extend the scope of problems covered by the former. Specifically, the hypothesis $\bar{F} \in RV_\infty(\beta)$ for *finite* $\beta < 0$ is equivalent to $\bar{F}^{-1} \in RV_0(1/\beta)$. Therefore we can view the hypothesis $\bar{F}^{-1} \in RV_0(0)$ in Theorem 2.6.3 as the analogue of $\bar{F} \in RV_\infty(\beta)$ in Theorem 2.6.2 as $\beta \to -\infty$; hence Theorem 2.6.3 is a natural extension of Theorem 2.6.2. Here are a couple of representative examples:

Example 2.6.5. Suppose $f(x) \sim ce^{\alpha x}$ as $x \to \infty$, where $c > 0$, $\alpha > 0$. Then

$$\bar{F}(x) \sim \frac{1}{c}\int_x^\infty e^{-\alpha u}\,du = \frac{1}{c\alpha} e^{-\alpha x} \text{ as } x \to \infty,$$

which implies

$$\bar{F}^{-1}(x) \sim \log\left(\frac{1}{x}\right)^{1/\alpha} \quad \text{as } x \to \infty,$$

and so $\bar{F}^{-1} \in RV_\infty(0)$, which satisfies the conditions of Theorem 2.6.3. So we have

$$\lim_{t\to T_\Delta^-}\left[\log\left(\frac{1}{T_\Delta - t}\right)\right]^{-1}\bar{X}_\Delta(t) = \frac{1}{\alpha}.$$

By Theorem 2.6.1, the solution of (0.1.6) obeys

$$\lim_{t\to T^-}\left[\log\left(\frac{1}{T - t}\right)\right]^{-1}x(t) = \frac{1}{\alpha},$$

and moreover $\lim_{\Delta\to 0^+} T_\Delta = T$.

Example 2.6.6. Suppose $f(x) \sim ce^{\alpha e^{\beta x}}e^{-\beta x}$ as $x \to \infty$, where $c > 0$, $\alpha > 0, \beta > 0$. Then

$$\bar{F}(x) \sim \frac{1}{c}\int_x^\infty e^{-\alpha e^{\beta u}}e^{\beta u}\,du = \frac{1}{c\alpha\beta}e^{-\alpha e^{\beta x}} \quad \text{as } x \to \infty,$$

which implies

$$\bar{F}^{-1}(x) \sim \log\left[\log\left(\frac{1}{x}\right)^{1/\alpha}\right]^{1/\beta} \quad \text{as } x \to \infty,$$

and so $\bar{F}^{-1} \in RV_\infty(0)$, which satisfies the conditions of Theorem 2.6.3. So we have

$$\lim_{t\to T_\Delta^-}\left[\log\log\left(\frac{1}{T_\Delta - t}\right)^{1/\alpha}\right]^{-1}\bar{X}_\Delta(t) = \frac{1}{\beta}.$$

By Theorem 2.6.1, the solution of (0.1.6) obeys

$$\lim_{t\to T^-}\left[\log\log\left(\frac{1}{T - t}\right)^{1/\alpha}\right]^{-1}x(t) = \frac{1}{\beta},$$

and moreover $\lim_{\Delta\to 0^+} T_\Delta = T$.

Of course (2.6.12) is not the exact analogue of (2.6.1), that is when we move beyond the scope of equations covered by Theorem 2.6.2 we lose a bit of information about the exact rate of explosion of the approximate solution. In Chapter 3, we investigate how to extend the numerical method described in Section 2.3 to deal with such equations. We suggest that an extra factor must be included in the step-size to create a “finer” mesh with which to discretise the solution. However, this extra factor depends on f itself, and so will vary within the class of equations for which $\bar{F}^{-1} \in RV_0(0)$. For instance to obtain the exact analogue of (2.6.1) for the equation given in Example 2.6.6 we must further modify the numerical method over what would be required to obtain the exact analogue of (2.6.1) for the equation given in Example 2.6.5.

2.7 Absence of Monotonicity

In this section we show that if f does *not* obey (2.2.3), that is we do not make any sort of monotonicity assumption on f, examples of f exist for which the existence of explosions in (0.1.6) is not detected by our adaptive mesh described in Section 2.3. To illustrate this, yet simplify the analysis, we do not consider the delayed component of (0.1.6), that is we construct examples of equations which grow at any rate described by an ordinary differential equation given by

$$y'(t) = \eta(y(t)), \quad t > 0; \quad y(0) = \xi,$$

where η obeys

$$\eta \in C^1((0,\infty);(0,\infty)), \quad \int_1^\infty \frac{1}{\eta(x)}\,dx < +\infty, \quad \eta \text{ is increasing.} \tag{2.7.1}$$

At the same time, the numerical method can be interpreted as returning the same rate of growth as the non-exploding equation

$$z'(t) = \theta(z(t)), \quad t > 0, \quad z(0) = \xi,$$

where

$$\theta \in C^1((0,\infty);(0,\infty)), \quad \int_1^\infty \frac{1}{\theta(x)}\,dx = +\infty, \quad \theta \text{ is increasing.} \tag{2.7.2}$$

Therefore, the essence of these results is that differential equations exist which have arbitrary explosion rates, but for which the numerical method incorrectly predicts that an explosion is absent, and that the rate of growth of that non-exploding solution is also arbitrary.

We make the mild additional assumption that

$$\eta(x) \geq \theta(x), \quad x > 0, \tag{2.7.3}$$

which is clearly consistent (but not a consequence of) (2.7.1) and (2.7.2).

Let $\Delta_0 > 0$ and define

$$\epsilon_n = \frac{1}{4}\frac{1}{n+2}\frac{\theta^2(\xi)}{\eta(\xi + n\Delta_0 - \Delta_0/2)\eta(\xi + n\Delta_0 + \Delta_0/2)}, \quad n \geq 1.$$

Notice that $\epsilon_n > 0$. Also, $\eta(\xi + n\Delta_0 + \Delta_0/2) \geq \eta(\xi) \geq \theta(\xi)$, and $\eta(\xi + n\Delta_0 - \Delta_0/2) \geq \eta(\xi) \geq \theta(\xi)$ so $\epsilon_n < 1/12$ for all $n \geq 1$. Since $1/\eta \in L^1((0,\infty);(0,\infty))$ and η is increasing, we have that

$$\sum_{n=1}^{\infty} \frac{1}{\eta(\xi + n\Delta_0 + \Delta_0/2)} < +\infty,$$

so we have that $(\epsilon_n)_{n\geq 1}$ is summable.

Define now $f : [0,\infty) \to \mathbb{R}$ by $f(x) = \eta(x)$ for $x \in [0, \xi + \Delta_0/2)$ and on the interval $[\xi + (n-1/2)\Delta_0, \xi + (n+1/2)\Delta_0)$ for $n \geq 1$ by

$$f(x) = \begin{cases} \eta(x), & x \in [\xi + (n-1/2)\Delta_0, \xi + n\Delta_0 - 2\epsilon_n\Delta_0), \\ l_-(x), & x \in [\xi + n\Delta_0 - 2\epsilon_n\Delta_0, \xi + n\Delta_0 - \epsilon_n\Delta_0), \\ \theta(x), & x \in [\xi + n\Delta_0 - \epsilon_n\Delta_0, \xi + n\Delta_0 + \epsilon_n\Delta_0), \\ l_+(x), & x \in [\xi + n\Delta_0 + \epsilon_n\Delta_0, \xi + n\Delta_0 + 2\epsilon_n\Delta_0), \\ \eta(x), & x \in [\xi + n\Delta_0 + 2\epsilon_n\Delta_0, \xi + (n+1/2)\Delta_0), \end{cases} \tag{2.7.4}$$

where

$$\begin{aligned} l_-(x) &:= \eta(\xi + n\Delta_0 - 2\epsilon_n\Delta_0) \\ &\quad + \frac{\theta(\xi + n\Delta_0 - \epsilon_n\Delta_0) - \eta(\xi + n\Delta_0 - 2\epsilon_n\Delta_0)}{\epsilon_n}(x - (\xi + n\Delta_0 - 2\epsilon_n\Delta_0)), \end{aligned}$$

and

$$\begin{aligned} l_+(x) &:= \theta(\xi + n\Delta_0 + \epsilon_n\Delta_0) \\ &\quad + \frac{\eta(\xi + n\Delta_0 + 2\epsilon_n\Delta_0) - \theta(\xi + n\Delta_0 + \epsilon_n\Delta_0)}{\epsilon_n}(x - (\xi + n\Delta_0 + \epsilon_n\Delta_0)). \end{aligned}$$

The condition that $\epsilon_n \in (0, 1/4)$ guarantees that the partition of $[\xi + (n-1/2)\Delta_0, \xi + (n+1/2)\Delta_0)$ into subintervals in (2.7.4) is well-defined, along with the function f itself. Also note that the linear functions l_+ and l_- are chosen to make f continuous on the interval $[\xi + (n-1/2)\Delta_0, \xi + (n+1/2)\Delta_0)$ for each $n \geq 1$. Also, the fact that f is equal to η on the first and last subinterval, as well as the continuity of η, ensures that f is a continuous and indeed a positive function on $(0,\infty)$. Moreover, f is locally Lipschitz continuous, because η and θ are in $C^1((0,\infty);(0,\infty))$ and the linear interpolants $l_\pm$ are of course locally Lipschitz continuous.

Using the monotonicity of η and θ and (2.7.3), it can be seen that f is not an increasing function. Indeed, if $\eta(x)/\theta(x) \to \infty$ as $x \to \infty$, we cannot have that f is asymptotic to an increasing function.

If we consider the initial value problem given by

$$x'(t) = f(x(t)), \quad t > 0; \quad x(0) = \xi, \tag{2.7.5}$$

where f is the function defined by (2.7.4), we can show that the solution of x of (2.7.5) obeys

$$\lim_{t \to T_\xi} x(t) = \infty \text{ where } T_\xi = \int_\xi^\infty 1/f(u)\,du. \tag{2.7.6}$$

Now, let $\Delta > 0$ and consider the following numerical approximation to the solution x of (2.7.5). Define $x_0(\Delta) = \xi$, $t_0(\Delta) = 0$ and

$$h_n(\Delta) = \frac{\Delta}{f(x_n(\Delta))}, \quad n \geq 0, \tag{2.7.7a}$$

$$x_{n+1}(\Delta) = x_n(\Delta) + h_n(\Delta) f(x_n(\Delta)), \quad n \geq 0, \tag{2.7.7b}$$

$$t_n(\Delta) = \sum_{j=0}^{n-1} h_j(\Delta), \quad n \geq 1. \tag{2.7.7c}$$

Clearly, $x_n(\Delta)$ is the forward Euler approximation to $x(t_n(\Delta))$ consistent with the method described in Section 2.3. However, the following results shows that the numerical method does not predict the presence of the explosion in (2.7.5).

Proposition 2.7.1. *Suppose that η and θ are functions which obey* (2.7.1), (2.7.2) *and* (2.7.3), *and let f be the function defined by* (2.7.4). *Then the solution x of* (2.7.5) *obeys* (2.7.6), *while the solution of* (2.7.7) *is such that $t_n(\Delta)$ defined by* (2.7.7c) *obeys*

$$T_{\Delta_0} := \lim_{n \to \infty} t_n(\Delta_0) = +\infty. \tag{2.7.8}$$

There are two other interesting properties of this example which does not detect the presence of an explosion. The first is that there exist *arbitrarily small* and at least *countably many* values of the control parameter Δ for which the presence of the explosion is not detected. The second is that the solution of the numerical method (2.7.7) grows at a rate consistent with the non-exploding differential equation $y'(t) = \theta(y(t))$ for $t > 0$, while the solution of the differential equation in fact grows at a rate consistent with the exploding differential equation $z'(t) = \eta(z(t))$ for $t > 0$. The rate for the ODE arises because the "troughs" in f (which represent departures from η) while deep, are sufficiently narrow to guarantee that the rate of growth of the explosion that would arise if the right hand side is η is retained. Therefore, as the numerical solution will appear quite well-behaved, and reducing the error control parameter may not assist in detecting the explosion, it can be seen that "false negatives" may be hard to spot for more complicated problems.

Proposition 2.7.2. *If f is defined by* (2.7.4)*, and x is the solution of* (2.7.5)*, then x obeys* (2.7.6) *and*

$$\lim_{t \to T_\xi^-} \frac{\bar{H}(x(t))}{T_\xi - t} = 1$$

where

$$\bar{H}(x) = \int_x^\infty \frac{1}{\eta(u)}\,du, \quad x \geq 0.$$

Proposition 2.7.3. *If f is defined by* (2.7.4)*, and x is the solution of* (2.7.5)*, then x obeys* (2.7.6)*. If T_{Δ_0} is defined by* (2.7.8)*, then $T_{\Delta_0} = +\infty$ and*

$$T_{\Delta_0/k} = +\infty, \quad \textit{for all } k \in \mathbb{N}.$$

Proof. Let $k \geq 2$. Clearly

$$T_{\Delta_0/k} = \sum_{n=0}^\infty \frac{\Delta_0/k}{f(\xi + n \cdot \Delta_0/k)} \geq \sum_{n=0\,:\,n/k \text{ is an non-negative integer}}^\infty \frac{\Delta_0/k}{f(\xi + n \cdot \Delta_0/k)},$$

so by (2.7.8) we have

$$T_{\Delta_0/k} \geq \frac{1}{k} \sum_{j=0}^\infty \frac{\Delta_0}{f(\xi + j\Delta_0)} = \frac{1}{k} T_{\Delta_0}.$$

By Proposition 2.7.1, we have $T_{\Delta_0} = +\infty$, the proof is complete. □

Proposition 2.7.4. *If f is defined by* (2.7.4)*, and $x_n(\Delta)$ is the solution of* (2.7.7)*, then $T_{\Delta_0} = +\infty$, $x_n(\Delta)$ is defined on $(0, \infty)$ and*

$$\lim_{n \to \infty} \frac{\Theta(x_n(\Delta_0))}{t_n(\Delta_0)} = 1$$

where

$$\Theta(x) = \int_\xi^x \frac{1}{\theta(u)}\,du, \quad x \geq \xi.$$

Proof. Note that since $x_n(\Delta_0) = \xi + n\Delta_0$ for $n \geq 0$ and f obeys (2.7.4),

$$f(x_n(\Delta_0)) = f(\xi + n\Delta_0) = \theta(\xi + n\Delta_0), \quad n \geq 1$$

and so

$$\begin{aligned} t_n(\Delta_0) &= \frac{\Delta_0}{f(x_0(\Delta_0))} + \sum_{j=1}^{n-1} \frac{\Delta_0}{f(x_j(\Delta_0))} \\ &= \frac{\Delta_0}{f(\xi)} + \sum_{j=1}^{n-1} \frac{\Delta_0}{\theta(\xi + j\Delta_0)} \\ &= \frac{\Delta_0}{\eta(\xi)} + \sum_{j=0}^{n-1} \frac{\Delta_0}{\theta(\xi + j\Delta_0)} - \frac{\Delta_0}{\theta(\xi)}. \end{aligned}$$

As $1/\theta$ is decreasing,

$$\begin{aligned} \sum_{j=0}^{n-1} \frac{\Delta_0}{\theta(\xi + j\Delta_0)} \geq \Theta(\xi + n\Delta_0)) &\geq \sum_{j=0}^{n-1} \frac{\Delta_0}{\theta(\xi + (j+1)\Delta_0)} \\ &= \sum_{j=0}^{n} \frac{\Delta_0}{\theta(\xi + j\Delta_0)} - \frac{\Delta_0}{\theta(\xi)} \\ &> \sum_{j=0}^{n-1} \frac{\Delta_0}{\theta(\xi + j\Delta_0)} - \frac{\Delta_0}{\theta(\xi)}. \end{aligned}$$

Therefore

$$t_n(\Delta_0) - \frac{\Delta_0}{\eta(\xi)} + \frac{\Delta_0}{\theta(\xi)} \geq \Theta(x_n(\Delta_0)) > t_n(\Delta_0) - \frac{\Delta_0}{\eta(\xi)}$$

and taking limits as $n \to \infty$ obtains $\Theta(x_n(\Delta_0))/t_n(\Delta_0) \to 1$, as required. □

These constructions can also be used to generate examples of equations for which the solution to the differential equation does *not* explode, yet the numerical method described by (2.7.7) spuriously detects the presence of an explosion. The analysis is very similar and can be obtained simply by switching the roles of the functions η and θ in the definition of f.

One solution to the problem illustrated by these examples is to consider inserting monotonicity into the numerical method itself. Consider the ODE given by

$$w'(t) = f(w(t)), \quad t > 0; \quad w(0) = \psi(0), \tag{2.7.9}$$

where f obeys (0.1.7) and (2.2.1). Let $\Delta > 0$ and $\varepsilon \in (0, 1/2)$. Define the sequence

$(w_n(\Delta))_{n\geq 0}$ by

$$w_0(\Delta) = \psi(0) \tag{2.7.10a}$$

$$\begin{aligned} w_{n+1}(\Delta) &= w_n(\Delta) \\ &\quad + \inf\{w \in [(1-\varepsilon)\Delta, \Delta] : f(w + w_n(\Delta)) = \max_{z\in[(1-\varepsilon)\Delta,\Delta]} f(z + w_n(\Delta))\}. \end{aligned} \tag{2.7.10b}$$

The continuity of f ensures that this sequence is well-defined. Moreover, it can be seen that $(w_n(\Delta))_{n\geq 0}$ is increasing.

Next, define the sequence $(l_n(\Delta))_{n\geq 0}$ by

$$l_n(\Delta) = \frac{\inf\{w \in [(1-\varepsilon)\Delta, \Delta] : f(w + w_n(\Delta)) = \max_{z\in[(1-\varepsilon)\Delta,\Delta]} f(z + w_n(\Delta))\}}{f(w_n(\Delta))}. \tag{2.7.11}$$

This is well-defined because $f(w_n(\Delta)) > 0$ for all $n \geq 0$, due to the positivity of f and $(w_n(\Delta))_{n\geq 0}$; moreover $l_n(\Delta) > 0$ for all $n \geq 0$. Also define the sequence $(s_n(\Delta))_{n\geq 0}$ by

$$s_0(\Delta) = 0; \quad s_n(\Delta) = \sum_{j=0}^{n-1} l_j(\Delta), \quad n \geq 1. \tag{2.7.12}$$

The fact that $l_n(\Delta)$ is positive for each $n \geq 0$ forces $(s_n(\Delta))_{n\geq 0}$ to be an increasing sequence. We mention in passing that the infima in (2.7.10b) and also in (2.7.11) can be replaced by minima, due to the continuity of f. In the case when f is increasing, we have that $w_{n+1}(\Delta) = w_n(\Delta) + \Delta$ for $n \geq 0$, and therefore that $l_n(\Delta) = \Delta/f(w_n(\Delta))$, which shows that this method reduces to the scheme describes in Section 2.3 when f is monotone.

Now,

$$\begin{aligned} &w_{n+1}(\Delta) - w_n(\Delta) \\ &= \inf\{w \in [(1-\varepsilon)\Delta, \Delta] : f(w + w_n(\Delta)) = \max_{z\in[(1-\varepsilon)\Delta,\Delta]} f(z + w_n(\Delta))\} \\ &= l_n(\Delta) f(w_n(\Delta)). \end{aligned}$$

Since $w_{n+1}(\Delta) \in [w_n(\Delta)+(1-\varepsilon)\Delta, w_n(\Delta)+\Delta]$, evidently we have $w_n(\Delta) \to \infty$ as $n \to \infty$.

We now show that the approximation $w_n(\Delta)$ does indeed detect the presence of an explosion. This amounts to showing that the sequence $l_n(\Delta)$ is summable.

Theorem 2.7.1. *Suppose that f obeys* (0.1.7) *and* (2.2.1). *Suppose that $(w_n(\Delta))_{n\geq 0}$ is defined by* (2.7.10a). *Then there exists a finite $T_\Delta \in (0,\infty)$ such that*

$$T_\Delta = \lim_{n\to\infty} s_n(\Delta) = \sum_{j=0}^{\infty} l_j(\Delta). \tag{2.7.13}$$

While always predicting an explosion when it is present, this method suffers from the disadvantage of needing to determine the maximum of f on an interval, a problem which requires information about the behaviour of f at every point on the interval which therefore makes it hard to implement precisely and somewhat impractical. Moreover, we do not know whether it falsely predicts the presence of an explosion when one is not present in the underlying continuous time equation (2.7.10). Therefore this method does not satisfactorily deal with equations which do not make any sort of monotonicity assumption on f.

As mentioned in the introduction to this chapter, by virtue of the counterexamples described in this section we wish to inform the reader that we will be assuming that the instantaneous coefficient of the differential equation obeys some sort of monotonicity assumption for the remainder of the thesis.

2.8 Proofs

Proof of Theorem 2.2.2 To prove part (i), note that since f is increasing and f obeys (2.2.5), we have that $f(x) \to \infty$ as $x \to \infty$. Therefore $(f(x) - f(\delta/2))/f(x) \to 1$ as $x \to \infty$. Also since f is monotone and we have $f(x) - f(\delta/2) > 0$ for all $x > \delta/2$ and so by (2.2.5) for any $\delta' > \delta/2$ we have

$$\int_{\delta'}^{\infty} \frac{1}{f(x) - f(\delta/2)}\, dx < +\infty.$$

Therefore $\psi(0) > \delta/2$ can be chosen sufficiently large so that

$$\int_{\psi(0)}^{\infty} \frac{1}{f(u) - f(\delta/2)}\, du < \frac{\tau}{2}. \tag{2.8.1}$$

Since f is in $C^1(0,\infty)$, (2.2.4) has a solution on $[-\tau, T_\psi)$. Let us assume that there is no explosion so that the solution is defined on $(0,\tau/2]$. Now for $t \in [0,\tau/2]$, $x(t-\tau) = \psi(t-\tau) \in (0,\delta/2]$, so $0 < f(x(t-\tau)) < f(\delta/2)$. Hence

$$x'(t) \geq f(x(t)) - f(\delta/2), \quad t \in [0,\tau/2]; \quad x(0) = \psi(0).$$

We now prove that there exists $T_\psi \le \tau/2$ such that $\lim_{t\to T_\psi^-} x(t) = \infty$. Since $x(0) > \delta/2$, $x'(0) > 0$ and x is increasing at 0. Moreover, if there exists a minimal $t' \in [0, \tau/2]$ such that $x'(t') = 0$, then $x(t') > x(0) > \delta/2$ and so $0 = x'(t') \ge f(x(t')) - f(\delta/2) > 0$, a contradiction. Therefore x is increasing on $[0, \tau/2]$. Therefore

$$\frac{x'(t)}{f(x(t)) - f(\delta/2)} \ge 1, \quad t \in [0, \tau/2].$$

This implies

$$\int_{\psi(0)}^{x(t)} \frac{1}{f(u) - f(\delta/2)}\,du = \int_0^t \frac{x'(s)}{f(x(s)) - f(\delta/2)}\,ds \ge t, \quad t \in [0, \tau/2].$$

By assumption $x(t)$ is finite for all $t \in [0, \tau/2]$. Therefore there exists $x^* \in (\psi(0), \infty)$ such that $x(\tau/2) = x^*$. Thus

$$\int_{\psi(0)}^{x^*} \frac{1}{f(u) - f(\delta/2)}\,du = \lim_{t\to\tau/2-} \int_{\psi(0)}^{x(t)} \frac{1}{f(u) - f(\delta/2)}\,du \ge \frac{\tau}{2}.$$

By (2.8.1) we have

$$\frac{\tau}{2} > \int_{\psi(0)}^{\infty} \frac{1}{f(u) - f(\delta/2)}\,du > \int_{\psi(0)}^{x^*} \frac{1}{f(u) - f(\delta/2)}\,du \ge \frac{\tau}{2},$$

a contradiction. Therefore there exists $T_\psi < \tau/2$ such that $\lim_{t\to T_\psi^-} x(t) = \infty$, as claimed.

To prove part (ii), note that since $f \in C^1([0,\infty),(0,\infty))$, for every $\delta > 0$ there exists $K_\delta \ge 0$ given by $K_\delta := \max_{0\le x\le\delta} |f'(x)|$. That is for $y, z \in [0, \delta]$,

$$|f(y) - f(z)| \le K_\delta |y - z|, \quad y, z \in [0, \delta].$$

Now extend f to be defined on $[-\delta, 0]$ according to

$$f(x) = 2f(0) - f(-x), \quad x \in [-\delta, 0].$$

Thus $f'(x) = f'(-x)$ for $x \in [-\delta, 0]$ and so for $y, z \in [-\delta, \delta]$,

$$|f(y) - f(z)| \le K_\delta |y - z|, \quad y, z \in [-\delta, \delta]. \tag{2.8.2}$$

Also since $f'(0) = 0$, we have $K_\delta \to 0$ as $\delta \to 0^+$. Let $\delta > 0$ be so small that

$$K_\delta \tau < \frac{1}{2}. \tag{2.8.3}$$

Let

$$N = \{y \in C([-\tau, \infty); \mathbb{R}) : y_0 = \psi, |y(t)| \le \delta,\, t \ge 0\}.$$

Define for $y \in N$

$$(\Omega y)(t) = \begin{cases} \psi(t), & t \in [-\tau, 0], \\ \psi(0) + \int_0^t f(y(s))\,ds - \int_{-\tau}^{t-\tau} f(\psi(s))\,ds, & t \in [0, \tau], \\ \psi(0) + \int_{t-\tau}^t f(y(s))\,ds - \int_{-\tau}^0 f(\psi(s))\,ds, & t \geq \tau. \end{cases}$$

We will show that $\Omega : N \to N$ and that Ω is a contraction on $(N, \|\cdot\|)$ where $\|\cdot\|$ is the supnorm on $[-\tau, \infty)$.

For $t \in [0, \tau]$,

$$\begin{aligned}
(\Omega y)(t) &= \psi(0) + \int_0^t f(y(s))\,ds - \int_{-\tau}^{t-\tau} f(\psi(s))\,ds \\
&= \psi(0) + \int_0^t \left(f(y(s)) - f(\psi(s-\tau))\right)\,ds \\
&\leq \psi(0) + \int_0^t \left|f(y(s)) - f(\psi(s-\tau))\right|\,ds \\
&\leq \psi(0) + \int_0^t \left|f(\delta) - f(\psi(s-\tau))\right|\,ds \\
&\leq \psi(0) + \int_0^t K_\delta \left|\delta - \psi(s-\tau)\right|\,ds \\
&\leq \psi(0) + K_\delta \delta t \\
&\leq \psi(0) + K_\delta \delta \tau \leq \delta.
\end{aligned}$$

Also for $t \in [0, \tau]$,

$$(\Omega y)(t) \geq \psi(0) + \int_0^t \left(f(-\delta) - f(\psi(s-\tau))\right)\,ds.$$

Now

$$|f(-\delta) - f(\psi(s-\tau))| \leq K_\delta \left|-\delta - \psi(s-\tau)\right| = K_\delta \left|\delta + \psi(s-\tau)\right| = K_\delta \left(\delta + \psi(s-\tau)\right).$$

So

$$\begin{aligned}
(\Omega y)(t) &\geq \psi(0) + \int_0^t \left(f(-\delta) - f(\psi(s-\tau))\right)\,ds \\
&\geq \psi(0) + \int_0^t -K_\delta \left(\delta + \psi(s-\tau)\right)\,ds
\end{aligned}$$

.

Since $\psi(t) \leq \delta$ for $t \in [-\tau, 0]$, we have $\delta/2 \leq \delta + \psi(s-\tau) \leq 3\delta/2$ for $s \in [0, \tau]$. So

$-K_\delta\left(\delta+\psi(s-\tau)\right)\geq -K_\delta 3\delta/2$. Thus for $t\in[0,\tau]$

$$\begin{aligned}(\Omega y)(t)&\geq \psi(0)+\int_0^t -K_\delta 3\delta/2\,ds\\&\geq \psi(0)-K_\delta t3\delta/2\\&\geq \psi(0)-K_\delta \tau 3\delta/2\\&> \psi(0)-3\delta/4\\&> -3\delta/4>-\delta,\end{aligned}$$

where we have used (2.8.3), $\psi(0)>0$ and $t\leq\tau$.

Now for $t\geq\tau$,

$$\begin{aligned}(\Omega y)(t)&=\psi(0)+\int_{t-\tau}^t f(y(s))\,ds-\int_{-\tau}^0 f(\psi(s))\,ds\\&\leq \psi(0)+\int_{t-\tau}^t K_\delta\left|\delta-\psi(s)\right|\,ds\\&\leq \psi(0)+K_\delta\delta\tau\leq\delta.\end{aligned}$$

Also for $t\geq\tau$,

$$\begin{aligned}(\Omega y)(t)&\geq \psi(0)+\int_{t-\tau}^t \left(f(-\delta)-f(\psi(s))\right)\,ds\\&\geq \psi(0)+\int_{t-\tau}^t -K_\delta\left(\delta+\psi(s)\right)\,ds\\&\geq \psi(0)+\int_{t-\tau}^t -K_\delta 3\delta/2\,ds\\&\geq \psi(0)-K_\delta\tau 3\delta/2>-\delta,\end{aligned}$$

using similar arguments as before.

Therefore $|(\Omega y)(t)|\leq\delta$ for $t\in[-\tau,\infty)$. Hence $\Omega:N\to N$ and moreover $(N,\|\cdot\|)$ is a Banach space.

Suppose that $y,z\in N$. Then for $t\in[-\tau,0]$ we have $(\Omega y)(t)-(\Omega z)(t)=0$. For $t\in[0,\tau]$ we have

$$(\Omega y)(t)-(\Omega z)(t)=\int_0^t f(y(s))-f(z(s))\,ds.$$

Therefore

$$\begin{aligned}|(\Omega y)(t)-(\Omega z)(t)|&\leq\int_0^t|f(y(s))-f(z(s))|\,ds\leq\int_0^\tau K_\delta|y(s)-z(s)|\,ds\\&\leq K_\delta\tau\sup_{0\leq s\leq\tau}|y(s)-z(s)|,\end{aligned}$$

and so $\sup_{0\le t\le\tau}|(\Omega y)(t)-(\Omega z)(t)|\le K_\delta\tau\sup_{0\le s\le\tau}|y(s)-z(s)|$. For $t\ge\tau$ we have

$$(\Omega y)(t)-(\Omega z)(t)=\int_{t-\tau}^{t} f(y(s))-f(z(s))\,ds,$$

so

$$\begin{aligned}|(\Omega y)(t)-(\Omega z)(t)| &\le \int_{t-\tau}^{t}|f(y(s))-f(z(s))|\,ds\le\int_{t-\tau}^{t}K_\delta|y(s)-z(s)|\,ds\\ &\le K_\delta\tau\sup_{t-\tau\le s\le t}|y(s)-z(s)|.\end{aligned}$$

Thus

$$\begin{aligned}\|\Omega y-\Omega z\| &= \sup_{t\ge-\tau}|(\Omega y)(t)-(\Omega z)(t)|\\ &=\sup_{t\ge0}|(\Omega y)(t)-(\Omega z)(t)|\\ &=\max\left(\sup_{0\le t\le\tau}|(\Omega y)(t)-(\Omega z)(t)|,\sup_{t\ge\tau}|(\Omega y)(t)-(\Omega z)(t)|\right)\\ &\le\max\left(K_\delta\tau\sup_{0\le s\le\tau}|y(s)-z(s)|,K_\delta\tau\sup_{t\ge\tau}\sup_{t-\tau\le s\le t}|y(s)-z(s)|\right)\\ &=K_\delta\tau\max\left(\sup_{0\le s\le\tau}|y(s)-z(s)|,\sup_{s\ge0}|y(s)-z(s)|\right)\\ &=K_\delta\tau\sup_{s\ge0}|y(s)-z(s)|=K_\delta\tau\sup_{s\ge-\tau}|y(s)-z(s)|=K_\delta\tau\|y-z\|\\ &\le\frac{1}{2}\|y-z\|.\end{aligned}$$

Therefore Ω is a contraction.

Hence by the contraction mapping theorem, Ω has a unique fixed point in N. This fixed point is a function $x\in C([-\tau,\infty),\mathbb{R})$ obeying

(i) $x(t)=\psi(t)$ for $t\in[-\tau,0]$;

(ii) $x(t)=(\Omega x)(t)$ for $t\ge-\tau$;

(iii) $|x(t)|\le\delta$ for all $t\ge0$.

Since $(\Omega x)'(t)=f(x(t))-f(x(t-\tau))$ for all $t>0$, by (i) and (ii), x is a solution of (0.1.6). Moreover it is the unique continuous solution of (0.1.6). As f is increasing and ψ is increasing, x must be increasing and so $x(t)\to L'$ as $t\to\infty$ where $L'\in(\psi(0),\delta)\subset[-\delta,\delta]$.

Now since x is continuous

$$L' = \lim_{t\to\infty} x(t) = \lim_{t\to\infty} (\Omega x)(t) = \lim_{t\to\infty} \left(\psi(0) - \int_{-\tau}^{0} f(\psi(s))\, ds + \int_{t-\tau}^{t} f(x(s))\, ds \right)$$

$$= \psi(0) - \int_{-\tau}^{0} f(\psi(s))\, ds + \tau f(L') = \Psi L'$$

where

$$\Psi x = \psi(0) - \int_{-\tau}^{0} f(\psi(s))\, ds + \tau f(x).$$

We claim that $\Psi : M \to M$ where $M = ([-\delta, \delta], |\cdot|)$ and $|\cdot|$ is the standard absolute value function on $\mathbb{R}$. Let $x \in [-\delta, \delta]$. As f is increasing, $f(-\delta) \le f(x) \le f(\delta)$. Note also that $f(0) \le f(\psi(t)) \le f(\delta/2)$. Thus,

$$f(-\delta) - f(\delta/2) \le f(x) - f(\psi(s)) \le f(\delta) - f(0).$$

Now $|f(\delta) - f(0)| \le K_\delta |\delta - 0| = K_\delta \delta$. Also,

$$|f(-\delta) - f(\delta/2)| \le K_\delta | - \delta - \delta/2| = K_\delta | - 3\delta/2| = K_\delta 3\delta/2.$$

Thus,

$$\begin{aligned} \Psi x &= \psi(0) + \int_{-\tau}^{0} \left(f(x) - f(\psi(s))\right)\, ds \\ &\le \psi(0) + \int_{-\tau}^{0} K_\delta \delta\, ds \\ &= \psi(0) + K_\delta \tau \delta < \delta/2 + \delta/2 = \delta. \end{aligned}$$

Also,

$$\begin{aligned} \Psi x &\ge \psi(0) + \int_{-\tau}^{0} -K_\delta 3\delta/2\, ds \\ &= \psi(0) - K_\delta \tau 3\delta/2 \\ &> \psi(0) - 3\delta/4 > -3\delta/4 > -\delta. \end{aligned}$$

Therefore $\Psi : M \to M$. Now suppose that $|x|, |y| \le \delta$. Then

$$|\Psi x - \Psi y| = \tau |f(x) - f(y)| \le \tau K_\delta |x - y| < \frac{1}{2}|x - y|.$$

Thus Ψ is a contraction and therefore must have a unique fixed point in M. This is the number L in (2.2.6). Since $\Psi L' = L'$, we have that $L' = L$. Therefore $\lim_{t\to\infty} x(t) = L$.

Proof of Theorem 2.2.3 Since ψ is positive and continuous, there exists $\delta > 0$ such that

$$0 < \psi(t) \leq \delta, \quad t \in [-\tau, -\tau/2].$$

Since g is continuous there is $g_\delta \geq 0$ such that $g_\delta := \max_{x\in(0,\delta]} g(x)$. Since f is increasing and f obeys (2.2.1), we have that $f(x) \to \infty$ as $x \to \infty$. Therefore $(f(x) - g_\delta)/f(x) \to 1$ as $x \to \infty$. Also, because f is increasing, we have $f(x) - g_\delta > 0$ for all $x > f^{-1}(g_\delta)$, so by (2.2.1) for any $\delta' > f^{-1}(g_\delta)$ we have

$$\int_{\delta'}^{\infty} \frac{1}{f(x) - g_\delta}\, dx < +\infty.$$

Therefore $\psi(0) > f^{-1}(g_\delta)$ can be chosen sufficiently large so that

$$\int_{\psi(0)}^{\infty} \frac{1}{f(u) - g_\delta}\, du < \frac{\tau}{2}. \tag{2.8.4}$$

Since f is in $C^1(0,\infty)$, and g is continuous, (2.2.7a) has a solution on $[-\tau, T_\psi)$. Let us assume that there is no explosion on $(0, \tau/2]$. Therefore $x'(\tau/2)$ is finite. Now for $t \in [0, \tau/2]$, $x(t-\tau) = \psi(t-\tau) \in (0, \delta]$, so $0 < g(x(t-\tau)) \leq \max_{x\in(0,\delta]} g(x) = g_\delta$. Hence

$$x'(t) \geq f(x(t)) - g_\delta, \quad t \in [0, \tau/2]; \quad x(0) = \psi(0).$$

We now prove that there exists $T_\psi < \tau/2$ such that $\lim_{t\to T_\psi^-} x(t) = \infty$. Since $x(0) > f^{-1}(g_\delta)$, $x'(0) > 0$ and x is increasing at 0. Moreover, if there exists a minimal $t' \in (0, \tau/2)$ such that $x'(t') = 0$, then $x(t') > x(0) > f^{-1}(g_\delta)$ and so $0 = x'(t') \geq f(x(t')) - g_\delta > 0$, a contradiction. Therefore x is increasing on $[0, \tau/2]$. Therefore

$$\frac{x'(t)}{f(x(t)) - g_\delta} > 1, \quad t \in [0, \tau/2].$$

This implies

$$\int_{\psi(0)}^{x(t)} \frac{1}{f(u) - g_\delta}\, du = \int_0^t \frac{x'(s)}{f(x(s)) - g_\delta}\, ds \geq t, \quad t \in [0, \tau/2].$$

By assumption $x(t)$ is finite for all $t \in [0, \tau/2]$. Therefore there exists $x^* \in (\psi(0), \infty)$ such that $x(\tau/2) = x^*$. Thus

$$\int_{\psi(0)}^{x^*} \frac{1}{f(u) - g_\delta}\, du = \lim_{t\to\tau/2-} \int_{\psi(0)}^{x(t)} \frac{1}{f(u) - g_\delta}\, du \geq \frac{\tau}{2}.$$

By (2.8.1) we have

$$\frac{\tau}{2} > \int_{\psi(0)}^{\infty} \frac{1}{f(u) - g_\delta}\, du > \int_{\psi(0)}^{x^*} \frac{1}{f(u) - g_\delta}\, du \geq \frac{\tau}{2},$$

a contradiction. Therefore there exists $T_\psi < \tau/2$ such that $\lim_{t\to T_\psi^-} x(t) = \infty$, as claimed.

Proof of Theorem 2.2.4 We have $x'(0) = f(\psi(0)) - g(\psi(-\tau))$. Since ψ is decreasing and g is increasing, $-g(\psi(-\tau)) < -g(\psi(0))$. Hence $x'(0) < f(\psi(0)) - g(\psi(0)) < 0$ by (2.2.9). Suppose there exists a minimal $t_1 > 0$ such that $x(t_1) > 0$ and $x'(t_1) = 0$; therefore x is decreasing on $[-\tau, t_1)$. Therefore as $-g(x(t_1)) > -g(x(t_1 - \tau))$ we have

$$0 = x'(t_1) = f(x(t_1)) - g(x(t_1 - \tau)) < f(x(t_1)) - g(x(t_1)) < 0$$

a contradiction. Therefore x is decreasing on $[-\tau, t_2)$ where $t_2 = \inf\{t > 0 : x(t) = 0\}$.

Suppose first that $x(t) > 0$ for all $t > 0$. Therefore either $x(t) \to L > 0$ as $t \to \infty$ or $x(t) \to 0$ as $t \to \infty$. If the former is true, then $x'(t) \to f(L) - g(L) < 0$ as $t \to \infty$. Therefore $x(t) \to -\infty$ as $t \to \infty$, a contradiction. Therefore either $x(t) \to 0$ as $t \to \infty$ or there is a minimal $t_2 > 0$ such that $x(t_2) = 0$. In the former case, we have (2.2.10) with $L = 0$.

In the latter case, we have $x'(t_2) = f(x(t_2)) - g(x(t_2 - \tau)) = -g(x(t_2 - \tau)) < 0$. Suppose now that there is a minimal $t_3 \in [t_2, t_2 + \tau)$ such that $x'(t_3) = 0$. Then $x'(t) < 0$ for $t_2 \le t < t_3$ and hence $x(t) < 0$ for $t \in [t_2, t_3]$. Hence

$$0 = x'(t_3) = f(x(t_3)) - g(x(t_3 - \tau)) = -g(x(t_3 - \tau)),$$

so $g(x(t_3 - \tau)) = 0$. But $t_3 < t_2 + \tau$ implies $t_3 - \tau < t_2$, so $x(t_3 - \tau) > 0$ and therefore $g(x(t_3 - \tau)) > 0$, a contradiction. Therefore x is decreasing on $[t_2, t_2 + \tau)$. Moreover $x(t) < 0$ for $t \in (t_2, t_2 + \tau]$. We claim that $x(t) = x(t_2 + \tau)$ for all $t \ge t_2 + \tau$. To see this, let $y(t) = x(t_2 + \tau)$ for $t \ge t_2 + \tau$ and $y(t) = x(t)$ for $t \in [t_2, t_2 + \tau]$. Then for $t > t_2 + \tau$ we have $y'(t) = 0$ and $y(t - \tau) = x(t - \tau)$, so $g(y(t - \tau)) = g(x(t - \tau)) = 0$, because $x(t - \tau) < 0$ for $t > t_2 + \tau$. Hence for $t > t_2 + \tau$ we have

$$y'(t) - f(y(t)) + g(y(t - \tau)) = -f(x(t_2 + \tau)) + g(x(t - \tau)) = 0,$$

so $y'(t) = f(y(t)) - g(y(t - \tau))$ for $t > t_2 + \tau$ and $y(t) = x(t)$ for $t \in [t_2, t_2 + \tau]$. Therefore $x(t) = y(t) = x(t_2 + \tau)$ for $t \ge t_2 + \tau$. Thus we have (2.2.10) with $L = x(t_2 + \tau) < 0$.

Proof of Lemma 2.4.1 Define $R(x) = \frac{f(x)}{\phi(x)}$ for $x > 0$. By (2.2.3), $R(x) \to 1$ as $x \to \infty$. Therefore there exist $\underline{R}, \overline{R} \in (0, \infty)$ such that for $x > 0$

$$\underline{R}\phi(x) \le f(x) \le \overline{R}\phi(x). \tag{2.8.5}$$

Therefore for $x > 0$

$$\frac{1}{\underline{R}}\frac{1}{\phi(x)} \geq \frac{1}{f(x)} \geq \frac{1}{\overline{R}}\frac{1}{\phi(x)}$$

and (2.2.1) holds if and only if $\int_1^\infty \frac{du}{\phi(u)} < \infty$. Since $x_n > 0$ for $n \geq 0$, by (2.8.5) we have for $n \geq 0$

$$\frac{1}{\underline{R}}\frac{1}{\phi(x_n(\Delta))} \geq \frac{1}{f(x_n(\Delta))} \geq \frac{1}{\overline{R}}\frac{1}{\phi(x_n(\Delta))}$$

So $\sum_{j=0}^\infty \frac{\Delta}{f(x_n(\Delta))} < \infty$ if and only if $\sum_{j=0}^\infty \frac{\Delta}{\phi(x_n(\Delta))} < \infty$.

Proof of Theorem 2.4.1 Notice that (2.3.4) and (2.3.5) imply for $n \geq 0$ that

$$x_{n+1}(\Delta) = x_n(\Delta) + \Delta + \frac{\Delta}{f(x_n(\Delta))} g(\bar{X}_\Delta(t_n(\Delta) - \tau)) > x_n(\Delta) + \Delta.$$

Hence

$$x_n(\Delta) \geq \psi(0) + n\Delta, \quad n \geq 0, \tag{2.8.6}$$

so $x_n(\Delta) \to \infty$ as $n \to \infty$. Next as $(t_n(\Delta))_{n\geq 0}$ is an increasing sequence, we notice that there exists $T_\Delta \in (0, \infty]$ such that

$$T_\Delta := \lim_{n\to\infty} t_n(\Delta).$$

Since $\bar{X}_\Delta(t) \geq x_n(\Delta)$ for all $t \in [t_n(\Delta), t_{n+1}(\Delta))$, we have

$$\lim_{t\to T_\Delta^-} \bar{X}_\Delta(t) = \infty.$$

Moreover, by (2.3.8c) and (2.4.3), the domain of definition of $\bar{X}_\Delta$ is $[-\tau, T_\Delta)$, $\bar{X}_\Delta$ is continuous on $[-\tau, T_\Delta)$ and $\bar{X}_\Delta$ is increasing on $[0, T_\Delta)$.

It remains to show that $T_\Delta < +\infty$. By (2.3.4), we have that

$$t_{n+1}(\Delta) = \sum_{j=0}^n \frac{\Delta}{f(x_j(\Delta))}, \quad n \geq 0. \tag{2.8.7}$$

Define $k_n(\Delta)$ for $n \geq 0$ by

$$k_{n+1}(\Delta) = \sum_{j=0}^n \frac{\Delta}{\phi(x_j(\Delta))}, \quad n \geq 0. \tag{2.8.8}$$

Now by (2.4.2), $t_n(\Delta)$ is summable if and only if $k_n(\Delta)$ is summable. Moreover

$$T_\Delta = \sum_{j=0}^\infty \frac{\Delta}{f(x_j(\Delta))} \leq \frac{1}{\underline{R}} \sum_{j=0}^\infty \frac{\Delta}{\phi(x_j(\Delta))}.$$

Since ϕ is monotone, by (2.8.6) we have that

$$\phi(x_j(\Delta)) \geq \phi(\psi(0) + j\Delta), \quad n \geq 0.$$

Hence

$$\begin{aligned} T_\Delta &\leq \frac{1}{\underline{R}} \sum_{j=0}^{\infty} \frac{\Delta}{\phi(\psi(0) + j\Delta)} \\ &= \frac{\Delta}{\underline{R}\phi(\psi(0))} + \frac{1}{\underline{R}} \sum_{j=0}^{\infty} \frac{\Delta}{\phi(\psi(0) + (j+1)\Delta)}. \end{aligned}$$

Next note that since ϕ is monotone, for $j \geq 0$

$$\frac{\Delta}{\phi(\psi(0) + (j+1)\Delta)} \leq \int_{\psi(0)+j\Delta}^{\psi(0)+(j+1)\Delta} \frac{1}{\phi(u)}\, du.$$

and so summing over $j \geq 0$ obtains

$$T_\Delta \leq \frac{\Delta}{\underline{R}\phi(\psi(0))} + \frac{1}{\underline{R}} \int_{\psi(0)}^{\infty} \frac{1}{\phi(u)}\, du.$$

Proof of Theorem 2.5.1 From (0.1.6), (2.3.7) and Remark 2.3.1 we have that

$$x(t) = \psi(0) + \int_0^t f(x(s))ds + \int_0^t g(x(s-\tau))ds, \tag{2.8.9}$$

$$\bar{X}_\Delta(t) = \psi(0) + \int_0^t f(X_\Delta(s))ds + \int_0^t g(X_\Delta(s-\tau))ds. \tag{2.8.10}$$

Notice moreover that

$$\bar{X}_\Delta(t) = \bar{X}_\Delta(t_n(\Delta)) + \int_{t_n(\Delta)}^t f(X_\Delta(s))ds + \int_{t_n(\Delta)}^t g(X_\Delta(s-\tau))ds. \tag{2.8.11}$$

The method of the proof is to develop a Gronwall-like inequality for $\sup_{t\in[0,\rho_M\wedge\bar{\rho}_{2M}(\Delta)]} \left|x(t) - \bar{X}_\Delta(t)\right|$, and then take the limit as $\Delta \to 0$.

Now, subtracting (2.8.9) from (2.8.10) gives for any $t \in [0, \rho_M \wedge \bar{\rho}_{2M}(\Delta)]$,

$$\begin{aligned} \left|x(t) - \bar{X}_\Delta(t)\right| &= \left| \int_0^t \left(f(x(s)) + g(x(s-\tau)) - f(X_\Delta(s)) - g(X_\Delta(s-\tau))\right) ds \right| \\ &\leq \int_0^t \left(|f(x(s)) - f(X_\Delta(s))| + |g(x(s-\tau)) - g(X_\Delta(s-\tau))|\right) ds. \end{aligned}$$

This implies that

$$\begin{aligned}
&\sup_{t\in[0,\rho_M\wedge\bar\rho_{2M}(\Delta)]}\left|x(t)-\bar X_\Delta(t)\right|\\
&\quad\le\int_0^{\rho_M\wedge\bar\rho_{2M}(\Delta)}\left(|f(x(s))-f(X_\Delta(s))|+|g(x(s-\tau))-g(X_\Delta(s-\tau))|\right)ds\\
&\quad\le\int_0^{\rho_M\wedge\bar\rho_{2M}(\Delta)}\Big(\left|f(x(s))-f(\bar X_\Delta(s))\right|+\left|g(x(s-\tau))-g(\bar X_\Delta(s-\tau))\right|\\
&\qquad\quad+\left|f(\bar X_\Delta(s))-f(X_\Delta(s))\right|+\left|g(\bar X_\Delta(s-\tau))-g(X_\Delta(s-\tau))\right|\Big)ds.
\end{aligned}$$

From (0.1.7) and (0.1.8), recall that f and g are locally Lipschitz. That is for all $M>\psi^*$ there exists c_M such that $|f(x)-f(y)|,|g(x)-g(y)|\le c_M\forall x,y\in[0,M]$. Therefore

$$\begin{aligned}
\sup_{t\in[0,\rho_M\wedge\bar\rho_{2M}(\Delta)]}\left|x(t)-\bar X_\Delta(t)\right|&\le c_{2M}\Big(\int_0^{\rho_M\wedge\bar\rho_{2M}(\Delta)}\left|\bar X_\Delta(s)-X_\Delta(s)\right|ds\\
&\qquad+\int_0^{\rho_M\wedge\bar\rho_{2M}(\Delta)}\left|\bar X_\Delta(s-\tau)-X_\Delta(s-\tau)\right|ds\Big)\\
&+\int_0^{\rho_M\wedge\bar\rho_{2M}(\Delta)}c_{2M}\left(\left|x(s)-\bar X_\Delta(s)\right|+\left|x(s-\tau)-\bar X_\Delta(s-\tau)\right|\right)ds. \quad (2.8.12)
\end{aligned}$$

In order to apply our integral inequality, we develop an estimate for the first two integrals on the r.h.s. of (2.8.12), which can be achieved by estimating $|\bar X_\Delta-X_\Delta|$. Given $s\in[0,\rho_M\wedge\bar\rho_{2M}(\Delta))$ let n be the integer for which $s\in[t_n(\Delta),t_{n+1}(\Delta))$. Then, because X_Δ is piecewise constant, by (2.8.11) we get

$$\begin{aligned}
&\left|\bar X_\Delta(s)-X_\Delta(s)\right|\\
&\quad=\left|\bar X_\Delta(t_n(\Delta))+\int_{t_n(\Delta)}^s f(X_\Delta(u))du+\int_{t_n(\Delta)}^s g(X_\Delta(u-\tau))du-X_\Delta(t_n(\Delta))\right|\\
&\quad=\left|\int_{t_n(\Delta)}^s f(X_\Delta(t_n(\Delta)))du+\int_{t_n(\Delta)}^s g(X_\Delta(u-\tau))du\right|\\
&\quad\le\int_{t_n(\Delta)}^s|f(X_\Delta(t_n(\Delta)))|\,du+\int_{t_n(\Delta)}^s|g(X_\Delta(u-\tau))|\,du\\
&\quad\le(s-t_n(\Delta))\,|f(X_\Delta(t_n(\Delta)))|+\int_{t_n(\Delta)}^s g(X_\Delta(u-\tau))du,
\end{aligned}$$

since g is positive. Now since $x_n(\Delta)\ge\psi(0)$ for $n\ge0$, we have that

$$\inf_{n\ge0}f(x_n(\Delta))\ge\inf_{x\ge\psi(0)}f(x)=:L(f,\psi)>0$$

by assumption and so $t_{n+1}(\Delta)-t_n(\Delta)=\Delta/f(x_n(\Delta))\le\Delta/L(f,\psi)$. Note that if f is monotone this implies $L(f,\psi)>0$. Using this and the fact that f is locally Lipschitz, we have

$$\begin{aligned}
&\left|\bar{X}_\Delta(s) - X_\Delta(s)\right| \\
&\quad \le \frac{\Delta}{L(f,\psi)} \left|f(X_\Delta(t_n(\Delta))) - f(\psi(0))\right| + \left|f(\psi(0))\right| + \int_{t_n(\Delta)}^{s} g(X_\Delta(u-\tau))du \\
&\quad \le \frac{\Delta}{L(f,\psi)} \left(c_{2M}\left|X_\Delta(t_n(\Delta)) - \psi(0)\right| + f(\psi(0))\right) + \int_{t_n(\Delta)}^{s} g(X_\Delta(u-\tau))du,
\end{aligned}$$

since $\psi(0) \le \psi^* < M < 2M$ and as $t_n(\Delta) \le s \le \rho_M \wedge \bar{\rho}_{2M}(\Delta)$, by (2.5.2) we have

$$0 < X_\Delta(t_n(\Delta)) = \bar{X}_\Delta(t_n(\Delta)) \le 2M.$$

Hence for $s \in [t_n(\Delta), t_{n+1}(\Delta))$

$$\begin{aligned}
&\left|\bar{X}_\Delta(s) - X_\Delta(s)\right| \\
&\quad \le \frac{\Delta}{L(f,\psi)} \left(2Mc_{2M} + f(\psi(0))\right) + \int_{t_n(\Delta)}^{s} g(X_\Delta(u-\tau))du. \qquad (2.8.13)
\end{aligned}$$

Next we estimate the integral on the right hand side of (2.8.13). Firstly

$$\begin{aligned}
\int_{t_n(\Delta)}^{s} \left|g(X_\Delta(u-\tau))\right| du &\le \int_{t_n(\Delta)}^{t_{n+1}(\Delta)} \left|g(X_\Delta(u-\tau))\right| du \\
&\le \left(t_{n+1}(\Delta) - t_n(\Delta)\right) \max_{t_n(\Delta) \le u \le t_{n+1}(\Delta)} \left|g(X_\Delta(u-\tau))\right| \\
&= \left(t_{n+1}(\Delta) - t_n(\Delta)\right) \left|g(X_\Delta(r_n(\Delta)-\tau))\right|
\end{aligned}$$

where $r_n(\Delta) \in [t_n(\Delta), t_{n+1}(\Delta))$ and we have used the fact that by Remark (2.3.2), $X_\Delta(s-\tau)$ can only assume finitely many distinct values for $s \in [t_n(\Delta), t_{n+1}(\Delta)]$. Since g is Lipschitz continuous and $r_n(\Delta) - \tau < \rho_{2M}$, we have

$$\begin{aligned}
\int_{t_n(\Delta)}^{s} \left|g(X_\Delta(u-\tau))\right| du &\le \frac{\Delta}{L(f,\psi)} \left|g(X_\Delta(r_n(\Delta)-\tau)) - g(\psi(0)) + g(\psi(0))\right| \\
&\le \frac{\Delta}{L(f,\psi)} \left(2Mc_{2M} + g(\psi(0))\right). \qquad (2.8.14)
\end{aligned}$$

Combining (2.8.13) and (2.8.14) gives an estimate for the first integral in (2.8.12), which is

$$\begin{aligned}
&\int_0^{\rho_M \wedge \bar{\rho}_{2M}(\Delta)} \left|\bar{X}_\Delta(s) - X_\Delta(s)\right| ds \\
&\quad \le (\rho_M \wedge \bar{\rho}_{2M}(\Delta)) \frac{\Delta}{L(f,\psi)} \left(4Mc_{2M} + f(\psi(0)) + g(\psi(0))\right) \\
&\quad \le \rho_M \frac{\Delta}{L(f,\psi)} \left(4Mc_{2M} + f(\psi(0)) + g(\psi(0))\right). \qquad (2.8.15)
\end{aligned}$$

For the second integral in (2.8.12), we have

$$\int_0^{\rho_M\wedge\bar{\rho}_{2M}(\Delta)} \left|\bar{X}_\Delta(s-\tau)-X_\Delta(s-\tau)\right| ds = \int_{-\tau}^{\rho_M\wedge\bar{\rho}_{2M}(\Delta)-\tau} \left|\bar{X}_\Delta(s)-X_\Delta(s)\right| ds$$

To evaluate this integral, we need to consider two distinct cases. Firstly if $\rho_M\wedge\bar{\rho}_{2M}(\Delta)-\tau\le 0$ then there exists $m\in\{-N_\Delta,-N_\Delta+1,\dots,-1\}$ such that $t_m(\Delta)\le\rho_M\wedge\bar{\rho}_{2M}(\Delta)-\tau$ and $t_{m+1}(\Delta)>\rho_M\wedge\bar{\rho}_{2M}(\Delta)-\tau$, and so

$$\begin{aligned}
&\int_{-\tau}^{\rho_M\wedge\bar{\rho}_{2M}(\Delta)-\tau} \left|\bar{X}_\Delta(s)-X_\Delta(s)\right| ds\\
&\quad=\sum_{j=-N_\Delta}^{m-1}\int_{t_j(\Delta)}^{t_{j+1}(\Delta)} \left|\bar{X}_\Delta(s)-X_\Delta(s)\right| ds+\int_{t_m(\Delta)}^{\rho_M\wedge\bar{\rho}_{2M}(\Delta)-\tau} \left|\bar{X}_\Delta(s)-X_\Delta(s)\right| ds\\
&\quad=\sum_{j=-N_\Delta}^{m-1}\int_{t_j(\Delta)}^{t_{j+1}(\Delta)} |\psi(s)-\psi(t_j(\Delta))|\, ds+\int_{t_m(\Delta)}^{\rho_M\wedge\bar{\rho}_{2M}(\Delta)-\tau} |\psi(s)-\psi(t_m(\Delta))|\, ds,
\end{aligned}$$

where we have used the fact that $\bar{X}_\Delta(s)=\psi(s)$ for $s\le\rho_M\wedge\bar{\rho}_{2M}(\Delta)-\tau\le 0$ and $X_\Delta(s)=\psi(t_j(\Delta))$ where $t_j(\Delta)\le s<t_{j+1}(\Delta)$.

Now since $|t_{j+1}(\Delta)-t_j(\Delta)|=t_{j+1}(\Delta)-t_j(\Delta)<2\Delta/f(\psi(0))$, for $s\in[t_j(\Delta),t_{j+1}(\Delta))$ we have for every $j\in\{-N_\Delta,-N_\Delta+1,\dots,-1\}$

$$\begin{aligned}
\sup_{t_j(\Delta)\le s\le t_{j+1}(\Delta)}|\psi(s)-\psi(t_j(\Delta))| &= \sup_{0\le s-t_j(\Delta)\le t_{j+1}(\Delta)-t_j(\Delta)}|\psi(s)-\psi(t_j(\Delta))|\\
&\le \sup_{0\le s-t_j(\Delta)<2\Delta/f(\psi(0))}|\psi(s)-\psi(t_j(\Delta))|\\
&\le \sup_{s,u\in[-\tau,0]:0\le s-u<2\Delta/f(\psi(0))}|\psi(s)-\psi(u)|\\
&=\omega_\psi\left(2\Delta/f(\psi(0))\right).
\end{aligned}$$

where ω_ψ is a modulus of continuity of the continuous function ψ.

Hence for $\rho_M\wedge\bar{\rho}_{2M}(\Delta)-\tau\le 0$

$$\begin{aligned}
&\int_{-\tau}^{\rho_M\wedge\bar{\rho}_{2M}(\Delta)-\tau} \left|\bar{X}_\Delta(s)-X_\Delta(s)\right| ds\\
&\quad\le\sum_{j=-N_\Delta}^{m-1}(t_{j+1}(\Delta)-t_j(\Delta))\omega_\psi\left(2\Delta/f(\psi(0))\right)+(t_{m+1}(\Delta)-t_m(\Delta))\omega_\psi\left(2\Delta/f(\psi(0))\right)\\
&\quad=\sum_{j=-N_\Delta}^{m}(t_{j+1}(\Delta)-t_j(\Delta))\omega_\psi\left(2\Delta/f(\psi(0))\right)\\
&\quad\le\frac{2N_\Delta\Delta}{f(\psi(0))}\omega_\psi\left(2\Delta/f(\psi(0))\right)\\
&\quad\le 2\tau\omega_\psi\left(2\Delta/f(\psi(0))\right). \qquad (2.8.16)
\end{aligned}$$

Secondly if $\rho_M \wedge \bar{\rho}_{2M}(\Delta) - \tau > 0$ then

$$\begin{aligned}\int_{-\tau}^{\rho_M \wedge \bar{\rho}_{2M}(\Delta)-\tau} &\left|\bar{X}_\Delta(s) - X_\Delta(s)\right| ds \\ &= \int_{-\tau}^{0} \left|\bar{X}_\Delta(s) - X_\Delta(s)\right| ds + \int_{0}^{\rho_M \wedge \bar{\rho}_{2M}(\Delta)-\tau} \left|\bar{X}_\Delta(s) - X_\Delta(s)\right| ds.\end{aligned}$$

The argument used to establish (2.8.16) yields

$$\int_{-\tau}^{0} \left|\bar{X}_\Delta(s) - X_\Delta(s)\right| ds \leq 2\tau\omega_\psi \left(2\Delta/f(\psi(0))\right),$$

and because (2.8.15) holds we get

$$\begin{aligned}\int_{0}^{\rho_M \wedge \bar{\rho}_{2M}(\Delta)-\tau} &\left|\bar{X}_\Delta(s) - X_\Delta(s)\right| ds \\ &\leq \int_{0}^{\rho_M \wedge \bar{\rho}_{2M}(\Delta)} \left|\bar{X}_\Delta(s) - X_\Delta(s)\right| ds \\ &\leq \rho_M \frac{\Delta}{L(f,\psi)} \left(4Mc_{2M} + f(\psi(0)) + g(\psi(0))\right).\end{aligned}$$

Together they give the estimate for $\rho_M \wedge \bar{\rho}_{2M}(\Delta) - \tau > 0$

$$\begin{aligned}\int_{-\tau}^{\rho_M \wedge \bar{\rho}_{2M}(\Delta)-\tau} &\left|\bar{X}_\Delta(s) - X_\Delta(s)\right| ds \\ &\leq 2\tau\omega_\psi \left(2\Delta/f(\psi(0))\right) + \rho_M \frac{\Delta}{L(f,\psi)} \left(4Mc_{2M} + f(\psi(0)) + g(\psi(0))\right).\end{aligned} \tag{2.8.17}$$

Note by combining (2.8.16) and (2.8.17) we get a bound that covers both cases for $\rho_M \wedge \bar{\rho}_{2M}(\Delta) - \tau$:

$$\begin{aligned}\int_{-\tau}^{\rho_M \wedge \bar{\rho}_{2M}(\Delta)-\tau} &\left|\bar{X}_\Delta(s) - X_\Delta(s)\right| ds \\ &\leq 2\tau\omega_\psi \left(2\Delta/f(\psi(0))\right) + \rho_M \frac{\Delta}{L(f,\psi)} \left(4Mc_{2M} + f(\psi(0)) + g(\psi(0))\right).\end{aligned} \tag{2.8.18}$$

Defining

$$\begin{aligned}\omega(\Delta, 2M) &:= c_{2M}\omega_\psi \left(2\Delta/f(\psi(0))\right) \\ K_{2M} &:= 2c_{2M} \frac{1}{L(f,\psi)} \left(4Mc_{2M} + f(\psi(0)) + g(\psi(0))\right)\end{aligned}$$

and by inserting (2.8.18), (2.8.15) into (2.8.12) we obtain

$$\begin{aligned}\sup_{t\in[0,\rho_M \wedge \bar{\rho}_{2M}(\Delta)]} &\left|x(t) - \bar{X}_\Delta(t)\right| \\ &\leq \Delta\rho_M K_{2M} + 2\tau\omega(\Delta, 2M) \\ &\quad + \int_{0}^{\rho_M \wedge \bar{\rho}_{2M}(\Delta)} c_{2M} \left(\left|x(s) - \bar{X}_\Delta(s)\right| + \left|x(s-\tau) - \bar{X}_\Delta(s-\tau)\right|\right) ds.\end{aligned}$$

Now, let $e^*(s) := \sup_{-\tau\le t\le s}|x(t)-\bar{X}_\Delta(t)|$ and note that

$$\begin{aligned} e^*(\rho_M \wedge \bar{\rho}_{2M}(\Delta)) &= \sup_{-\tau\le t\le \rho_M\wedge\bar{\rho}_{2M}(\Delta)} |x(t)-\bar{X}_\Delta(t)| \\ &= \sup_{0\le t\le\ \rho_M\wedge\bar{\rho}_{2M}(\Delta)} |x(t)-\bar{X}_\Delta(t)|. \end{aligned}$$

Therefore

$$\begin{aligned} &e^*(\rho_M \wedge \bar{\rho}_{2M}(\Delta)) \\ &\quad\le \Delta\rho_M K_{2M} + 2\tau\omega(\Delta,2M) + \int_0^{\rho_M\wedge\bar{\rho}_{2M}(\Delta)} c_{2M}\left(e^*(s)+e^*(s)\right)ds \\ &\quad\le \Delta\rho_M K_{2M} + 2\tau\omega(\Delta,2M) + 2c_{2M}\int_0^{\rho_M\wedge\bar{\rho}_{2M}(\Delta)} e^*(s)ds. \end{aligned}$$

Now, by Gronwall's inequality

$$\begin{aligned} e^*(\rho_M \wedge \bar{\rho}_{2M}(\Delta)) &\le (\Delta\rho_M K_{2M} + 2\tau\omega(\Delta,2M))e^{(\rho_M\wedge\bar{\rho}_{2M}(\Delta))2c_{2M}} \\ &\le (\Delta\rho_M K_{2M} + 2\tau\omega(\Delta,2M))e^{\rho_M 2c_{2M}}. \end{aligned}$$

Since ω_ψ is a modulus of continuity of ψ, $\lim_{\delta\to0}\omega_\psi(\delta)=0$ and so $\lim_{\Delta\to0}\omega(\Delta,2M)=0$. Hence taking limits as $\Delta\to0$ yields the desired result.

Proof of Lemma 2.5.1 First we show that there exists a $\Delta_1(M)$ such that $\bar{\rho}_{2M}(\Delta)\ge\rho_M$ for $\Delta<\Delta_1(M)$.

Given any $\Delta\in(0,\tau f(\psi(0)))$, either

$$\bar{X}_\Delta(\bar{\rho}_{2M}(\Delta)) \le x(\bar{\rho}_{2M}(\Delta)) \quad \text{or} \quad \bar{X}_\Delta(\bar{\rho}_{2M}(\Delta)) > x(\bar{\rho}_{2M}(\Delta)).$$

In the former case, this implies that $\bar{\rho}_{2M}(\Delta)\ge\rho_{2M}>\rho_M$.

Now if $\bar{X}_\Delta(\bar{\rho}_{2M}(\Delta)) > x(\bar{\rho}_{2M}(\Delta))$,

$$\bar{X}_\Delta(\bar{\rho}_{2M}(\Delta)) - x(\bar{\rho}_{2M}(\Delta)) \le \sup_{0\le t\le\bar{\rho}_{2M}(\Delta)}\left|\bar{X}_\Delta(t)-x(t)\right|,$$

and using the techniques of Theorem 2.5.1, it can be readily shown that since $\bar{X}_\Delta(t)\le 2M$ on $t\in[0,\bar{\rho}_{2M}(\Delta)]$,

$$\sup_{0\le t\le\bar{\rho}_{2M}(\Delta)}\left|\bar{X}_\Delta(t)-x(t)\right| \le (\Delta\bar{\rho}_{2M}(\Delta)K_{2M}+2\tau\omega(\Delta,2M))e^{\bar{\rho}_{2M}(\Delta)(c^f_{2M}+c^g_{2M})}.$$

Now since $\bar{X}_\Delta$ is increasing, $\bar{\rho}_{2M}(\Delta) < T_\Delta$. Hence

$$\sup_{0\le t\le \bar{\rho}_{2M}(\Delta)} \left|\bar{X}_\Delta(t) - x(t)\right| \le (\Delta T_\Delta K_{2M} + 2\tau\omega(\Delta, 2M))e^{T_\Delta(c^f_{2M}+c^g_{2M})}.$$

By (2.4.6), given any $\Delta > 0$ there exists $C > 0$ such that $T_\Delta < C$. Thus

$$\sup_{0\le t\le \bar{\rho}_{2M}(\Delta)} \left|\bar{X}_\Delta(t) - x(t)\right| \le (\Delta C K_{2M} + 2\tau\omega(\Delta, 2M))e^{C(c^f_{2M}+c^g_{2M})},$$

and since $\omega(\Delta, M) \to 0$ as $\Delta \to 0$, $\sup_{0\le t\le \bar{\rho}_{2M}(\Delta)} \left|\bar{X}_\Delta(t) - x(t)\right| \to 0$ as $\Delta \to 0$ for any $M > \psi^*$. Therefore for any $M > \psi^*$ we can choose $\Delta_1 := \Delta_1(M)$ such that

$$\bar{X}_{\Delta_1}(\bar{\rho}_{2M}(\Delta_1)) - x(\bar{\rho}_{2M}(\Delta_1)) \le M.$$

Since we have $\bar{X}_{\Delta_1}(\bar{\rho}_{2M}(\Delta_1)) = 2M$, we have $x(\bar{\rho}_{2M}(\Delta_1)) \ge M$ and so $\bar{\rho}_{2M}(\Delta_1) \ge \rho_M$.

Now for any $\Delta < \Delta_1$, either $\bar{X}_\Delta(\rho_{2M}) \le x(\rho_{2M})$, which implies $\bar{\rho}_{2M}(\Delta) \ge \rho_{2M} > \rho_M$, or $\bar{X}_\Delta(\rho_{2M}) > x(\rho_{2M})$. In the latter case, we have

$$\begin{aligned}\sup_{0\le t\le \bar{\rho}_{2M}(\Delta)} \left|\bar{X}_\Delta(t) - x(t)\right| &\le (\Delta C K_{2M} + 2\tau\omega(\Delta, 2M))e^{C(c^f_{2M}+c^g_{2M})}\\ &< (\Delta_1 C K_{2M} + 2\tau\omega(\Delta_1, 2M))e^{C(c^f_{2M}+c^g_{2M})}\\ &\le M,\end{aligned}$$

since ω_ψ is a modulus of continuity of ψ and is non-decreasing, hence $\omega(\Delta, M)$ is non-decreasing in Δ. Combining both cases, we see that for any $M > \psi^*$ we can find $\Delta_1(M) > 0$ such that

$$\bar{\rho}_{2M}(\Delta) \ge \rho_M \text{ for all } \Delta < \Delta_1(M). \tag{2.8.19}$$

We can similarly show that there exists a $\Delta_2(M)$ such that $\rho_{2M} \ge \bar{\rho}_M(\Delta)$ for $\Delta < \Delta_2(M)$. Given any $\Delta \in (0, \tau f(\psi(0)))$, either $x(\rho_{2M}) \le \bar{X}_\Delta(\rho_{2M})$ or $x(\rho_{2M}) > \bar{X}_\Delta(\rho_{2M})$. In the former case, this imples that $\rho_{2M} \ge \bar{\rho}_{2M}(\Delta) > \bar{\rho}_M(\Delta)$.

Now if $x(\rho_{2M}) > \bar{X}_\Delta(\rho_{2M})$

$$x(\rho_{2M}) - \bar{X}_\Delta(\rho_{2M}) \le \sup_{0\le t\le \rho_{2M}} \left|x(t) - \bar{X}_\Delta(t)\right|$$

Using Theorem 2.5.1 it can be readily shown that since $x(\rho_{2M}) > \bar{X}_\Delta(\rho_{2M})$ implies $\bar{X}_\Delta(\rho_{2M}) < 2M$ and hence $\bar{X}_\Delta(t) < 2M$ on $t \in [0, \rho_{2M}]$,

$$\sup_{0\le t\le \rho_{2M}} \left|x(t) - \bar{X}_\Delta(t)\right| \le (\Delta\rho_{2M} K_{2M} + 2\tau\omega(\Delta, 2M))e^{\rho_{2M}(c^f_{2M}+c^g_{2M})},$$

and since $\omega(\Delta, M) \to 0$ as $\Delta \to 0$, $\sup_{0\le t\le\rho_{2M}} \left|\bar{X}_\Delta(t) - x(t)\right| \to 0$ as $\Delta \to 0$ for any $M > \psi^*$. Therefore for any $M > \psi^*$ we can choose $\Delta_2 := \Delta_2(M)$ such that

$$x(\rho_{2M}) - \bar{X}_{\Delta_2}(\rho_{2M}) \le M.$$

Hence $\rho_{2M} \ge \bar{\rho}_M(\Delta_2)$.

Now for any $\Delta < \Delta_2$, either $x(\rho_{2M}) \le \bar{X}_\Delta(\rho_{2M})$, which implies $\rho_{2M} \ge \bar{\rho}_{2M}(\Delta) > \bar{\rho}_M(\Delta)$, or $x(\rho_{2M}) > \bar{X}_\Delta(\rho_{2M})$. In the latter case, we have

$$\begin{aligned}\sup_{0\le t\le\rho_{2M}(\Delta)} \left|x(t) - \bar{X}_\Delta(t)\right| &\le (\Delta\rho_{2M}K_{2M} + 2\tau\omega(\Delta, 2M))e^{\rho_{2M}(c^f_{2M}+c^g_{2M})} \\ &< (\Delta_2\rho_{2M}K_{2M} + 2\tau\omega(\Delta_2, 2M))e^{\rho_{2M}(c^f_{2M}+c^g_{2M})} \\ &\le M,\end{aligned}$$

Combining both cases, we see that for any $M > \psi^*$ we can find $\Delta_2(M) > 0$ such that

$$\rho_{2M} \ge \bar{\rho}_M(\Delta) \text{ for all } \Delta < \Delta_2(M). \tag{2.8.20}$$

Taking $\Delta(M) = \min(\Delta_1(M), \Delta_2(M))$ yields the desired result.

Proof of Theorem 2.6.2 By Lemma 2.6.1, for every $\varepsilon \in (0, 1)$ there exists $L_1(\varepsilon) > 0$ such that

$$(1-\varepsilon)j\Delta < x_j(\Delta) < (1+\varepsilon)j\Delta, \quad j > L_1(\varepsilon). \tag{2.8.21}$$

Also by condition (2.2.3), given any $\varepsilon \in (0, 1)$ there exists $x_\varepsilon > 0$ such that

$$(1-\varepsilon)\phi(x) < f(x) < (1+\varepsilon)\phi(x), \quad x > x_\varepsilon,$$

and therefore as $x_n(\Delta) \to \infty$ as $x \to \infty$ there exists $L_2(\varepsilon) > 0$ such that

$$(1-\varepsilon)\phi(x_j(\Delta)) < f(x_j(\Delta)) < (1+\varepsilon)\phi(x_j(\Delta)), \quad j > L_2(\varepsilon). \tag{2.8.22}$$

Combining (2.8.21) and (2.8.22) and using the monotonicity of ϕ we have

$$(1-\varepsilon)\phi((1-\varepsilon)j\Delta) < f(x_j(\Delta)) < (1+\varepsilon)\phi((1+\varepsilon)j\Delta), \quad j > L_3(\varepsilon),$$

where $L_3(\varepsilon) := \max(L_1(\varepsilon), L_2(\varepsilon))$ and so

$$\frac{1}{(1-\varepsilon)}\frac{\Delta}{\phi((1-\varepsilon)j\Delta)} > \frac{\Delta}{f(x_j(\Delta))} > \frac{1}{(1+\varepsilon)}\frac{\Delta}{\phi((1+\varepsilon)j\Delta)}, \quad j > L_3(\varepsilon). \tag{2.8.23}$$

Now let $n > L_3(\varepsilon)$. Since

$$T_\Delta - t_n(\Delta) = \sum_{j=0}^{\infty} \frac{\Delta}{f(x_j(\Delta))} - \sum_{j=0}^{n-1} \frac{\Delta}{f(x_j(\Delta))} = \sum_{j=n}^{\infty} \frac{\Delta}{f(x_j(\Delta))}$$

we have

$$\frac{1}{(1-\varepsilon)} \sum_{j=n}^{\infty} \frac{\Delta}{\phi((1-\varepsilon)j\Delta)} \geq T_\Delta - t_n(\Delta) \geq \frac{1}{(1+\varepsilon)} \sum_{j=n}^{\infty} \frac{\Delta}{\phi((1+\varepsilon)j\Delta)}.$$

By the monotonicity of ϕ on $[(1-\varepsilon)j\Delta, (1-\varepsilon)(j+1)\Delta]$, we have

$$(1-\varepsilon)\Delta \frac{1}{\phi((1-\varepsilon)j\Delta)} \geq \int_{(1-\varepsilon)j\Delta}^{(1-\varepsilon)(j+1)\Delta} \frac{1}{\phi(x)}\,dx \geq (1-\varepsilon)\Delta \frac{1}{\phi((1-\varepsilon)(j+1)\Delta)}.$$

Thus for $n > L_3(\varepsilon)$, we get

$$(1-\varepsilon) \sum_{j=n}^{\infty} \frac{\Delta}{\phi((1-\varepsilon)j\Delta)} \geq \int_{(1-\varepsilon)n\Delta}^{\infty} \frac{1}{\phi(x)}\,dx \geq (1-\varepsilon) \sum_{j=n}^{\infty} \frac{\Delta}{\phi((1-\varepsilon)(j+1)\Delta)}.$$

Define $\bar{\Phi}(x) := \int_x^\infty 1/\phi(u)\,du$. Now for $n > L_3(\varepsilon) + 1$ we have

$$\bar{\Phi}((1-\varepsilon)(n-1)\Delta) \geq (1-\varepsilon) \sum_{j=n}^{\infty} \frac{\Delta}{\phi((1-\varepsilon)j\Delta)} \geq (1-\varepsilon)^2 (T_\Delta - t_n(\Delta)). \tag{2.8.24}$$

In a similar manner we obtain

$$(1+\varepsilon)^2 (T_\Delta - t_n) \geq (1+\varepsilon) \sum_{j=n}^{\infty} \frac{\Delta}{\phi((1+\varepsilon)j\Delta)} \geq \bar{\Phi}((1+\varepsilon)n\Delta). \tag{2.8.25}$$

Combining (2.8.24) and (2.8.25) we get

$$\frac{1}{(1-\varepsilon)^2} \bar{\Phi}((1-\varepsilon)(n-1)\Delta) \geq T_\Delta - t_n(\Delta) \geq \frac{1}{(1+\varepsilon)^2} \bar{\Phi}((1+\varepsilon)n\Delta), \quad n > L_3(\varepsilon) + 1. \tag{2.8.26}$$

By (2.8.21) and the monotonicity of $\bar{\Phi}$ we get

$$\frac{1}{\bar{\Phi}((1-\varepsilon)n\Delta)} \leq \frac{1}{\bar{\Phi}(x_n(\Delta))} \leq \frac{1}{\bar{\Phi}((1+\varepsilon)n\Delta)}. \tag{2.8.27}$$

Using the first member of (2.8.27) and the second of (2.8.26) gives

$$\frac{T_\Delta - t_n(\Delta)}{\bar{\Phi}(x_n(\Delta))} \geq \frac{T_\Delta - t_n(\Delta)}{\bar{\Phi}((1-\varepsilon)n\Delta)} \geq \frac{1}{(1+\varepsilon)^2} \frac{\bar{\Phi}((1+\varepsilon)n\Delta)}{\bar{\Phi}((1-\varepsilon)n\Delta)}.$$

By (2.2.3) and L'Hôpital's rule we have

$$\lim_{x\to\infty} \frac{\bar{\Phi}(x)}{\bar{F}(x)} = 1, \tag{2.8.28}$$

and since $\bar{F}$ is in $\mathrm{RV}_\infty(\beta)$, we have that $\bar{\Phi}$ is in $\mathrm{RV}_\infty(\beta)$, since

$$\lim_{x\to\infty}\frac{\bar{\Phi}(\lambda x)}{\bar{\Phi}(x)}=\lim_{x\to\infty}\frac{\bar{\Phi}(\lambda x)}{\bar{F}(\lambda x)}\frac{\bar{F}(\lambda x)}{\bar{F}(x)}\frac{\bar{F}(x)}{\bar{\Phi}(x)}=\lambda^\beta,\quad \text{for all } \lambda>0.$$

So we obtain

$$\liminf_{n\to\infty}\frac{T_\Delta-t_n(\Delta)}{\bar{\Phi}(x_n(\Delta)}\geq\frac{1}{(1+\varepsilon)^2}\cdot\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^\beta.$$

Letting $\varepsilon\to 0^+$ yields

$$\liminf_{n\to\infty}\frac{T_\Delta-t_n(\Delta)}{\bar{\Phi}(x_n(\Delta)}\geq 1.$$

Thus by (2.8.28),

$$\liminf_{n\to\infty}\frac{T_\Delta-t_n(\Delta)}{\bar{F}(x_n(\Delta)}\geq 1 \tag{2.8.29}$$

Similarly, using the second member of (2.8.27) and the first of (2.8.26) gives

$$\begin{aligned}\frac{T_\Delta-t_n(\Delta)}{\bar{\Phi}(x_n(\Delta))}&\leq\frac{T_\Delta-t_n(\Delta)}{\bar{\Phi}((1+\varepsilon)n\Delta)}\leq\frac{1}{(1-\varepsilon)^2}\frac{\bar{\Phi}((1-\varepsilon)(n-1)\Delta)}{\bar{\Phi}((1+\varepsilon)n\Delta)}\\&=\frac{1}{(1-\varepsilon)^2}\frac{\bar{\Phi}((1-\varepsilon)n\Delta-(1-\varepsilon)\Delta))}{\bar{\Phi}((1-\varepsilon)n\Delta)}\frac{\bar{\Phi}((1-\varepsilon)n\Delta)}{\bar{\Phi}((1+\varepsilon)n\Delta)}.\end{aligned}$$

Since $\bar{\Phi}$ is regularly varying, the second term on the right-hand side as limit equal to unity as $n\to\infty$. The third term has limit $((1-\varepsilon)/(1+\varepsilon))^\beta$ as $n\to\infty$. Therefore

$$\limsup_{n\to\infty}\frac{T_\Delta-t_n(\Delta)}{\bar{\Phi}(x_n(\Delta))}\leq\frac{1}{(1-\varepsilon)^2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^\beta.$$

Letting $\varepsilon\to 0^+$ yields

$$\limsup_{n\to\infty}\frac{T_\Delta-t_n(\Delta)}{\bar{\Phi}(x_n(\Delta)}\leq 1.$$

and so by (2.8.28),

$$\limsup_{n\to\infty}\frac{T_\Delta-t_n(\Delta)}{\bar{F}(x_n(\Delta)}\leq 1. \tag{2.8.30}$$

Combining (2.8.29) and (2.8.30) gives (2.6.9), as required.

Given that (2.6.9), we prove (2.6.10). We have $x_n(\Delta)\leq\bar{X}_\Delta(t)\leq x_{n+1}(\Delta)$ for $t\in[t_n(\Delta),t_{n+1}(\Delta)]$. Since $\bar{\Phi}$ is decreasing, for $t_n(\Delta)\leq t\leq t_{n+1}(\Delta)$ we have

$$\frac{T_\Delta-t_{n+1}(\Delta)}{\bar{\Phi}(x_n(\Delta))}\leq\frac{T_\Delta-t}{\bar{\Phi}(x_n(\Delta))}\leq\frac{T_\Delta-t}{\bar{\Phi}(\bar{X}_\Delta(t))}\leq\frac{T_\Delta-t}{\bar{\Phi}(x_{n+1}(\Delta))}\leq\frac{T_\Delta-t_n(\Delta)}{\bar{\Phi}(x_{n+1}(\Delta))}. \tag{2.8.31}$$

Since $x_n(\Delta)\to\infty$ as $n\to\infty$,

$$x_{n+1}(\Delta)=x_n(\Delta)+\Delta+\varepsilon_n(\Delta)$$

and $\varepsilon_n(\Delta) \to 0$ as $n \to \infty$, the fact that $\bar{\Phi} \in \mathrm{RV}_\infty(\beta)$ implies that

$$\lim_{n\to\infty} \frac{\bar{\Phi}(x_{n+1}(\Delta))}{\bar{\Phi}(x_n(\Delta))} = 1. \tag{2.8.32}$$

By (2.8.32), (2.6.9) and (2.8.28) we have

$$\lim_{n\to\infty} \frac{T_\Delta - t_{n+1}(\Delta)}{\bar{\Phi}(x_n(\Delta))} = 1 \text{ and } \lim_{n\to\infty} \frac{T_\Delta - t_n(\Delta)}{\bar{\Phi}(x_{n+1}(\Delta))} = 1.$$

Using these limits in conjunction with (2.8.31) and (2.8.28) gives

$$\lim_{t\to T_\Delta^-} \frac{\bar{\Phi}(\bar{X}_\Delta(t))}{T_\Delta - t} = 1. \tag{2.8.33}$$

and using (2.8.28) in conjunction with (2.8.33) yields (2.6.10).

Proof of Theorem 2.6.3 We note that (2.8.26) in the proof of Theorem 2.6.2 can be deduced without making any assumptions concerning the regular variation of $\bar{F}$. Recall that (2.8.26) reads

$$\frac{1}{(1-\varepsilon)^2}\bar{\Phi}((1-\varepsilon)(n-1)\Delta) \geq T_\Delta - t_n(\Delta) \geq \frac{1}{(1+\varepsilon)^2}\bar{\Phi}((1+\varepsilon)n\Delta), \quad n > L(\varepsilon)+1.$$

The first member of this implies $(1-\varepsilon)(n-1)\Delta \leq \bar{\Phi}^{-1}((1-\varepsilon)^2(T_\Delta - t_n(\Delta)))$ for $n > L(\varepsilon)+1$. Now,

$$\begin{aligned}&\frac{x_n(\Delta)}{\bar{\Phi}^{-1}(T_\Delta - t_n(\Delta))}\\ &= \frac{x_n(\Delta)}{(1-\varepsilon)(n-1)\Delta}\cdot\frac{(1-\varepsilon)(n-1)\Delta}{\bar{\Phi}^{-1}((1-\varepsilon)^2(T_\Delta - t_n(\Delta)))}\cdot\frac{\bar{\Phi}^{-1}((1-\varepsilon)^2(T_\Delta - t_n(\Delta)))}{\bar{\Phi}^{-1}(T_\Delta - t_n(\Delta))}.\end{aligned} \tag{2.8.34}$$

We now look to show $\bar{F}^{-1}(x)/\bar{\Phi}^{-1}(x) \to 1$ as $x \to 0^+$ and so $\bar{\Phi}^{-1} \in RV_0(0)$. Recall that by (2.2.3), given any $\varepsilon \in (0,1)$ there exists $x_\varepsilon > 0$ such that

$$(1-\varepsilon)\phi(x) < f(x) < (1+\varepsilon)\phi(x), \quad x > x_\varepsilon.$$

For $t > 0$ define $z'(t) = f(z(t))$ and $z(0) = z_0 > x_\varepsilon$. As f is positive we have $z(t) > x_\varepsilon$ for all $t > 0$ and therefore by (2.2.3)

$$(1-\varepsilon)\phi(z(t)) < f(z(t)) < (1+\varepsilon)\phi(z(t)), \quad t > 0.$$

Since $z'(t) = f(z(t))$, by (2.2.1) it is easily shown that $z(t) = \bar{F}^{-1}(T-t), t \in (0,T)$ where $T = \bar{F}(z_0)$. Thus with $\bar{\Phi}(x) := \int_x^\infty 1/\phi(u)\,du$ we have

$$(1-\varepsilon)(T-t) < \bar{\Phi}(z(t)) < (1+\varepsilon)(T-t), \quad t > 0,$$

and since $\bar{\Phi}$ and therefore $\bar{\Phi}^{-1}$ are decreasing,

$$\bar{\Phi}^{-1}(1+\varepsilon)(T-t) < \bar{F}^{-1}(T-t) < \bar{\Phi}^{-1}(1-\varepsilon)(T-t), \quad t>0.$$

Putting $x=(1+\varepsilon)(T-t)$ and $x=(1-\varepsilon)(T-t)$ for the first and second inequalities respectively we obtain

$$\bar{F}^{-1}((1-\varepsilon)^{-1}x) < \bar{\Phi}^{-1}(x) < \bar{F}^{-1}((1+\varepsilon)^{-1}x), \quad x \in (0,T). \tag{2.8.35}$$

Thus,

$$\frac{\bar{F}^{-1}((1+\varepsilon)^{-1}x)}{\bar{F}^{-1}(x)} < \frac{\bar{\Phi}^{-1}(x)}{\bar{F}^{-1}(x)} < \frac{\bar{F}^{-1}((1-\varepsilon)^{-1}x)}{\bar{F}^{-1}(x)}, \quad x \in (0,T),$$

and so

$$\liminf_{x\to 0^+} \frac{\bar{\Phi}^{-1}(x)}{\bar{F}^{-1}(x)} \geq \liminf_{x\to 0^+} \frac{\bar{F}^{-1}((1+\varepsilon)^{-1}x)}{\bar{F}^{-1}(x)} = \left(\frac{1}{1+\varepsilon}\right)^0 = 1,$$
$$\limsup_{x\to 0^+} \frac{\bar{\Phi}^{-1}(x)}{\bar{F}^{-1}(x)} \leq \liminf_{x\to 0^+} \frac{\bar{F}^{-1}((1-\varepsilon)^{-1}x)}{\bar{F}^{-1}(x)} = \left(\frac{1}{1-\varepsilon}\right)^0 = 1.$$

Combining these two equations we have

$$\lim_{x\to 0^+} \frac{\bar{\Phi}^{-1}(x)}{\bar{F}^{-1}(x)} = 1, \tag{2.8.36}$$

and since $\bar{F}^{-1} \in RV_0(0)$ and

$$\lim_{x\to\infty} \frac{\bar{\Phi}^{-1}(\lambda x)}{\bar{\Phi}^{-1}(x)} = \lim_{x\to\infty} \frac{\bar{\Phi}^{-1}(\lambda x)}{\bar{F}^{-1}(\lambda x)} \frac{\bar{F}^{-1}(\lambda x)}{\bar{F}^{-1}(x)} \frac{\bar{F}^{-1}(x)}{\bar{\Phi}^{-1}(x)} = 1, \quad \text{for all } \lambda>0,$$

this implies that $\bar{\Phi}^{-1} \in RV_0(0)$.

So for (2.8.34), using the facts that $\bar{\Phi}^{-1} \in \mathrm{RV}_0(0)$ and $x_n(\Delta)/n \to \Delta$ as $n\to\infty$, we have

$$\limsup_{n\to\infty} \frac{x_n(\Delta)}{\bar{\Phi}^{-1}(T_\Delta - t_n(\Delta))} \leq \frac{1}{1-\varepsilon} \cdot 1 \cdot ((1-\varepsilon)^2)^0 = \frac{1}{1-\varepsilon}.$$

Letting $\varepsilon \to 0^+$ yields

$$\limsup_{n\to\infty} \frac{x_n(\Delta)}{\bar{\Phi}^{-1}(T_\Delta - t_n(\Delta))} \leq 1.$$

Thus by (2.8.36),

$$\limsup_{n\to\infty} \frac{x_n(\Delta)}{\bar{F}^{-1}(T_\Delta - t_n(\Delta))} \leq 1. \tag{2.8.37}$$

The second member of (2.8.26) can be rewritten as $\bar{\Phi}^{-1}((1+\varepsilon)^2(T_\Delta - t_n(\Delta))) \leq (1+\varepsilon)n\Delta$ for $n > L(\varepsilon)+1$. Since

$$\frac{x_n(\Delta)}{\bar{\Phi}^{-1}(T_\Delta - t_n(\Delta))}$$
$$= \frac{x_n(\Delta)}{(1+\varepsilon)n\Delta} \cdot \frac{(1+\varepsilon)n\Delta}{\bar{\Phi}^{-1}((1+\varepsilon)^2(T_\Delta - t_n(\Delta)))} \cdot \frac{\bar{\Phi}^{-1}((1+\varepsilon)^2(T_\Delta - t_n(\Delta)))}{\bar{\Phi}^{-1}(T_\Delta - t_n(\Delta))},$$

$\bar{\Phi}^{-1} \in \mathrm{RV}_0(0)$, and $x_n(\Delta)/n \to \Delta$ as $n \to \infty$, we have

$$\liminf_{n\to\infty} \frac{x_n(\Delta)}{\bar{\Phi}^{-1}(T_\Delta - t_n(\Delta))} \geq \frac{1}{1+\varepsilon} \cdot 1 \cdot ((1+\varepsilon)^2)^0 = \frac{1}{1+\varepsilon}.$$

Letting $\varepsilon \to 0^+$ yields

$$\liminf_{n\to\infty} \frac{x_n(\Delta)}{\bar{\Phi}^{-1}(T_\Delta - t_n(\Delta))} \geq 1,$$

and by (2.8.36),

$$\liminf_{n\to\infty} \frac{x_n(\Delta)}{\bar{F}^{-1}(T_\Delta - t_n(\Delta))} \geq 1, \tag{2.8.38}$$

Combining (2.8.37) and (2.8.38) yields (2.6.11).

Given that (2.6.11), we prove (2.6.12). We have $x_n(\Delta) \leq \bar{X}_\Delta(t) \leq x_{n+1}(\Delta)$ for $t \in [t_n(\Delta), t_{n+1}(\Delta)]$. Since $\bar{\Phi}^{-1}$ is decreasing, for $t_n(\Delta) \leq t \leq t_{n+1}(\Delta)$ we have

$$\bar{\Phi}^{-1}(T_\Delta - t_n(\Delta)) \leq \bar{\Phi}^{-1}(T_\Delta - t) \leq \bar{\Phi}^{-1}(T_\Delta - t_{n+1}(\Delta)).$$

Hence

$$\begin{aligned}\frac{x_{n+1}(\Delta)}{\bar{\Phi}^{-1}(T_\Delta - t_n(\Delta))} \geq \frac{\bar{X}_\Delta(t)}{\bar{\Phi}^{-1}(T_\Delta - t_n(\Delta))} \geq \frac{\bar{X}_\Delta(t)}{\bar{\Phi}^{-1}(T_\Delta - t)} &\geq \frac{\bar{X}_\Delta(t)}{\bar{\Phi}^{-1}(T_\Delta - t_{n+1}(\Delta))} \\ &\geq \frac{x_n(\Delta)}{\bar{\Phi}^{-1}(T_\Delta - t_{n+1}(\Delta))}\end{aligned} \tag{2.8.39}$$

Since $x_n(\Delta)/n \to \Delta$ as $n \to \infty$, $x_{n+1}(\Delta)/x_n(\Delta) \to 1$ as $n \to \infty$. Using this fact, (2.6.11) and (2.8.36) implies that

$$\lim_{n\to\infty} \frac{x_{n+1}(\Delta)}{\bar{F}^{-1}(T_\Delta - t_n(\Delta)} = 1 \text{ and } \lim_{n\to\infty} \frac{x_n(\Delta)}{\bar{F}^{-1}(T_\Delta - t_{n+1}(\Delta))} = 1.$$

Using these limits in conjunction with (2.8.39) yields (2.6.12).

Proof of Proposition 2.7.1 We first prove that

$$T_\xi = \int_\xi^\infty \frac{1}{f(x)}\,dx < +\infty.$$

To see this, write

$$\int_\xi^\infty \frac{1}{f(x)}\,dx = \int_\xi^{\xi+\Delta_0/2} \frac{1}{f(x)}\,dx + \sum_{n=1}^\infty \int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(x)}\,dx.$$

By (2.7.4), we have

$$\begin{aligned}\int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(x)}\,dx &= \int_{\xi+(n-1/2)\Delta_0}^{\xi+n\Delta_0-2\epsilon_n\Delta_0} \frac{1}{\eta(x)}\,dx + \int_{\xi+n\Delta_0-2\epsilon_n\Delta_0}^{\xi+n\Delta_0-\epsilon_n\Delta_0} \frac{1}{l_-(x)}\,dx \\ &+ \int_{\xi+n\Delta_0-\epsilon_n\Delta_0}^{\xi+n\Delta_0+\epsilon_n\Delta_0} \frac{1}{\theta(x)}\,dx + \int_{\xi+n\Delta_0+\epsilon_n\Delta_0}^{\xi+n\Delta_0+2\epsilon_n\Delta_0} \frac{1}{l_+(x)}\,dx + \int_{\xi+n\Delta_0+2\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(x)}\,dx.\end{aligned}$$

By the monotonicity of η and θ and (2.7.3), we notice that $l_+(x) \geq \theta(x)$ and $l_-(x) \geq \theta(x)$ on their domains, so

$$\int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(x)}\,dx \leq \int_{\xi+(n-1/2)\Delta_0}^{\xi+n\Delta-2\epsilon_n\Delta_0} \frac{1}{\eta(x)}\,dx + \int_{\xi+n\Delta_0-2\epsilon_n\Delta_0}^{\xi+n\Delta_0+2\epsilon_n\Delta_0} \frac{1}{\theta(x)}\,dx + \int_{\xi+n\Delta_0+2\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(x)}\,dx,$$

and since η is positive, we have

$$\int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(x)}\,dx \leq \int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(x)}\,dx + \int_{\xi+n\Delta_0-2\epsilon_n\Delta_0}^{\xi+n\Delta_0+2\epsilon_n\Delta_0} \frac{1}{\theta(x)}\,dx.$$

Using the monotonicity of θ and the fact that $\epsilon_n < 1/4$, for $x \in [\xi + n\Delta_0 - 2\epsilon_n\Delta_0, \xi + n\Delta_0 + 2\epsilon_n\Delta_0]$, we have

$$\theta(x) \geq \theta(\xi + n\Delta_0 - 2\epsilon_n\Delta_0) \geq \theta(\xi + n\Delta_0 - \Delta_0/2).$$

Hence

$$\int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(x)}\,dx \leq \int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(x)}\,dx + 4\epsilon_n\Delta_0 \frac{1}{\theta(\xi + (n-1/2)\Delta_0)}. \tag{2.8.40}$$

Using the monotonicity of θ, for $n \geq 1$, we also have the estimate

$$\int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(x)}\,dx \leq \int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(x)}\,dx + 4\epsilon_n\Delta_0 \frac{1}{\theta(\xi + 1/2\Delta_0)}.$$

Since $1/\eta \in L^1(0,\infty)$ and $(\epsilon_n)_{n\geq 0}$ is summable, it follows that

$$\sum_{n=1}^{\infty} \int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(x)}\,dx < +\infty,$$

and therefore that T_ξ is finite.

On the other hand, if we consider

$$\lim_{n\to\infty} \sum_{n=0}^{\infty} \frac{\Delta_0}{f(\xi + n\Delta_0)},$$

we see from (2.7.4) that $f(\xi+n\Delta_0) = \theta(\xi+n\Delta_0)$ for $n \geq 1$. However, since $\int_1^\infty dx/\theta(x) = +\infty$ and θ is increasing, we have that

$$\lim_{n\to\infty} t_n(\Delta_0) = \frac{\Delta}{f(\xi)} + \sum_{n=1}^{\infty} \frac{\Delta}{\theta(\xi + n\Delta_0)} = +\infty,$$

as desired.

Proof of Proposition 2.7.2 We will establish

$$\lim_{x\to\infty} \frac{\int_x^\infty \frac{1}{f(u)}\,du}{\int_x^\infty \frac{1}{\eta(u)}\,du} = 1. \tag{2.8.41}$$

Since $x(t) \to \infty$ as $t \to T_\xi$, and

$$\int_{x(t)}^\infty \frac{1}{f(u)}\,du = T_\xi - t, \quad t < T_\xi,$$

we have

$$\lim_{t\uparrow T_\xi} \frac{\bar{H}(x(t))}{T_\xi - t} = \lim_{t\uparrow T_\xi} \frac{\int_{x(t)}^\infty \frac{1}{\eta(u)}\,du}{\int_{x(t)}^\infty \frac{1}{f(u)}\,du} = 1,$$

using (2.8.41), and completing the proof.

We now turn to the proof of (2.8.41). Since $f(x) \geq \theta(x)$ we obtained (2.8.40), that is

$$\int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(x)}\,dx \leq \int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(x)}\,dx + \frac{\Delta_0}{n+2}\frac{1}{\eta(\xi+n\Delta_0+\Delta_0/2)}\frac{\theta(\xi)}{\theta(\xi+(n-1/2)\Delta_0)}.$$

Since θ is increasing, for $n \geq 1$ we have $\theta(\xi+(n-1/2)\Delta_0) \geq \theta(\xi)$, so

$$\int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(x)}\,dx \leq \int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(x)}\,dx + \frac{\Delta_0}{n+2}\frac{1}{\eta(\xi+n\Delta_0+\Delta_0/2)}.$$

Now $\eta(x) \leq \eta(\xi+n\Delta_0+\Delta_0/2)$ for $x \in [\xi+(n-1/2)\Delta_0, \xi+(n+1/2)\Delta_0]$, so

$$\int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(x)} \geq \frac{\Delta_0}{\eta(\xi+n\Delta_0+\Delta_0/2)}.$$

Thus

$$\int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(x)}\,dx \leq \left(1+\frac{1}{n+2}\right)\int_{\xi+(n-1/2)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(x)}\,dx, \quad n \geq 1. \tag{2.8.42}$$

We now obtain an estimate for

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \text{ in terms of } \int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du$$

for $x \in [\xi+(n-1/2)\Delta_0, \xi+(n+1/2)\Delta_0]$.

For $x \in [\xi+n\Delta_0+2\epsilon_n\Delta_0, \xi+(n+1/2)\Delta_0]$

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du = \int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du. \tag{2.8.43}$$

For $x \in [\xi+n\Delta_0+\epsilon_n\Delta_0, \xi+n\Delta_0+2\epsilon_n\Delta_0]$, we have

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \leq \int_{\xi+n\Delta_0+\epsilon_n\Delta_0}^{\xi+n\Delta_0+2\epsilon_n\Delta_0} \frac{1}{\theta(u)}\,du + \int_{\xi+n\Delta_0+2\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du$$

using the fact that $f(x) \geq \theta(x)$. Therefore

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \leq \epsilon_n\Delta_0 \frac{1}{\theta(\xi+n\Delta_0+\epsilon_n\Delta_0)} + \int_{\xi+n\Delta_0+2\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du,$$

so as θ is increasing, using the definition of ϵ_n we have

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \leq \frac{1}{4}\frac{1}{n+2}\frac{\theta(\xi)}{\eta(\xi+(n-1/2)\Delta_0)\eta(\xi+(n+1/2)\Delta_0)}\Delta_0$$

$$+ \int_{\xi+n\Delta_0+2\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du.$$

For $x \in [\xi+n\Delta_0+2\epsilon_n\Delta_0, \xi+(n+1/2)\Delta_0]$, we have

$$\eta(x) \leq \eta(\xi+(n+1/2)\Delta_0).$$

Hence as $\epsilon_n < 1/12$, we have

$$\int_{\xi+n\Delta_0+2\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du \geq \frac{\Delta_0(1/2-2\epsilon_n)}{\eta(\xi+(n+1/2)\Delta_0)} \geq \frac{\Delta_0/3}{\eta(\xi+(n+1/2)\Delta_0)}.$$

Therefore as $\theta(\xi) \leq \eta(\xi+(n-1/2)\Delta_0)$, we have

$$\frac{1}{4}\frac{1}{n+2}\frac{\theta(\xi)}{\eta(\xi+(n-1/2)\Delta_0)\eta(\xi+(n+1/2)\Delta_0)}\Delta_0 \leq \frac{3}{4}\frac{1}{n+2}\frac{\Delta_0/3}{\eta(\xi+(n+1/2)\Delta_0)},$$

and so

$$\frac{1}{4}\frac{1}{n+2}\frac{\theta(\xi)}{\eta(\xi+(n-1/2)\Delta_0)\eta(\xi+(n+1/2)\Delta_0)}\Delta_0 \leq \frac{3}{4}\frac{1}{n+2}\int_{\xi+n\Delta_0+2\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du. \tag{2.8.44}$$

Thus

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \leq \left(1+\frac{3}{4}\frac{1}{n+2}\right)\int_{\xi+n\Delta_0+2\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du.$$

Now since $x \in [\xi+(n+\epsilon_n)\Delta_0, \xi+(n+2\epsilon_n)\Delta_0]$ we have

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du \geq \int_{\xi+n\Delta_0+2\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du,$$

so

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \leq \left(1+\frac{3}{4}\frac{1}{n+2}\right)\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du,$$

$$x \in [\xi+(n+\epsilon_n)\Delta_0, \xi+(n+2\epsilon_n)\Delta_0]. \tag{2.8.45}$$

For $x \in [\xi + n\Delta_0 - \epsilon_n\Delta_0, \xi + n\Delta_0 + \epsilon_n\Delta_0]$, we have

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \int_{\xi+n\Delta_0-\epsilon_n\Delta_0}^{\xi+n\Delta_0+\epsilon_n\Delta_0} \frac{1}{\theta(u)}\,du + \int_{\xi+n\Delta_0+\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du,$$

using the fact that $f(x) \ge \theta(x)$. By (2.8.45), we have

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \int_{\xi+n\Delta_0-\epsilon_n\Delta_0}^{\xi+n\Delta_0+\epsilon_n\Delta_0} \frac{1}{\theta(u)}\,du + \left(1 + \frac{3}{4}\frac{1}{n+2}\right) \int_{\xi+n\Delta_0+\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du.$$

Next, as θ is increasing, we have

$$\int_{\xi+n\Delta_0-\epsilon_n\Delta_0}^{\xi+n\Delta_0+\epsilon_n\Delta_0} \frac{1}{\theta(u)}\,du \le 2\epsilon_n\Delta_0 \frac{1}{\theta(\xi+n\Delta_0-\epsilon_n\Delta_0)}.$$

Since θ is increasing, by the definition of ϵ_n we have

$$\int_{\xi+n\Delta_0-\epsilon_n\Delta_0}^{\xi+n\Delta_0+\epsilon_n\Delta_0} \frac{1}{\theta(u)}\,du \le 2\frac{1}{4}\frac{1}{n+2}\frac{\theta(\xi)}{\eta(\xi+(n+1/2)\Delta_0)\eta(\xi+(n-1/2)\Delta_0)}\Delta_0.$$

Therefore by (2.8.44) we have

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le 2\frac{3}{4}\frac{1}{n+2} \int_{\xi+n\Delta_0+2\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du + \left(1 + \frac{3}{4}\frac{1}{n+2}\right) \int_{\xi+n\Delta_0+\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du,$$

or

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \left(1 + \frac{9}{4}\frac{1}{n+2}\right) \int_{\xi+n\Delta_0+\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du, \tag{2.8.46}$$

Now since $x \in [\xi + (n-\epsilon_n)\Delta_0, \xi + (n+\epsilon_n)\Delta_0]$ we have

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \left(1 + \frac{9}{4}\frac{1}{n+2}\right) \int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du, \quad x \in [\xi + (n-\epsilon_n)\Delta_0, \xi + (n+\epsilon_n)\Delta_0]. \tag{2.8.47}$$

For $x \in [\xi + n\Delta_0 - 2\epsilon_n\Delta_0, \xi + n\Delta_0 - \epsilon_n\Delta_0]$, we have

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \int_{\xi+n\Delta_0-2\epsilon_n\Delta_0}^{\xi+n\Delta_0-\epsilon_n\Delta_0} \frac{1}{\theta(u)}\,du + \int_{\xi+n\Delta_0-\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du,$$

using the fact that $f(x) \ge \theta(x)$. By (2.8.46), we have

$$\int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \int_{\xi+n\Delta_0-2\epsilon_n\Delta_0}^{\xi+n\Delta_0-\epsilon_n\Delta_0} \frac{1}{\theta(u)}\,du + \left(1 + \frac{9}{4}\frac{1}{n+2}\right) \int_{\xi+n\Delta_0-\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du.$$

Next, as θ is increasing, we have

$$\int_{\xi+n\Delta_0-2\epsilon_n\Delta_0}^{\xi+n\Delta_0-\epsilon_n\Delta_0} \frac{1}{\theta(u)}\,du \le \epsilon_n\Delta_0 \frac{1}{\theta(\xi+n\Delta_0-2\epsilon_n\Delta_0)}.$$

Since θ is increasing, by the definition of ϵ_n we have

$$\int_{\xi+n\Delta_0-2\epsilon_n\Delta_0}^{\xi+n\Delta_0-\epsilon_n\Delta_0} \frac{1}{\theta(u)}\,du \le \frac{1}{4}\frac{1}{n+2}\frac{\theta(\xi)}{\eta(\xi+(n+1/2)\Delta_0)\eta(\xi+(n-1/2)\Delta_0)}\Delta_0.$$

Therefore by (2.8.44) we have

$$\int_{x}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \frac{3}{4}\frac{1}{n+2}\int_{\xi+n\Delta_0+2\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du + \left(1+\frac{9}{4}\frac{1}{n+2}\right)\int_{\xi+n\Delta_0-\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du,$$

or

$$\int_{x}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \left(1+\frac{12}{4}\frac{1}{n+2}\right)\int_{\xi+n\Delta_0-\epsilon_n\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du,$$

and since $x \in [\xi+(n-2\epsilon_n)\Delta_0, \xi+(n-\epsilon_n)\Delta_0]$ we have

$$\int_{x}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \left(1+\frac{12}{4}\frac{1}{n+2}\right)\int_{x}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du,$$

$$x \in [\xi+(n-2\epsilon_n)\Delta_0, \xi+(n-\epsilon_n)\Delta_0]. \quad (2.8.48)$$

For $x \in [\xi+(n-1/2)\Delta_0, \xi+(n-2\epsilon_n)\Delta_0]$, we have

$$\int_{x}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du = \int_{x}^{\xi+(n-2\epsilon_n)\Delta_0} \frac{1}{\eta(u)}\,du + \int_{\xi+(n-2\epsilon_n)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du.$$

Therefore by (2.8.48), we have

$$\int_{x}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \int_{x}^{\xi+(n-2\epsilon_n)\Delta_0} \frac{1}{\eta(u)}\,du + \left(1+3\frac{1}{n+2}\right)\int_{\xi+(n-2\epsilon_n)\Delta_0}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du.$$

Hence

$$\int_{x}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \left(1+3\frac{1}{n+2}\right)\int_{x}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du,$$

$$x \in [\xi+(n-1/2)\Delta_0, \xi+(n-2\epsilon_n)\Delta_0]. \quad (2.8.49)$$

Combining (2.8.43), (2.8.45), (2.8.47), (2.8.48) and (2.8.49), we have that

$$\int_{x}^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du \le \left(1+3\frac{1}{n+2}\right)\int_{x}^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du,$$

$$x \in [\xi+(n-1/2)\Delta_0, \xi+(n+1/2)\Delta_0]. \quad (2.8.50)$$

Thus for $x \in [\xi + (n-1/2)\Delta_0, \xi + (n+1/2)\Delta_0]$, by (2.8.42) and (2.8.50), we have

$$\begin{aligned}
\int_x^\infty \frac{1}{f(u)}\,du &= \int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{f(u)}\,du + \int_{\xi+(n+1/2)\Delta_0}^\infty \frac{1}{f(u)}\,du \\
&\le \left(1 + 3\frac{1}{n+2}\right) \int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du + \int_{\xi+(n+1/2)\Delta_0}^\infty \frac{1}{f(u)}\,du \\
&\le \left(1 + 3\frac{1}{n+2}\right) \int_x^{\xi+(n+1/2)\Delta_0} \frac{1}{\eta(u)}\,du \\
&\quad + \left(1 + \frac{1}{n+2}\right) \int_{\xi+(n+1/2)\Delta_0}^\infty \frac{1}{\eta(u)}\,du \\
&\le \left(1 + 3\frac{1}{n+2}\right) \int_x^\infty \frac{1}{\eta(u)}\,du.
\end{aligned}$$

Therefore we have

$$\limsup_{x\to\infty} \frac{\int_x^\infty \frac{1}{f(u)}\,du}{\int_x^\infty \frac{1}{\eta(u)}} \le 1.$$

On the other hand, we have that $f(x) \le \eta(x)$ for all $x \ge 0$ so

$$\int_x^\infty \frac{1}{f(u)}\,du \ge \int_x^\infty \frac{1}{\eta(u)}\,du.$$

Combining these inequalities yields (2.8.41).

Proof of Theorem 2.7.1 For all $z \in [w_n(\Delta) + (1-\varepsilon)\Delta, w_n(\Delta) + \Delta]$ and by (2.7.10b), we have

$$\frac{1}{f(z)} \ge \min_{z\in[w_n(\Delta)+(1-\varepsilon)\Delta, w_n(\Delta)+\Delta]} \frac{1}{f(z)} = \frac{1}{f(w_{n+1}(\Delta))}.$$

Therefore

$$\int_{w_n(\Delta)+(1-\varepsilon)\Delta}^{w_n(\Delta)+\Delta} \frac{1}{f(z)}\,dz \ge \frac{\varepsilon\Delta}{f(w_{n+1}(\Delta))}.$$

Also, as $\varepsilon \in (0, 1/2)$, we have $w_{n+1}(\Delta) \ge w_n(\Delta) + (1-\varepsilon)\Delta > w_n(\Delta) + \varepsilon\Delta$. Therefore $w_{n+1}(\Delta) + (1-\varepsilon)\Delta \ge w_n(\Delta) + \Delta$. Hence

$$\int_{w_n(\Delta)+(1-\varepsilon)\Delta}^{w_{n+1}(\Delta)+(1-\varepsilon)\Delta} \frac{1}{f(z)}\,dz \ge \int_{w_n(\Delta)+(1-\varepsilon)\Delta}^{w_n(\Delta)+\Delta} \frac{1}{f(z)}\,dz \ge \frac{\varepsilon\Delta}{f(w_{n+1}(\Delta))}.$$

Therefore we have

$$\int_{\psi(0)+(1-\epsilon)\Delta}^{w_n(\Delta)+(1-\varepsilon)\Delta} \frac{1}{f(z)}\,dz = \sum_{j=0}^{n-1} \int_{w_j(\Delta)+(1-\varepsilon)\Delta}^{w_{j+1}(\Delta)+(1-\varepsilon)\Delta} \frac{1}{f(z)}\,dz \ge \sum_{j=0}^{n-1} \frac{\varepsilon\Delta}{f(w_{j+1}(\Delta))}.$$

Taking the limit as $n \to \infty$ and using (2.2.1) now establishes that

$$\sum_{j=0}^{\infty} \frac{\Delta}{f(w_{j+1}(\Delta))} < +\infty.$$

By (2.7.11), we see that $l_n \leq \Delta/f(w_n(\Delta))$ for $n \geq 0$. Therefore we have

$$s_n(\Delta) = \sum_{j=0}^{n-1} l_j(\Delta) \leq \sum_{j=0}^{n-1} \frac{\Delta}{f(w_j(\Delta))},$$

and so $\limsup_{n\to\infty} s_n(\Delta) < +\infty$. Since $(s_n(\Delta))_{n\geq 0}$ is an increasing sequence, we have that $\lim_{n\to\infty} s_n(\Delta)$ exists and is finite, as required.

Chapter 3

Replicating Rates of Highly Explosive Equations

3.1 Introduction

In the previous chapter, we noted that the numerical method described in Section 2.3 did not replicate the *exact* rate of explosion for equations for which $\bar{F}^{-1} \in \mathrm{RV}_0(0)$ where $\bar{F}$ is defined by (0.1.12). That is, the numerical approximation $\bar{X}_\Delta$ given by (2.3.7) obeyed $\bar{X}_\Delta(t)/\bar{F}^{-1}(T_\Delta - t) \to 1$ as $t \to \infty$ whereas the unique solution x to the delay differential equation (0.1.6) obeyed $F(x(t))/(T-t) \to 1$. Note that if $\bar{F}^{-1} \in \mathrm{RV}_0(0)$, then f is growing more rapidly than regularly varying (see e.g., Beirlant and Willekens [9]). This result indicated that an alternative discretisation is needed in order to pick up the exact asymptotics. In this chapter, we illustrate how to refine our state–dependent method for the purpose of precisely replicating the explosion rates of equations which grow more rapidly than regularly varying.

Theorem 2.6.2 showed the explosion rates of regularly varying equations could be mimicked precisely by a mesh proportional to $1/f$. It is therefore natural to assume that a mesh which will recover the exact asymptotics of rapidly varying equations will involve an extra factor in the denominator.

In this section, we modify the step-size of the state–dependent method given in Section 2.3 to

$$t_{n+1}(\Delta) = t_n(\Delta) + \frac{\Delta}{f(x_n(\Delta))\theta(x_n(\Delta))}, \quad n \geq 0 \tag{3.1.1}$$

where $\Delta > 0$, $(x_n(\Delta))_{n\geq 0}$ is defined by (2.3.5),

$$f \in C^1((0,\infty);(0,\infty)), \quad f'(x) > 0 \text{ for all } x > 0 \tag{3.1.2}$$

and f obeys (2.2.1) and

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{1/f'(x)} = 1, \tag{3.1.3}$$

and θ is given by

$$\theta(x) = \frac{(f'(x))^2}{f(x)}, \quad x > 0. \tag{3.1.4}$$

3.2 Auxiliary Functions and Discussion of Hypotheses

We now provide some commentary on above assumptions.

Introduce the function

$$\Theta(x) = \frac{1}{\bar{F}(x)}, \quad x > 0. \tag{3.2.1}$$

Since f is non-zero, $\bar{F}(x) > 0$ for all $x > 0$, and so Θ is well–defined and positive. By (0.1.12) and (3.1.2), it follows that $\Theta \in C^2((0,\infty);(0,\infty))$. Similarly, θ is well-defined and positive by (3.1.2).

We note, even for relatively complicated functions f, that θ is computable in closed form, contingent of course on f being continuously differentiable. On the other hand, an explicit formula for Θ may be complicated, or even impossible to obtain, because such a formula is unavailable for $\bar{F}$ even though asymptotic information about $\bar{F}$ (and hence Θ) can often be readily ascertained. Therefore, when constructing a state–dependent mesh for the solution of the delay differential equation, it can be appropriate to use f' to determine the step size, but sometimes not $\bar{F}$.

Example 3.2.1. If $f(x) = e^{x^2/2}$ for $x > 0$, we have that $f'(x) = xe^{x^2/2}$, but no closed form formula is available for $\bar{F}(x)$. However, by l'Hôpital's rule we can compute

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{1/x \cdot e^{-x^2/2}} = \lim_{x\to\infty} \frac{-1/f(x)}{-1/x^2 \cdot e^{-x^2/2} - e^{-x^2/2}} = 1.$$

On the other hand, if $f(x) = e^x$ for $x > 0$, we have that $f'(x) = e^x$ and $\bar{F}(x) = e^{-x}$, so both functions are computable.

3.2.1 Assumption on asymptotic behaviour of $\bar{F}$

In Example 3.2.1, the functions f considered grow rapidly (and in particular grow more rapidly than any regularly varying function), we notice that $\bar{F}(x) \sim 1/f'(x)$ as $x \to \infty$. In order to cover the case of very rapidly growing f, we assume throughout that

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{1/f'(x)} = 1. \tag{3.2.2}$$

This condition appears restrictive, but as our discussion now shows, it covers many rapidly growing functions like $f(x) = e^{x^\eta}$ for $\eta > 0$, $f(x) = \exp_n(x)$ where $\exp_n$ is the n-th fold composition of exponential functions. Note that by l'Hôpital,

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{1/f'(x)} = \lim_{x\to\infty} \frac{(f'(x))^2}{f(x)f''(x)}, \tag{3.2.3}$$

which under the additional assumption that $f \in C^2$, gives a condition that in practice is much more easily verified than (3.2.2), since the second derivative of f is generally more straightforward to obtain than $\bar{F}$.

Remark 3.2.1. We now show that if $\log f$ is a smoothly regularly varying function with index $\eta > 0$, and f satisfies (2.2.1), then (3.2.2) holds. Notice that $f(x) = e^{x^\eta}$ for $\eta > 0$ is an example of such a function.

This hypothesis on f implies that $\phi(x) := \log f(x)$ obeys

$$\lim_{x\to\infty} \frac{x\phi'(x)}{\phi(x)} = \eta, \quad \lim_{x\to\infty} \frac{x\phi''(x)}{\phi'(x)} = \eta - 1,$$

in contrast to standard regularly variation which only assumes the existence of a function asymptotic to $\log f$ which obeys these limits. Since we require that $f(x) \to \infty$ as $x \to \infty$, this forces $\phi(x) \to \infty$ as $x \to \infty$. Therefore

$$\frac{\phi''(x)}{\phi'(x)^2} = \frac{x\phi''(x)}{\phi'(x)} \cdot \frac{\phi(x)}{x\phi'(x)} \cdot \frac{1}{\phi(x)}$$

tends to 0 as $x \to \infty$. Notice that $f(x) = e^{\phi(x)}$. Then $f'(x) = \phi'(x)e^{\phi(x)}$. Thus, if ϕ is increasing, we have that f is increasing. By l'Hôpital's rule we have

$$\begin{aligned}
\lim_{x\to\infty} \frac{\bar{F}(x)}{1/f'(x)} &= \lim_{x\to\infty} \frac{\int_x^\infty e^{-\phi(u)}\,du}{1/\phi'(x)\cdot e^{-\phi(x)}} \\
&= \lim_{x\to\infty} \frac{e^{-\phi(x)}}{e^{-\phi(x)} + \phi''(x)/\phi'(x)^2 e^{-\phi(x)}} \\
&= \lim_{x\to\infty} \frac{1}{1 + \phi''(x)/\phi'(x)^2} = 1,
\end{aligned}$$

which is (3.2.2), as required.

Remark 3.2.2. Moreover if $\log_2 f$ is a smoothly regularly varying function with index $\eta > 0$ and f satisfies (2.2.1), then (3.2.2) holds. Indeed if $\log_n f$, $n \geq 2$ is a smoothly regularly varying function with index $\eta > 0$ and f satisfies (2.2.1), then (3.2.2) holds. This family of functions include the n-th fold composition of exponential functions such as $f(x) = \exp(\exp(\exp \ldots \exp(x^\eta)))$ for $\eta > 0$. The proofs are deferred to Appendix A.

Remark 3.2.3. In the case when $\log f$ is smoothly regularly varying at infinity with index $\eta = 0$, if we assume f grows faster than any regularly varying function in such a manner that

$$\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = \infty, \tag{3.2.4}$$

that is f is *rapidly* varying, then once again f obeys (3.2.2).

The condition (3.2.4) implies that $x\phi'(x) \to \infty$ as $x \to \infty$. The smooth regular variation of $\phi = \log f$ ensures

$$\lim_{x\to\infty} \frac{x\phi''(x)}{\phi'(x)} = -1.$$

Therefore

$$\frac{\phi''(x)}{\phi'(x)^2} = \frac{x\phi''(x)}{\phi'(x)} \cdot \frac{1}{x\phi'(x)} \to 0 \quad \text{as } x \to \infty.$$

The argument above in Remark 3.2.1 now guarantees that f obeys (3.2.2). Similarly if $\log_n f$, $n \geq 2$ is smoothly regularly varying at infinity with index $\eta = 0$ and f obeys (3.2.4), then (3.2.2) holds.

Remark 3.2.4. We note now that (3.2.2) *does not hold* if f is a smoothly regularly varying function of index $\beta > 1$. If this is the case, then

$$\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = \beta,$$

and since $1/f \in \mathrm{RV}_\infty(-\beta)$, and $\beta > 1$, we have

$$\lim_{x\to\infty} \frac{\int_x^\infty \frac{1}{f(u)}\, du}{\frac{1}{\beta-1}\frac{x}{f(x)}} = 1.$$

Therefore

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{1/f'(x)} = \lim_{x\to\infty} \frac{\frac{1}{\beta-1}\frac{x}{f(x)}}{\frac{1}{\beta}\frac{x}{f(x)}} = \frac{\beta}{\beta-1}, \tag{3.2.5}$$

so f does not obey (3.2.2). We notice however, that as $\beta \to \infty$ (in other words, as we consider regularly varying functions which grow more rapidly to infinity), then f comes "closer" to satisfying the condition (3.2.2), in the sense that the right hand side of (3.2.5) tends to unity, the right hand side of (3.2.2). This suggests that (3.2.2) is a condition associated with very rapidly growing functions f.

3.2.2 Asymptotic properties of Θ

We now deduce some useful asymptotic properties of Θ. Notice that

$$\Theta'(x) = \frac{1}{f(x)\bar{F}(x)^2} > 0,$$

so by (3.2.2), we have

$$\lim_{x\to\infty} \frac{\Theta'(x)}{\theta(x)} = \lim_{x\to\infty} \frac{\frac{1/f(x)}{\bar{F}(x)^2}}{f'(x)^2/f(x)} = 1. \tag{3.2.6}$$

Analogously, we have

$$\Theta''(x) = \frac{1}{f(x)^2\bar{F}(x)^2}\left(-f'(x) + \frac{2}{\bar{F}(x)}\right),$$

and so it follows from (3.2.2) that

$$\lim_{x\to\infty} \frac{\Theta''(x)}{\frac{1}{f(x)^2\bar{F}(x)^3}} = 1. \tag{3.2.7}$$

One implication of this limit is that

$$\text{There exists } x^* > 0 \text{ such that } \Theta''(x) > 0 \text{ for all } x \geq x^*. \tag{3.2.8}$$

We now show that these hypotheses lead to

$$\lim_{x\to\infty} \theta(x) = +\infty. \tag{3.2.9}$$

This fact leads to our new discretisation being more computationally intensive than the method introduced in Chapter 2, and it is this, we conjecture, that more readily enables us to demonstrate that our new method recovers the precise rate of growth of solutions of (0.1.6).

To prove (3.2.9), first note that (3.2.8) implies Θ' is increasing on (x^*, ∞) and therefore we have that $\Theta'(x)$ tends to a limit $L \in (0, \infty]$ as $x \to \infty$. Therefore, by (3.2.6), we have that $\theta(x) \to L \in (0, \infty]$. Suppose $L \in (0, \infty)$. Then $f'(x)^2/f(x) \to L \in (0, \infty)$ as $x \to \infty$. Since $f'(x) > 0$ by hypothesis, we have $f'(x)/f(x)^{1/2} \to \sqrt{L}$ as $x \to \infty$. Hence for every $\varepsilon \in (0, 1)$, there exists $x(\varepsilon) > 0$ such that

$$\sqrt{L}(1-\varepsilon) \leq \frac{f'(x)}{f(x)^{1/2}} \leq \sqrt{L}(1+\varepsilon), \quad x \geq x(\varepsilon).$$

Integrating on both sides of the inequality over $[x(\varepsilon), x]$ yields

$$\sqrt{L}(1-\varepsilon)(x - x(\varepsilon)) \leq 2(f(x)^{1/2} - f(x(\varepsilon))^{1/2}) \leq \sqrt{L}(1+\varepsilon)(x - x(\varepsilon)), \quad x \geq x(\varepsilon).$$

Thus

$$\sqrt{L}(1-\varepsilon) \le \liminf_{x\to\infty} \frac{2f(x)^{1/2}}{x} \le \limsup_{x\to\infty} \frac{2f(x)^{1/2}}{x} \le \sqrt{L}(1+\varepsilon),$$

and letting $\varepsilon \to 0$ yields

$$\lim_{x\to\infty} \frac{f(x)}{x^2} = \frac{L}{4}. \tag{3.2.10}$$

Since $f'(x)/f(x)^{1/2} \to \sqrt{L}$ as $x \to \infty$, we have

$$\lim_{x\to\infty} \frac{f'(x)}{x} = \lim_{x\to\infty} \frac{\sqrt{L}f(x)^{1/2}}{x} = \frac{L}{2}. \tag{3.2.11}$$

Now by l'Hôpital's rule and (3.2.10), we have

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{1/x} = \lim_{x\to\infty} \frac{\int_x^\infty \frac{1}{f(u)}\,du}{1/x} = \lim_{x\to\infty} \frac{\frac{1}{f(x)}}{1/x^2} = \frac{4}{L}.$$

Therefore by this limit and (3.2.11), we have

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{1/f'(x)} = \lim_{x\to\infty} \frac{\frac{4}{L}\frac{1}{x}}{\frac{2}{L}\frac{1}{x}} = 2,$$

which contradicts (3.2.2). Therefore, we must have $L = \infty$, and so (3.2.9) holds.

3.3 A Refined Mesh

Let $\Delta \in (0, \tau f(\psi(0)))$, and define $N_\Delta \in \mathbb{N}$ so that

$$N_\Delta \frac{\Delta}{f(\psi(0))} \le \tau, \quad \frac{(N_\Delta+1)\Delta}{f(\psi(0))} > \tau.$$

Now define $t_{-N_\Delta}(\Delta) = -\tau$ and

$$t_n(\Delta) = \frac{n\Delta}{f(\psi(0))}, \quad n = -N_\Delta + 1, \dots, 0.$$

Note that $\Delta < \tau f(\psi(0))$ ensures that $N_\Delta \ge 1$, that is we have at least one mesh point on the initial interval $[-\tau, 0]$. Define $(t_n(\Delta))_{n\ge 0}$ by (3.1.1), $x_n(\Delta)$ by (2.3.1) and (2.3.5), X_Δ by (2.3.2) and (2.3.6) and $\bar{X}_\Delta$ by (2.3.3) and (2.3.7). The sequence $t_n(\Delta)$ is well-defined and increasing, because f and θ are positive functions on $(0, \infty)$. Thus x_n, X_Δ and $\bar{X}_\Delta$ are well-defined.

We remark that the existence of the scheme relies on f being differentiable; moreover, in order for the scheme to be implemented in practice, it is necessary to have a formula for θ, and therefore for f'. However, this does not amount to a serious limitation in most cases of interest.

If we compare this scheme with that outlined in Chapter 2, we see that the new step size $t_{n+1}(\Delta) - t_n(\Delta)$ is modified by a factor $1/\theta(x_n(\Delta))$. It is instructive to compare the asymptotic relative size of the step size

$$h^{\text{new}} := \frac{\Delta}{f(x)\theta(x)} = \frac{\Delta}{f'(x)^2},$$

with that in the (old) scheme in Chapter 2, which is given by

$$h^{\text{old}} := \frac{\Delta}{f(x)}.$$

If the solutions for the two schemes were extended by one step when both solutions lie at the same level x, we see that the ratio of the step lengths are:

$$\frac{h^{\text{new}}}{h^{\text{old}}} = \frac{1}{\theta(x)} \to 0 \quad \text{as } x \to \infty$$

by (3.2.9). Thus, the new scheme requires (asymptotically) a much finer mesh than the scheme in Chapter 2, because the new step size becomes asymptotically negligible in relation to the old step size.

3.4 Explosions of Rapidly Varying Equations

We are now in a position to state our main result of this chapter. Notice that our hypotheses on f and g lead to the solution x of (0.1.6) obeying

$$\lim_{t\to T^-} x(t) = \infty \text{ for some } T \in (0,\infty) \tag{3.4.1}$$

and

$$\lim_{t\to T^-} \frac{\bar{F}(x(t))}{T-t} = 1. \tag{3.4.2}$$

Theorem 3.4.1. *Suppose that f obeys* (3.1.2) *and* (2.2.1)*, g is a positive and continuous function and let $\tau > 0$ and ψ obey* (0.1.9)*. Further suppose that f obeys* (3.2.2)*. Let $\Delta \in (0, \tau f(\psi(0)))$ and let the sequences $x_n(\Delta)$, $t_n(\Delta)$ be as defined by* (3.1.1), (2.3.1), (2.3.5) *and the functions X_Δ and $\bar{X}_\Delta$ be as defined by* (2.3.2), (2.3.6) *and* (2.3.3), (2.3.7)*. Then there exists $T_\Delta := \lim_{n\to\infty} t_n(\Delta) > 0$ such that*

$$T_\Delta < \infty, \quad \lim_{n\to\infty} x_n(\Delta) = \infty.$$

and if $\bar{F}$ *is the function defined by* (0.1.12), *then*

$$\lim_{n\to\infty} \frac{\bar{F}(x_n(\Delta))}{T_\Delta - t_n(\Delta)} = 1.$$

Moreover,

$$\lim_{t\to T_\Delta^-} \bar{X}_\Delta(t) = \infty$$

and

$$\lim_{t\to T_\Delta^-} \frac{\bar{F}(X_\Delta(t))}{T_\Delta - t} = 1.$$

Part (a) of Theorem 3.4.1 shows that the solution of the difference equation inherits the salient asymptotic properties (3.4.1) and (3.4.2), when suitably interpreted. Part (b) of the Theorem shows that the continuous extension of the discrete scheme to continuous time obeys the appropriate continuous–time analogues of (3.4.1) and (3.4.2).

3.5 Proof of Theorem 3.4.1

3.5.1 Preservation of the explosion

In this subsection, we show that there is a finite $T_\Delta > 0$ such that

$$\lim_{n\to\infty} t_n = T_\Delta.$$

Since g is a positive function, it follows for $n \ge 0$

$$x_{n+1}(\Delta) \ge x_n(\Delta) + \frac{\Delta}{f(x_n(\Delta))\theta(x_n(\Delta))}.$$

Hence $(x_n(\Delta))_{n\ge 0}$ is an increasing sequence. Therefore we have $x_n(\Delta) \to L \in (0, \infty]$. If L is finite, using the continuity of f and θ, by taking limits we get

$$L \ge L + \frac{\Delta}{f(L)\theta(L)},$$

which implies $f(L)\theta(L) \le 0$, contradicting the positivity of f and θ. Hence $x_n(\Delta) \to \infty$ as $n \to \infty$.

We define

$$\gamma_n(\Delta) := \int_{t_n(\Delta)}^{t_{n+1}(\Delta)} g(X_\Delta(s-\tau))\, ds > 0 \tag{3.5.1}$$

Then by (3.1.4) and (2.3.5) we have

$$x_{n+1}(\Delta) = x_n(\Delta) + \frac{\Delta}{\theta(x_n(\Delta))} + \frac{\Delta}{f(x_n(\Delta))\theta(x_n(\Delta))}\gamma_n(\Delta).$$

Since $x_n(\Delta)$ is increasing and $\Theta \in C^2((0,\infty);(0,\infty))$, by Taylor's theorem there exists $\xi_n(\Delta) \in [x_n(\Delta), x_{n+1}(\Delta)]$ such that

$$\begin{aligned}\Theta(x_{n+1}(\Delta)) = \Theta(x_n(\Delta)) + \Theta'(x_n(\Delta))\frac{\Delta}{\theta(x_n(\Delta))} + \Theta'(x_n(\Delta))\frac{\Delta}{f(x_n(\Delta))\theta(x_n(\Delta))}\gamma_n(\Delta) \\ + \frac{1}{2}\Theta''(\xi_n(\Delta))\left(\frac{\Delta}{\theta(x_n(\Delta))} + \frac{\Delta}{f(x_n(\Delta))\theta(x_n(\Delta))}\gamma_n(\Delta)\right)^2.\end{aligned}$$

Recall that $\Theta'(x) > 0$ for all $x > 0$, so $\Theta'(x_n(\Delta)) > 0$ for all $n \geq 0$. Since $x_n(\Delta) \to \infty$ as $n \to \infty$, there exists $N^* > 0$ such that $x_n(\Delta) > x^*$ for all $n \geq N^*$. Therefore, as $\xi_n(\Delta) \geq x_n(\Delta)$, we have $\Theta''(\xi_n(\Delta)) > 0$ for all $n \geq N^*$. Since $f(x_n(\Delta))\theta(x_n(\Delta)) > 0$ and $\gamma_n(\Delta) > 0$, for $n \geq N^*$, we have

$$\Theta(x_{n+1}(\Delta)) \geq \Theta(x_n(\Delta)) + \Delta\frac{\Theta'(x_n(\Delta))}{\theta(x_n(\Delta))}.$$

Now, recalling (3.2.6), we have

$$\liminf_{n\to\infty} \frac{\Theta(x_n(\Delta))}{n\Delta} \geq 1. \tag{3.5.2}$$

Next, we show that $t_n \to T_\Delta$ as $n \to \infty$. Since f is monotone, $\Theta'(x) \sim \theta(x)$ as $x \to \infty$ and $x_n(\Delta) \to \infty$ as $n \to \infty$, it follows that t_n tends to a finite limit whenever

$$\tau_n = \sum_{j=0}^{n-1} \frac{\Delta}{f(x_j(\Delta))\Theta'(x_j(\Delta))}$$

tends to a finite limit. By (3.5.2), it follows that for every $\varepsilon \in (0,1)$, we have

$$\Theta(x_j(\Delta)) > (1-\varepsilon)j\Delta, \quad j \geq j^*(\varepsilon)$$

for some integer j^*. Since Θ is increasing, we have

$$x_j(\Delta) > \Theta^{-1}((1-\varepsilon)j\Delta), \quad j \geq j^*(\varepsilon).$$

Since f and Θ' are monotone on (x^*,∞), $x_j(\Delta) > x^*$ for $j \geq N^*$, if we take $j_2(\varepsilon) = \max(N^*, j^*) + 1$, then

$$f(x_j(\Delta))\Theta'(x_j(\Delta)) > f(\Theta^{-1}((1-\varepsilon)j\Delta))\Theta'(\Theta^{-1}((1-\varepsilon)j\Delta)), \quad j \geq j_2(\varepsilon).$$

Therefore, using the monotonicity of $x \mapsto f(\Theta^{-1}(x))\Theta'(\Theta^{-1}(x))$, we arrive at

$$\begin{aligned}
&\sum_{j=j_2(\varepsilon)}^{n-2} \int_{(1-\varepsilon)j\Delta}^{(1-\varepsilon)(j+1)\Delta} \frac{1}{f(\Theta^{-1}(x))\Theta'(\Theta^{-1}(x))}\,dx \\
&\geq \sum_{j=j_2(\varepsilon)}^{n-2} \frac{(1-\varepsilon)\Delta}{f(\Theta^{-1}((1-\varepsilon)(j+1)\Delta))\Theta'(\Theta^{-1}((1-\varepsilon)(j+1)\Delta))} \\
&= \sum_{j=j_2(\varepsilon)+1}^{n-1} \frac{(1-\varepsilon)\Delta}{f(\Theta^{-1}((1-\varepsilon)j\Delta))\Theta'(\Theta^{-1}((1-\varepsilon)j\Delta))} \\
&\geq \sum_{j=j_2(\varepsilon)+1}^{n-1} \frac{(1-\varepsilon)\Delta}{f(x_j(\Delta))\Theta'(x_j(\Delta))}.
\end{aligned}$$

Hence for $n \geq j_2(\varepsilon)+1$, we have

$$\int_{(1-\varepsilon)j_2\Delta}^{(1-\varepsilon)(n-1)\Delta} \frac{1}{f(\Theta^{-1}(x))\Theta'(\Theta^{-1}(x))}\,dx \geq \sum_{j=j_2(\varepsilon)+1}^{n-1} \frac{(1-\varepsilon)\Delta}{f(x_j(\Delta))\Theta'(x_j(\Delta))}.$$

The integral on the left hand side is equal to

$$\int_{\Theta^{-1}((1-\varepsilon)j_2\Delta)}^{\Theta^{-1}((1-\varepsilon)(n-1)\Delta)} \frac{1}{f(x)}\,dx.$$

Since f obeys (2.2.1), we have that

$$\limsup_{n\to\infty} \sum_{j=j_2(\varepsilon)+1}^{n-1} \frac{(1-\varepsilon)\Delta}{f(x_j(\Delta))\Theta'(x_j(\Delta))} < +\infty$$

for every $\varepsilon \in (0,1)$. Hence τ_n tends to a finite limit, and so t_n must tend to a finite limit as $n \to \infty$.

3.5.2 Asymptotic analysis of $\Theta(x_n(\Delta))$

Define

$$\tilde{\gamma}_n(\Delta) := \frac{1}{t_{n+1}(\Delta)-t_n(\Delta)} \int_{t_n(\Delta)}^{t_{n+1}(\Delta)} g(X_\Delta(s-\tau))\,ds.$$

Since $t_n(\Delta) \to T_\Delta$ as $n \to \infty$, we have that

$$\tilde{\gamma}_n(\Delta) \leq \sup_{t\in[-\tau, T_\Delta-\tau]} g(X_\Delta(t)) < +\infty,$$

because $X_\Delta(t)$ is finite on $T_\Delta - \tau$. Hence $\gamma_n(\Delta)$ given by (3.5.1) is finite. Now, by (2.3.5) we have

$$x_{n+1}(\Delta) = x_n(\Delta) + \frac{\Delta}{\theta(x_n(\Delta))} + \frac{\Delta}{f(x_n(\Delta))\theta(x_n(\Delta))}\gamma_n(\Delta), \quad n \geq 0.$$

Define

$$\Delta_n = \Delta\left(1 + \frac{\gamma_n(\Delta)}{f(x_n(\Delta))}\right) > \Delta, \quad n \geq 0.$$

Since $\gamma_n(\Delta)$ is bounded above, and $f(x_n(\Delta)) \to \infty$ as $x \to \infty$, we have that

$$\lim_{n\to\infty} \Delta_n = \Delta. \tag{3.5.3}$$

The definition of Δ_n leads to

$$x_{n+1}(\Delta) = x_n(\Delta) + \frac{\Delta_n}{\theta(x_n(\Delta))}, \quad n \geq 0. \tag{3.5.4}$$

By Taylor's theorem, there exists $\xi_n(\Delta) \in [x_n(\Delta), x_{n+1}(\Delta)]$ such that

$$\Theta(x_{n+1}(\Delta)) = \Theta(x_n(\Delta)) + \frac{\Delta_n}{\theta(x_n(\Delta))}\Theta'(x_n(\Delta)) + \frac{1}{2}\Theta''(\xi_n(\Delta))\frac{\Delta_n^2}{\theta^2(x_n(\Delta))}.$$

Since $\Theta'(x)/\theta(x) \to 1$ as $x \to \infty$ and $x_n(\Delta) \to \infty$, the second term on the righthand side tends to Δ as $n \to \infty$. Define

$$\rho_n = \frac{\Theta''(\xi_n(\Delta))}{\theta^2(x_n(\Delta))}.$$

If $\rho_n \to 0$ as $n \to \infty$, we have that $\Theta(x_{n+1}(\Delta)) = \Theta(x_n(\Delta)) + \delta_n$ where $\delta_n \to \Delta$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \frac{\Theta(x_n(\Delta))}{n\Delta} = 1. \tag{3.5.5}$$

It remains to show that $\rho_n \to 0$ as $n \to \infty$. Since $\xi_n(\Delta) \in [x_n(\Delta), x_n(\Delta)+\Delta_n/\theta(x_n(\Delta))]$, $x_n(\Delta) \to \infty$ as $n \to \infty$ and $0 < \Delta_n \to \Delta$ as $n \to \infty$, it suffices to prove that

$$\lim_{x\to\infty} \frac{\sup_{c\in[0,2\Delta]} \Theta''(x + c/\theta(x))}{\theta^2(x)} = 0,$$

recalling that $\Theta''(x) > 0$ for $x > x^*$. By (3.2.7), we see that the last limit holds if

$$\lim_{x\to\infty} \frac{\sup_{c\in[0,2\Delta]} \frac{1}{f^2(x+c/\theta(x))\bar{F}^3(x+c/\theta(x))}}{\theta^2(x)} = 0. \tag{3.5.6}$$

Recalling that f is increasing and $\bar{F}$ is decreasing means that (3.5.6) is implied by

$$\lim_{x\to\infty} \frac{\frac{1}{f^2(x)\bar{F}^3(x+2\Delta/\theta(x))}}{\theta^2(x)} = 0.$$

By the definition of θ in (3.1.4), this is equivalent to

$$\lim_{x\to\infty} \frac{\frac{1}{\bar{F}^3(x+2\Delta/\theta(x))}}{f'(x)^4} = 0,$$

and in turn to

$$\lim_{x\to\infty} \frac{\bar{F}(x)^4}{\bar{F}^3(x+2\Delta/\theta(x))} = 0, \tag{3.5.7}$$

due to (3.2.2). Since $\bar{F}(x) \to 0$ as $x \to \infty$, (3.5.7) is implied by

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{\bar{F}(x+2\Delta/\theta(x))} = 1, \tag{3.5.8}$$

for instance.

Since $\bar{F}$ is decreasing, we have

$$\liminf_{x\to\infty} \frac{\bar{F}(x)}{\bar{F}(x+2\Delta/\theta(x))} \geq 1. \tag{3.5.9}$$

To prove (3.5.8), note by Taylor's theorem that for every $x > 0$ there exists $c(x) \in [0, 2\Delta]$ such that

$$\bar{F}(x+2\Delta/\theta(x)) = \bar{F}(x) - \frac{2\Delta}{\theta(x)} \frac{1}{f(x+c(x)/\theta(x))}.$$

Since f is increasing, we have

$$\bar{F}(x+2\Delta/\theta(x)) \geq \bar{F}(x) - \frac{2\Delta}{\theta(x)} \frac{1}{f(x)}.$$

By (3.1.4) and (3.2.2) and the fact that $\bar{F}(x) \to 0$ as $x \to \infty$, we have that

$$\lim_{x\to\infty} \theta(x) f(x) \bar{F}(x) = \lim_{x\to\infty} \frac{f'(x)^2}{f(x)} f(x)\bar{F}(x) = \lim_{x\to\infty} \frac{1}{\bar{F}(x)^2} \bar{F}(x) = \infty.$$

Therefore, for $x > 0$ sufficiently large, we may write

$$\frac{\bar{F}(x)}{\bar{F}(x+2\Delta/\theta(x))} \leq \frac{1}{1 - 2\Delta \frac{1}{\theta(x)f(x)\bar{F}(x)}},$$

and by taking limits we deduce that

$$\limsup_{x\to\infty} \frac{\bar{F}(x)}{\bar{F}(x+2\Delta/\theta(x))} \leq \limsup_{x\to\infty} \frac{1}{1 - 2\Delta \frac{1}{\theta(x)f(x)\bar{F}(x)}} = 1.$$

Combining this with (3.5.9), we get (3.5.8), as required. Since (3.5.8) implies (3.5.6), and therefore in turn that $\rho_n \to 0$ as $n \to \infty$, we have established (3.5.5).

3.5.3 Conclusion of the proof of Theorem 3.4.1

By virtue of (3.5.5), and the monotonicity of Θ, for every $\varepsilon \in (0, 1)$ there exists an $N(\varepsilon) \in \mathbb{N}$ such that

$$\Theta^{-1}((1-\varepsilon)n\Delta) < x_n(\Delta) < \Theta^{-1}((1+\varepsilon)n\Delta), \quad n \geq N(\varepsilon). \tag{3.5.10}$$

Since $t_n(\Delta) \to T_\Delta$ as $n \to \infty$, and

$$t_n(\Delta) = \sum_{j=0}^{n-1} \frac{\Delta}{f(x_j(\Delta))\theta(x_j(\Delta))}, \quad n \geq 1,$$

we have that

$$T_\Delta - t_n(\Delta) = \sum_{j=n}^{\infty} \frac{\Delta}{f(x_j(\Delta))\theta(x_j(\Delta))}.$$

Define

$$\bar{\tau}_n := \sum_{j=n}^{\infty} \frac{\Delta}{f(x_j(\Delta))\Theta'(x_j(\Delta))}. \tag{3.5.11}$$

Then by (3.2.6)

$$\lim_{n\to\infty} \frac{T_\Delta - t_n(\Delta)}{\bar{\tau}_n} = 1. \tag{3.5.12}$$

We now estimate the asymptotic behaviour of $\bar{\tau}_n$ as $n \to \infty$. Recall that $\Theta''(x) > 0$ for $x > x^*$. Since $\Theta^{-1}(x) \to \infty$ as $x \to \infty$, it follows that $\Theta^{-1}((1-\varepsilon)n\Delta) > x^*$ for all $n \geq N_1(\varepsilon)$. Let $N_2(\varepsilon) = \max(N_1(\varepsilon), N^*)$. Then for $j \geq N_2(\varepsilon)$, by the monotonicity of f and Θ' we have

$$\begin{aligned} f(\Theta^{-1}((1+\varepsilon)j\Delta))\Theta'(\Theta^{-1}((1+\varepsilon)j\Delta)) &> f(x_j(\Delta))\Theta'(x_j(\Delta)) \\ &> f(\Theta^{-1}((1-\varepsilon)j\Delta))\Theta'(\Theta^{-1}((1-\varepsilon)j\Delta)). \end{aligned}$$

Hence for $j \geq N_2(\varepsilon)$, we have

$$\begin{aligned} \frac{\Delta}{f(\Theta^{-1}((1+\varepsilon)j\Delta))\Theta'(\Theta^{-1}((1+\varepsilon)j\Delta))} &< \frac{\Delta}{f(x_j(\Delta))\Theta'(x_j(\Delta))} \\ &< \frac{\Delta}{f(\Theta^{-1}((1-\varepsilon)j\Delta))\Theta'(\Theta^{-1}((1-\varepsilon)j\Delta))}. \end{aligned} \tag{3.5.13}$$

Let $c > 0$ and define $S_n^{(c)}$ by

$$S_n^{(c)} = \sum_{j=n}^{\infty} \frac{\Delta}{f(\Theta^{-1}(c\Delta))\Theta'(\Theta^{-1}(cj\Delta))}. \tag{3.5.14}$$

Then by (3.5.11), (3.5.14) and (3.5.13), we have

$$S_n^{(1+\varepsilon)} < \bar{\tau}_n < S_n^{(1-\varepsilon)}, \quad n \geq N_2(\varepsilon). \tag{3.5.15}$$

The asymptotic behaviour of $S_n^{(1\pm\varepsilon)}$ will now be ascertained, and hence the asymptotic behaviour of $T_\Delta - t_n(\Delta)$, by using (3.5.15) and (3.5.12).

For every $c > 0$, there exists $N_3(c) \in \mathbb{N}$ such that $\Theta^{-1}(cj\Delta) > x^*$ for $j \geq N_3(c)$. Let $j \geq N_3(c)$. Then, for $x \in [cj\Delta, c(j+1)\Delta]$, by the monotonicity of Θ^{-1}, we have

$$\Theta^{-1}(cj\Delta) \leq \Theta^{-1}(x) \leq \Theta^{-1}(c(j+1)\Delta),$$

so the monotonicity of Θ' and f then yield

$$\begin{aligned} f(\Theta^{-1}(cj\Delta))\Theta'(\Theta^{-1}(cj\Delta)) &\leq f(\Theta^{-1}(x))\Theta'(\Theta^{-1}(x))) \\ &\leq f(\Theta^{-1}(c(j+1)\Delta))\Theta'(\Theta^{-1}(c(j+1)\Delta)), \quad j \geq N_3(c). \end{aligned}$$

Therefore integrating over $x \in [cj\Delta, c(j+1)\Delta]$ we get

$$\begin{aligned} \frac{c\Delta}{f(\Theta^{-1}(cj\Delta))\Theta'(\Theta^{-1}(cj\Delta))} &\leq \int_{cj\Delta}^{c(j+1)\Delta} \frac{1}{f(\Theta^{-1}(x))\Theta'(\Theta^{-1}(x)))}\,dx \\ &\leq \frac{c\Delta}{f(\Theta^{-1}(c(j+1)\Delta))\Theta'(\Theta^{-1}(c(j+1)\Delta))}, \quad j \geq N_3(c), \end{aligned}$$

and so, by summing across the inequality for $j \geq n \geq N_3(c)$, and using (3.5.14), we have

$$\begin{aligned} cS_n^{(c)} = \sum_{j=n}^{\infty} \frac{c\Delta}{f(\Theta^{-1}(cj\Delta))\Theta'(\Theta^{-1}(cj\Delta))} &\leq \int_{cn\Delta}^{\infty} \frac{1}{f(\Theta^{-1}(x))\Theta'(\Theta^{-1}(x)))}\,dx \\ &\leq \sum_{j=n}^{\infty} \frac{c\Delta}{f(\Theta^{-1}(c(j+1)\Delta))\Theta'(\Theta^{-1}(c(j+1)\Delta))} = cS_{n+1}^{(c)}. \end{aligned}$$

Since $\lim_{x\to\infty} \Theta^{-1}(x) = \infty$, by (0.1.12), we have that

$$\int_{cn\Delta}^{\infty} \frac{1}{f(\Theta^{-1}(x))\Theta'(\Theta^{-1}(x)))}\,dx = \int_{\Theta^{-1}(cn\Delta)}^{\infty} \frac{\Theta'(u)}{f(u)\Theta'(u)}\,du = \bar{F}(\Theta^{-1}(cn\Delta)).$$

Thus

$$cS_n^{(c)} \leq \bar{F}(\Theta^{-1}(cn\Delta)) \leq cS_{n+1}^{(c)}, \quad n \geq N_3(c).$$

Since $\Theta(x) = 1/\bar{F}(x)$, we have that $\bar{F}(\Theta^{-1}(x)) = 1/x$. Hence

$$c^2 S_n^{(c)} \leq \frac{1}{n\Delta} \leq c^2 S_{n+1}^{(c)}, \quad n \geq N_3(c). \tag{3.5.16}$$

Also observe by (3.5.5) and the definition of Θ that we have

$$\lim_{n\to\infty} \bar{F}(x_n(\Delta))n\Delta = \lim_{n\to\infty} \frac{1}{\Theta(x_n(\Delta))} n\Delta = 1. \tag{3.5.17}$$

Thus by (3.5.15) and the first member of (3.5.16), we get for $n \geq \max(N_2(\varepsilon), N_3(1-\varepsilon))$

$$\frac{\bar{\tau}_n}{\bar{F}(x_n(\Delta))} < \frac{S_n^{(1-\varepsilon)}}{\bar{F}(x_n(\Delta))} \leq \frac{1}{(1-\varepsilon)^2} \frac{1}{n\Delta\bar{F}(x_n(\Delta))},$$

and so by (3.5.17)

$$\limsup_{n\to\infty} \frac{\bar{\tau}_n}{\bar{F}(x_n(\Delta))} \leq \frac{1}{(1-\varepsilon)^2}.$$

Letting $\varepsilon \to 0$, and recalling (3.5.12) yields

$$\limsup_{n\to\infty} \frac{T_\Delta - t_n(\Delta)}{\bar{F}(x_n(\Delta))} \leq 1. \tag{3.5.18}$$

By (3.5.15) and the second member of (3.5.16), we have for $n \geq \max(N_2(\varepsilon), N_3(1+\varepsilon))$

$$\frac{\bar{\tau}_n}{\bar{F}(x_n(\Delta))} > \frac{S_n^{(1+\varepsilon)}}{\bar{F}(x_n(\Delta))} \geq \frac{1}{(1+\varepsilon)^2} \frac{1}{n\Delta \bar{F}(x_n(\Delta))} \frac{n}{n+1},$$

and so by (3.5.17)

$$\liminf_{n\to\infty} \frac{\bar{\tau}_n}{\bar{F}(x_n(\Delta))} \geq \frac{1}{(1+\varepsilon)^2}.$$

Letting $\varepsilon \to 0$, and recalling (3.5.12) yields

$$\liminf_{n\to\infty} \frac{T_\Delta - t_n(\Delta)}{\bar{F}(x_n(\Delta))} \geq 1.$$

Combining this limit with (3.5.18) yields

$$\lim_{n\to\infty} \frac{\bar{F}(x_n(\Delta))}{T_\Delta - t_n(\Delta)} = 1. \tag{3.5.19}$$

This completes the proof of part (a) of Theorem 3.4.1.

It remains to prove part (b) of Theorem 3.4.1. Since $\bar{X}_\Delta(t) \in [x_n(\Delta), x_{n+1}(\Delta)]$ for $t \in [t_n(\Delta), t_{n+1}(\Delta)]$, $t_n(\Delta) \uparrow T_\Delta$ as $n \to \infty$ and $x_n(\Delta) \to \infty$ as $n \to \infty$, we have that $\bar{X}_\Delta \to \infty$ as $t \uparrow T_\Delta$.

Since $\bar{X}_\Delta(t) \in [x_n(\Delta), x_{n+1}(\Delta)]$ for $t \in [t_n(\Delta), t_{n+1}(\Delta)]$ and $\bar{F}$ is decreasing, we get

$$\frac{T_\Delta - t_{n+1}(\Delta)}{\bar{F}(x_n(\Delta))} \leq \frac{T_\Delta - t}{\bar{F}(x_n(\Delta))} \leq \frac{T_\Delta - t}{\bar{F}(\bar{X}_\Delta(t))} \leq \frac{T_\Delta - t}{\bar{F}(x_{n+1}(\Delta))} \leq \frac{T_\Delta - t_n(\Delta)}{\bar{F}(x_{n+1}(\Delta))}.$$

Therefore for $t \in [t_n(\Delta), t_{n+1}(\Delta)]$, we have

$$\frac{T_\Delta - t_{n+1}(\Delta)}{\bar{F}(x_{n+1}(\Delta))} \frac{\bar{F}(x_{n+1}(\Delta))}{\bar{F}(x_n(\Delta))} \leq \frac{T_\Delta - t}{\bar{F}(\bar{X}_\Delta(t))} \leq \frac{T_\Delta - t_n(\Delta)}{\bar{F}(x_n(\Delta))} \frac{\bar{F}(x_n(\Delta))}{\bar{F}(x_{n+1}(\Delta))}. \tag{3.5.20}$$

Now, notice from (3.5.17) that

$$\lim_{n\to\infty} \frac{\bar{F}(x_{n+1}(\Delta))}{\bar{F}(x_n(\Delta))} = \lim_{n\to\infty} \frac{(n+1)\Delta \bar{F}(x_{n+1}(\Delta))}{n\Delta \bar{F}(x_n(\Delta))} \cdot \frac{n}{n+1} = 1. \tag{3.5.21}$$

Taking limits as $t \uparrow T_\Delta$ is equivalent to letting $n \to \infty$; taking the former limit across (3.5.20) and employing (3.5.21) yields

$$\lim_{t\to T_\Delta^-} \frac{T_\Delta - t}{\bar{F}(\bar{X}_\Delta(t))} = 1,$$

as required.

Chapter 4

Non-explosive Growth in Equations with Strictly Constrained Delay Coefficients

4.1 Introduction

In Chapter 1, we showed that a uniform step-size Euler discretisation resulted in a numerical method which underestimated the growth rate of a superlinear, non-explosive differential equation. This suggested that special meshes are needed to replicate the growth rates of such equations. In the following chapters we extend the numerical method described in Chapter 2 (where it was utilised to replicate the behaviour of explosive equations) for the purpose of constructing continuous–time approximations which inherit the exact growth rates of superlinear, non-explosive equations.

In this chapter, we begin by verifying that under condition (1.1.1) the solution to the differential equation grows unboundedly and the state-dependent method described in Section 2.3 does indeed replicate this behaviour. During this verification, which is featured in Section 4.2, we make no additional assumptions on f or g outside of what is necessary and sufficient to ensure that the solutions do *not* explode. Then, we consider equations for which the instantaneous feedback function f is both superlinear and dominant, in the sense that g/f is bounded. This has the effect of ensuring that the growth rate of the solution to the delay differential equation is determined solely by f. The exact rates of growth of the solutions to both the continuous and discrete–time equations with constrained delay coefficients are determined in Section 4.3, and the convergence of the numerical method is demonstrated in Section 4.4. In later chapters, we determine the asymptotics of equations for which g/f tends to infinity, investigating what determines the growth rates of the differential equation and what numerical methods are needed to replicate these rates.

4.2 Unbounded Growth

Recall that under condition (1.1.1), the solution to the delay differential equation given by (0.1.6) *cannot* explode in finite–time. Now, consider the state-dependent discretisation introduced in Section 2.3. We now show that the function $\bar{X}_\Delta$ defined by (2.3.7) cannot explode in finite–time and mimics other properties of the solution x of (0.1.6) (cf. Theorem 1.1.1).

Theorem 4.2.1. *Let f obey* (0.1.7)*,* (1.1.1) *and*

$$f \text{ is non-decreasing on } [0,\infty), \tag{4.2.1}$$

g obey (0.1.8)*, and ψ obey* (0.1.9) *where $\tau > 0$. Let $\Delta \in (0, \tau f(\psi(0)))$ and $\bar{X}_\Delta$ be defined by* (2.3.7)*. Then $\bar{X}_\Delta \in C([-\tau,\infty);(0,\infty))$ is increasing on $[0,\infty)$ and*

$$\lim_{t\to\infty} \bar{X}_\Delta(t) = \infty. \tag{4.2.2}$$

Proof. Define

$$I_n(\Delta) = \int_{t_n(\Delta)}^{t_{n+1}(\Delta)} g(X_\Delta(s-\tau))\, ds.$$

where $t_n(\Delta)$ and X_Δ are defined by (2.3.4) and (2.3.6). Notice that (2.3.4) and (2.3.5) imply for $n \geq 0$ that

$$x_{n+1}(\Delta) = x_n(\Delta) + \Delta + I_n(\Delta) > x_n(\Delta) + \Delta.$$

Hence

$$x_n(\Delta) \geq \psi(0) + n\Delta, \quad n \geq 0,$$

so $x_n(\Delta) \to \infty$ as $n \to \infty$. Next as $(t_n(\Delta))_{n\geq 0}$ is an increasing sequence, we notice that there exists $T_\Delta \in (0,\infty]$ which obeys (2.4.3), that is

$$T_\Delta := \lim_{n\to\infty} t_n(\Delta).$$

Since $\bar{X}_\Delta(t) \geq x_n(\Delta)$ for all $t \in [t_n(\Delta), t_{n+1}(\Delta))$, we have

$$\lim_{t\to T_\Delta^-} \bar{X}_\Delta(t) = \infty.$$

Moreover, by (2.4.3), the domain of definition of $\bar{X}_\Delta$ is $[-\tau, T_\Delta)$, and $\bar{X}_\Delta$ is continuous on $[-\tau, T_\Delta)$. We have already shown in Section 2.3 that $\bar{X}_\Delta$ is increasing on $[0, t_n(\Delta))$ for all $n \geq 0$ and therefore $\bar{X}_\Delta$ is increasing on $[0, T_\Delta)$.

It remains to show that when f obeys (1.1.1), $T_\Delta = +\infty$. Suppose the converse is true. There exists $T_n(\Delta) \in [t_n(\Delta), t_{n+1}(\Delta)]$ such that $g(X_\Delta(s-\tau)) \le g(X_\Delta(T_n(\Delta)-\tau))$ for all $s \in [t_n(\Delta), t_{n+1}(\Delta)]$. Therefore

$$\begin{aligned} I_n(\Delta) &= \int_{t_n(\Delta)}^{t_{n+1}(\Delta)} g(X_\Delta(s-\tau))\,ds \le (t_{n+1}(\Delta)-t_n(\Delta))g(X_\Delta(T_n(\Delta)-\tau)) \\ &= \frac{\Delta}{f(x_n(\Delta))} g(X_\Delta(T_n(\Delta)-\tau)). \end{aligned}$$

If $T_\Delta < \infty$, then $T_n(\Delta)-\tau \le T_\Delta - \tau$. Since X_Δ is finite on $[-\tau, T_\Delta-\tau]$, and $f(x_n(\Delta)) \to \infty$ as $n \to \infty$, it follows that $I_n(\Delta) \to 0$ as $n \to \infty$. Therefore, $x_n(\Delta)/n \to \Delta$ as $n \to \infty$. Hence there exists $N \in \mathbb{N}$ such that

$$x_n(\Delta) < 2n\Delta, \quad n > N.$$

Therefore as f is non-decreasing

$$\frac{\Delta}{f(x_n(\Delta))} \ge \frac{\Delta}{f(2n\Delta)}, \quad n > N.$$

By (2.3.4), we have that

$$t_n(\Delta) = \sum_{j=0}^{n-1} \frac{\Delta}{f(x_j(\Delta))}, \quad n \ge 0,$$

and so for $n > N+1$ we get

$$t_n(\Delta) = \sum_{j=0}^{N} \frac{\Delta}{f(x_j(\Delta))} + \sum_{j=N+1}^{n-1} \frac{\Delta}{f(x_j(\Delta))} \ge \sum_{j=0}^{N} \frac{\Delta}{f(x_j(\Delta))} + \sum_{j=N+1}^{n-1} \frac{\Delta}{f(2j\Delta)}$$

Therefore $T_\Delta = +\infty$ provided

$$\sum_{j=N+1}^{\infty} \frac{2\Delta}{f(2j\Delta)} = +\infty. \tag{4.2.3}$$

(4.2.1) implies for $x \in [2j\Delta, 2(j+1)\Delta]$ that

$$\frac{1}{f(2j\Delta)} \ge \frac{1}{f(x)}.$$

Hence

$$\frac{2\Delta}{f(2j\Delta)} \ge \int_{2j\Delta}^{2(j+1)\Delta} \frac{1}{f(x)}\,dx.$$

Therefore for $n > N$

$$\sum_{j=N+1}^{n} \frac{2\Delta}{f(2j\Delta)} \ge \sum_{j=N+1}^{n} \int_{2j\Delta}^{2(j+1)\Delta} \frac{1}{f(x)}\,dx = \int_{2(N+1)\Delta}^{2(n+1)\Delta} \frac{1}{f(x)}\,dx.$$

Taking limits as $n \to \infty$ and using (1.1.1) establishes (4.2.3), which proves $T_\Delta = \infty$. □

4.3 Determination of Growth Rates

In this section, we determine the growth rate of the solution to the state-dependent numerical scheme in the case where f is superlinear and g/f is bounded. Firstly, we state Theorem 1.2.4 from Chapter 1, where we showed that the growth rate to the delay differential equation given by (0.1.6) grows at a rate consistent with the related ODE given by $y'(t) = f(y(t))$.

Statement of Theorem 1.2.4 Suppose that f obeys (0.1.7), (1.1.1) and let f obey (1.2.19), that is

$$\lim_{x\to\infty} f(x)/x = \infty, f \in \text{RV}_\infty(1); \quad \text{and}$$
$$\text{there exists } \Lambda \in [0,\infty) \text{ such that } \Lambda := \limsup_{x\to\infty} \frac{g(x)}{f(x)}.$$

Let g obey (0.1.8) and let $\tau > 0$ and ψ obey (0.1.9). Then the unique continuous solution x of (0.1.6) satisfies (1.2.20), that is

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1,$$

where F is defined by (1.2.4).

Next, we prove the discrete–time analogue of Theorem 1.2.4 and use this to show that the continuous–time interpolant of the numerical scheme described in Section 2.3 inherits the same rate of growth.

Before starting, we state and prove an auxiliary lemma.

Lemma 4.3.1. *Suppose f obeys* (4.2.1) *and* (1.2.19). *Then*

$$\lim_{x\to\infty} \frac{f(F^{-1}(F(x)-\tau))}{f(x)} = 0. \tag{4.3.1}$$

Proof. Define y by $y'(t) = f(y(t))$, $t > 0$ and $y(0) = \xi$. Then $y(t) = F^{-1}(t)$. Since $f(x)/x \to \infty$ as $x \to \infty$, we have $y'(t)/y(t) \to \infty$ as $t \to \infty$. Therefore, for every $M > 0$ there exists $T_M > 0$ such that $y'(t)/y(t) > M$ for all $t > T_M$. Then for $t > T_M + \tau =: T'_M$ we have

$$\log\left(\frac{y(t)}{y(t-\tau)}\right) = \int_{t-\tau}^{t} \frac{y'(s)}{y(s)}\,ds \geq M\tau.$$

Hence $y(t)/y(t-\tau) \geq e^{M\tau}$, so $y(t-\tau)/y(t) \leq e^{-M\tau}$ for $t > T'_M$. Therefore $y(t-\tau)/y(t) \to 0$ as $t \to \infty$. Hence

$$\lim_{t\to\infty} \frac{F^{-1}(t-\tau)}{F^{-1}(t)} = 0.$$

Since $F(x) \to \infty$ as $x \to \infty$, we have

$$\lim_{x\to\infty} \frac{F^{-1}(F(x)-\tau)}{x} = 0.$$

Thus for every $M > 0$ there is an $x_M > 0$ such that

$$F^{-1}(F(x)-\tau)/x < 1/M, \quad x > x_M.$$

Hence $F^{-1}(F(x-\tau)) < x/M$ for $x > x_M$. Since f is non-decreasing, we have

$$\frac{f(F^{-1}(F(x-\tau)))}{f(x)} \leq \frac{f(x/M)}{f(x)}, \quad x > x_M.$$

Thus as $f \in \mathrm{RV}_\infty(1)$, we have

$$\limsup_{x\to\infty} \frac{f(F^{-1}(F(x-\tau)))}{f(x)} \leq \lim_{x\to\infty} \frac{f(x/M)}{f(x)} = \frac{1}{M}.$$

Letting $M \to \infty$ establishes the result. $\square$

Theorem 4.3.1. *Let f obey* (0.1.7), (1.1.1), (4.2.1) *and* (1.2.19). *Let g obey* (0.1.8) *and let $\tau > 0$ and ψ obey* (0.1.9). *Let $\Delta \in (0, \tau f(\psi(0)))$ and suppose $t_n(\Delta)$, $x_n(\Delta)$, X_Δ and $\bar{X}_\Delta$ are given by* (2.3.4), (2.3.5), (2.3.6) *and* (2.3.7). *Then*

$$\lim_{n\to\infty} \frac{F(x_n(\Delta))}{t_n(\Delta)} = 1, \tag{4.3.2}$$

and

$$\lim_{t\to\infty} \frac{F(\bar{X}_\Delta(t))}{t} = 1. \tag{4.3.3}$$

Proof. Note that since f and g are positive, $x_{n+1}(\Delta) > x_n(\Delta) + \Delta$ and so $x_n(\Delta) \geq \psi(0) + n\Delta$ for $n \geq 0$. As F is increasing, using the same arguments used to prove Theorem 4.2.1 we have

$$\begin{aligned} F(x_n(\Delta)) \geq F(\psi(0)+n\Delta) &\geq \sum_{j=0}^{n-1} \frac{\Delta}{f(\psi(0)+(j+1)\Delta)} \\ &= \sum_{j=0}^{n} \frac{\Delta}{f(\psi(0)+j\Delta)} - \frac{\Delta}{f(\psi(0))} \\ &= t_{n+1}(\Delta) - \frac{\Delta}{f(\psi(0))} > t_n(\Delta) - \frac{\Delta}{f(\psi(0))}. \end{aligned}$$

So

$$\liminf_{n\to\infty} \frac{F(x_n(\Delta))}{t_n(\Delta) - \Delta/f(\psi(0))} \geq 1$$

and as $t_n(\Delta) \to \infty$ as $n \to \infty$

$$\liminf_{n\to\infty} \frac{F(x_n(\Delta))}{t_n(\Delta)} \geq 1. \tag{4.3.4}$$

If we can prove a similar result for an upper estimate of the solution we will have (4.3.2). Note that since f obeys (1.2.19b), there exists $\bar{\Lambda} > \Lambda$

$$I_n(\Delta) = \int_{t_n(\Delta)}^{t_{n+1}(\Delta)} g(X_\Delta(s-\tau))\,ds \leq \bar{\Lambda} \int_{t_n(\Delta)}^{t_{n+1}(\Delta)} f(X_\Delta(s-\tau))\,ds.$$

Since X_Δ is non-decreasing on $[0,\infty)$, there exists N_Δ^* such that for $n > N_\Delta^*$ and $s \in [t_n(\Delta), t_{n+1}(\Delta))$ we have $X_\Delta(s-\tau) \leq X_\Delta(t_{n+1}(\Delta)-\tau)$. Therefore for $n > N_\Delta^*$

$$\begin{aligned} I_n(\Delta) &\leq \bar{\Lambda}(t_{n+1}(\Delta) - t_n(\Delta)) f(X_\Delta(t_{n+1}(\Delta)-\tau)) \\ &= \bar{\Lambda}\Delta \frac{f(X_\Delta(t_{n+1}(\Delta)-\tau))}{f(x_n(\Delta))}. \end{aligned}$$

Next define $N(n) \in \mathbb{N}$ such that $t_{N(n)}(\Delta) \leq t_{n+1}(\Delta) - \tau < t_{N(n)+1}(\Delta)$. This implies $x_{N(n)}(\Delta) = X_\Delta(t_{n+1}(\Delta)-\tau)$. So for $n > N_\Delta^*$

$$I_n(\Delta) \leq \bar{\Lambda}\Delta \frac{f(x_{N(n)}(\Delta))}{f(x_n(\Delta))}.$$

Clearly $N(n) \leq n$ as $X_\Delta(t_{n+1}(\Delta)-\tau) \leq x_n(\Delta)$ in accordance with Remark 2.3.2. Therefore as $x_n(\Delta)$ is increasing and f is non-decreasing, $f(x_{N(n)}(\Delta)) \leq f(x_n(\Delta))$ and $x_n(\Delta) \leq \psi(0) + (1+\bar{\Lambda})n\Delta$ for $n > N_\Delta^*$, thus

$$\begin{aligned} \frac{f(x_{N(n)}(\Delta))}{f(x_n(\Delta))} &\leq \frac{f(\psi(0) + (1+\bar{\Lambda})N(n)\Delta)}{f(n\Delta)}, \\ &= \frac{f(\psi(0) + (1+\bar{\Lambda})N(n)\Delta)}{f(N(n)\Delta)} \cdot \frac{f(N(n)\Delta)}{f(n\Delta)}, \quad n > N_\Delta^*. \end{aligned}$$

Therefore as $f \in RV_\infty(1)$ and $N(n) \to \infty$ as $n \to \infty$,

$$\limsup_{n\to\infty} \frac{f(x_{N(n)}(\Delta))}{f(x_n(\Delta))} \leq (1+\bar{\Lambda}) \limsup_{n\to\infty} \frac{f(N(n)\Delta)}{f(n\Delta)}. \tag{4.3.5}$$

Now for any $c > 0$,

$$\frac{f(N(n)\Delta)}{f(n\Delta)} = \frac{f(N(n)\Delta)}{f(F^{-1}(F(n\Delta)-c))} \cdot \frac{f(F^{-1}(F(n\Delta)-c))}{f(n\Delta)}.$$

The second factor tends to zero as $n \to \infty$ by (4.3.1). For the first factor, note that

$$t_{n+1}(\Delta) - t_{N(n)}(\Delta) = \sum_{j=N(n)}^{n} \frac{\Delta}{f(x_j(\Delta)}$$

and $x_n(\Delta) > n\Delta$, so

$$\sum_{j=N(n)}^{n} \frac{\Delta}{f(j\Delta)} \geq \sum_{j=N(n)}^{n} \frac{\Delta}{f(x_j(\Delta)} = t_{n+1}(\Delta) - t_{N(n)}(\Delta) \geq \tau. \tag{4.3.6}$$

Now since f is non-decreasing,

$$\sum_{j=N(n)}^{n} \frac{\Delta}{f(j\Delta)} \leq \frac{\Delta}{f(N(n)\Delta)} + \int_{N(n)\Delta}^{(n-1)\Delta} \frac{1}{f(u)}\,du.$$

and so

$$\sum_{j=N(n)}^{n} \frac{\Delta}{f(j\Delta)} \leq \frac{\Delta}{f(N(n)\Delta)} + F((n-1)\Delta) - F(N(n)\Delta).$$

Now as $N(n) \to \infty$ as $n \to \infty$, there exists $n_\psi > 0$ such that $N(n)\Delta > \psi(0)$ for $n > n_\psi$. Using this and the fact that f and F are non-decreasing we have for $n > n_\psi$,

$$\sum_{j=N(n)}^{n} \frac{\Delta}{f(j\Delta)} \leq \frac{\Delta}{f(\psi(0))} + F(n\Delta) - F(N(n)\Delta)$$

and by (4.3.6),

$$F(n\Delta) - F(N(n)\Delta) \geq \sum_{j=N(n)}^{n} \frac{\Delta}{f(j\Delta)} - \frac{\Delta}{f(\psi(0))} \geq \tau - \frac{\Delta}{f(\psi(0))} =: c, \quad n > n_\psi$$

So $F(n\Delta) - c \geq F(N(n)\Delta)$ and since F^{-1} and f are non-decreasing, $f(F^{-1}(F(n\Delta)-c)) \geq f(N(n)\Delta)$ for $n > n_\psi$. Therefore

$$\limsup_{n\to\infty} \frac{f(N(n)\Delta)}{f(n\Delta)} \leq \limsup_{n\to\infty} \frac{f(N(n)\Delta)}{f(F^{-1}(F(n\Delta)-c))} \cdot \limsup_{n\to\infty} \frac{f(F^{-1}(F(n\Delta)-c))}{f(n\Delta)} \leq 1 \cdot 0 = 0,$$

and inserting this into (4.3.5) we have

$$\limsup_{n\to\infty} \frac{f(x_{N(n)}(\Delta))}{f(x_n(\Delta))} = 0.$$

Therefore for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $f(x_{N(n)}(\Delta))/f(x_n(\Delta)) < \epsilon/\bar{\Lambda}$ and $x_{N_\epsilon}(\Delta) > \psi^*$ for $n > N_\epsilon - 1$. Thus

$$I_n(\Delta) \leq \bar{\Lambda}\Delta\frac{\epsilon}{\bar{\Lambda}} = \epsilon\Delta, \quad n > N_\epsilon - 1$$

and so

$$x_{n+1}(\Delta) \leq x_n(\Delta) + (1+\epsilon)\Delta, \quad n > N_\epsilon - 1,$$

yielding

$$F(x_n(\Delta)) < F(x_{N_\epsilon}(\Delta) + (n - N_\epsilon)(1 + \epsilon)\Delta), \quad n > N_\epsilon \tag{4.3.7}$$

as $F(x)$ is increasing for $x > \psi^*$. Next for $n > N_\epsilon + 1$,

$$\begin{aligned}\int_{x_{N_\epsilon}(\Delta)+\Delta(1+\epsilon)}^{x_{N_\epsilon}(\Delta)+(n-N_\epsilon)\Delta(1+\epsilon)} \frac{1}{f(u)}\,du &\leq \sum_{j=N_\epsilon+1}^{n-1} \frac{\Delta(1+\epsilon)}{f(x_{N_\epsilon}(\Delta) + (j - N_\epsilon)(1+\epsilon)\Delta)}, \\ &\leq (1+\epsilon) \sum_{j=N_\epsilon+1}^{n-1} \frac{\Delta}{f(x_j(\Delta))}, \\ &= (1+\epsilon)(t_n(\Delta) - t_{N_\epsilon+1}(\Delta)).\end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \frac{F(x_{N_\epsilon}(\Delta) + (n - N_\epsilon)\Delta(1+\epsilon))}{t_n(\Delta) - t_{N_\epsilon+1}(\Delta)} \leq 1 + \epsilon$$

and by (4.3.7) we obtain

$$\limsup_{n\to\infty} \frac{F(x_n(\Delta))}{t_n(\Delta)} \leq 1 + \epsilon.$$

Letting $\epsilon \to 0^+$ and combining this result with (4.3.4) yields (4.3.2).

(4.3.3) follows from (4.3.2). Note that for every $t > 0$ there is an $n(t) \in \mathbb{N}$ such that $t \in [n(t)h, (n(t)+1)h)$, so $x_{n(t)}(h) \leq \bar{X}_\Delta(t) < x_{n(t)+1}(h)$. As F is increasing,

$$\frac{n(t)\Delta}{t}\frac{1}{n(t)\Delta}F(x_{n(t)}(\Delta)) \leq \frac{1}{t}F(\bar{X}_\Delta(t)) \leq \frac{(n(t)+1)\Delta}{t}\frac{1}{(n(t)+1)\Delta}F(x_{n(t)+1}(\Delta)).$$

As $n(t)\Delta/t \to 1$ as $t \to \infty$, (4.3.2) implies (4.3.3). □

4.4 Controlling the Approximation Error

In Section 2.5, we showed that the numerical approximation $\bar{X}_\Delta$ could be made arbitrarily close to the true solution x on a compact interval where both $\bar{X}_\Delta$ and x remain finite. It was necessary to consider such an interval in order to ensure that both functions were well-defined, as Theorem 2.5.1 makes no assumptions on the integrability of $1/f$ and so must include the possibility of a finite–time explosion. However, if f obeys (1.1.1) and the solutions to the differential equation and the numerical scheme do not explode, both x and $\bar{X}_\Delta$ are well-defined for all $t \geq 0$. Therefore if (1.1.1) holds the error can be controlled on any compact interval.

Theorem 4.4.1. *Let f obey (0.1.7), (1.1.1), (4.2.1) and (1.2.19). Let g obey (0.1.8) and let $\tau > 0$ and ψ obey (0.1.9). Then there exists a unique continuous solution $x \in C([-\tau, \infty))$ of (0.1.6) which obeys $\lim_{t\to\infty} x(t) = \infty$.*

Let $\Delta \in (0, \tau f(\psi(0)))$ and suppose $t_n(\Delta)$, $x_n(\Delta)$, X_Δ and $\bar{X}_\Delta$ are given by (2.3.4), (2.3.5), (2.3.6) and (2.3.7). Let $T > 0$. Then

$$\lim_{\Delta\to 0} \sup_{t\in[0,T]} \left|x(t) - \bar{X}_\Delta(t)\right| = 0. \tag{4.4.1}$$

Proof. The proof is very similar to the proof of Theorem 2.5.1. However, recall that Theorem 2.5.1 proved that

$$\sup_{0\le t\le\ \rho_M \wedge \bar{\rho}_{2M}(\Delta)} |x(t) - \bar{X}_\Delta(t)| \le (\Delta \rho_M K_{2M} + 2\tau\omega(\Delta, 2M))e^{\rho_M(c^f_{2M}+c^g_{2M})}.$$

The Lipschitz constants c^f_{2M} and c^g_{2M} were easily determined as given $M > \psi^*$ both $x, \bar{X}_\Delta \in [0, 2M]$ for $t \in \rho_M \wedge \bar{\rho}_{2M}(\Delta)$. For this Theorem however, determining the Lipschitz constants is not quite as straightforward.

For $t > 0$, $x'(t)/f(x(t)) = 1 + g(x(t-\tau))/f(x(t))$. Now if $t \le \tau$,

$$g(x(t-\tau)) = g(\psi(t-\tau)) \le \max_{x\in(0,\psi^*]} g(x) =: \bar{g}_\psi.$$

Also since f is monotone and x is non-decreasing on $[0, \infty)$, $f(x(t)) \ge f(x(0)) = f(\psi(0))$. Thus

$$\frac{x'(t)}{f(x(t))} \le 1 + \frac{\bar{g}_\psi}{f(\psi(0))} =: 1 + L_\psi, \quad t \le \tau.$$

If $t > \tau$, since f obeys (4.2.1) and (1.2.19) and x is non-decreasing on $[0, \infty)$, by (1.2.19b) there exists $\bar{\Lambda} > \Lambda$ such that

$$\frac{x'(t)}{f(x(t))} \le 1 + \frac{g(x(t-\tau))}{f(x(t))} \le 1 + \bar{\Lambda} \cdot \frac{f(x(t-\tau))}{f(x(t))} \le 1 + \bar{\Lambda}.$$

Setting $L^* = \max(L_\psi, \bar{\Lambda})$ we have $x'(t)/f(x(t)) \le 1 + L^*$ for all $t > 0$. Integrating over $(0, t]$ yields

$$F(x(t)) \le F(\psi(0)) + (1 + L^*)t$$

and so as F^{-1} is increasing, for any $T > 0$

$$x(T) \le F^{-1}(F(\psi(0)) + (1 + L^*)T) =: M(T). \tag{4.4.2}$$

Since $x_n(\Delta)$ obeys (2.3.5), for $n \geq 0$

$$\begin{aligned}
x_{n+1}(\Delta) &= x_n(\Delta) + \Delta + \int_{t_n(\Delta)}^{t_{n+1}(\Delta)} g(X_\Delta(s-\tau))\,ds \\
&\leq x_n(\Delta) + \Delta + ((t_{n+1}(\Delta) - t_n(\Delta)) \max_{t_n(\Delta)\leq u \leq t_{n+1}(\Delta)} g(\bar{X}_\Delta(u-\tau)) \\
&= x_n(\Delta) + \Delta + \Delta \cdot \frac{g(\bar{X}_\Delta(r_n(\Delta)-\tau))}{f(x_n(\Delta))}, \quad r_n(\Delta) \in [t_n(\Delta), t_{n+1}(\Delta)].
\end{aligned}$$

If $r_n(\Delta) \leq \tau$,

$$x_{n+1}(\Delta) \leq x_n(\Delta) + \Delta(1 + \bar{g}_\psi), \quad n \text{ such that } r_n(\Delta) \leq \tau.$$

If $r_n(\Delta) > \tau$, then $t \geq \tau$. Since $\bar{X}_\Delta$ is non-decreasing on $(0,\infty)$ and since f is non-decreasing and obeys (1.2.19),

$$g(\bar{X}_\Delta(r_n(\Delta)-\tau)) \leq \Lambda f(\bar{X}_\Delta(r_n(\Delta)-\tau)) \leq \Lambda f(\bar{X}_\Delta(t_{n+1}(\Delta)-\tau)).$$

Since $t_{n+1}(\Delta) > t_n(\Delta) \geq \tau$ we have $f(x_n(\Delta)) \geq f(\psi(0))$. As $\Delta < \tau f(\psi(0))$ we get

$$0 < t_{n+1}(\Delta) - \tau = t_n(\Delta) - \tau + \Delta/f(x_n(\Delta)) \leq t_n(\Delta) - \tau + \Delta/f(\psi(0)) < t_n(\Delta).$$

Therefore $g(\bar{X}_\Delta(r_n(\Delta)-\tau)) \leq \Lambda f(\bar{X}(t_n(\Delta))) = \Lambda f(x_n(\Delta))$. Hence

$$\begin{aligned}
x_{n+1}(\Delta) &\leq x_n(\Delta) + \Delta + \Delta \cdot \frac{g(\bar{X}_\Delta(r_n(\Delta)-\tau))}{f(x_n(\Delta))} \\
&\leq x_n(\Delta) + \Delta(1+\Lambda), \quad n \text{ such that } r_n(\Delta) > \tau.
\end{aligned}$$

Therefore $x_{n+1}(\Delta) \leq x_n(\Delta) + \Delta(1+L^*)$ for all $n \geq 0$, and so

$$x_n(\Delta) \leq \psi(0) + n\Delta(1+L^*), \quad n \geq 0. \tag{4.4.3}$$

Now since $t_n(\Delta)$ defined by (2.3.4) is increasing, for any $T > 0$ there exists $n_\Delta(T) > 0$ such that

$$t_{n_\Delta(T)}(\Delta) \geq T > t_{n_\Delta(T)-1}(\Delta).$$

Note that $\bar{X}_\Delta(T) \leq x_{n_\Delta(T)}(\Delta)$. Thus

$$\bar{X}_\Delta(T) \leq \psi(0) + n_\Delta(T)\Delta(1+L^*) =: M_\Delta(T).$$

We wish to bound $M_\Delta(T)$ uniformly in Δ. Since $n_\Delta(T)$ is such that $T > t_{n_\Delta(T)-1}(\Delta)$, f is non-decreasing and $x_n(\Delta)$ obeys (4.4.3),

$$\begin{aligned}
T > t_{n_\Delta(T)-1}(\Delta) &= \sum_{j=0}^{n_\Delta(T)-2} \frac{\Delta}{f(\psi(0)+j\Delta(1+L^*))} \\
&\geq \frac{1}{1+L^*}\int_{\psi(0)}^{\psi(0)+(n_\Delta(T)-1)\Delta(1+L^*)} \frac{1}{f(u)}\,du \\
&= \frac{1}{1+L^*}\left(F(\psi(0)+(n_\Delta(T)-1)\Delta(1+L^*)) - F(\psi(0))\right) \\
&= \frac{1}{1+L^*}\left(F(M_\Delta(T)-\Delta(1+L^*)) - F(\psi(0))\right).
\end{aligned}$$

Therefore as F^{-1} is increasing,

$$M_\Delta(T) < \Delta(1+L^*) + F^{-1}((1+L^*)T + F(\psi(0)))$$

and so for any $\Delta_0 \in (0, \tau f(\psi(0)))$,

$$M_\Delta(T) < \Delta_0(1+L^*) + F^{-1}((1+L^*)T + F(\psi(0))), \quad \Delta < \Delta_0.$$

Thus

$$\bar{X}_\Delta(T) < M_{\Delta_0}(T), \quad \Delta < \Delta_0. \tag{4.4.4}$$

Setting $M^*(T) = \max(\psi^*, M(T), M_{\Delta_0}(T))$ we have that $x(t), \bar{X}_\Delta(t), X_\Delta(t) \in [0, M^*(T)]$ for $t \in [-\tau, T]$ and $\Delta < \Delta_0$. Define the Lipschitz constants as $c^f_{M^*(T)}$ and $c^g_{M^*(T)}$. The rest of the proof is similar to that of Theorem 2.5.1, leading to an error estimate of the form

$$\sup_{t\in[0,T]} \left|x(t) - \bar{X}_\Delta(t)\right| \leq (\Delta T K_{M^*(T)} + 2\tau\omega(\Delta, M^*(T)))e^{T(c^f_{M^*(T)}+c^g_{M^*(T)})}, \quad \Delta < \Delta_0.$$

Taking limits as $\Delta \to 0$ and noting $\omega(\Delta, M^*(T)) \to 0$ yields the desired result. □

Chapter 5

Instantaneously-Dominated Growth Rates

5.1 Introduction

In the previous chapter, we investigated the growth rate of superlinear delay differential equations when the delay term is constrained, that is $\limsup_{x\to\infty} g(x)/f(x) < \infty$. We saw in Section 4.3 that the asymptotics of the solution were determined solely by the instantaneous feedback function f. However if the delay term is no longer constrained in this manner, things become more complex. Now the magnitude of the delayed feedback g and the size of the delay τ can be such that the growth rate of the solution is no longer determined by f, that is the equation will no longer grow at a rate identical to that of the equivalent ODE given by $y'(t) = f(y(t))$. In the forthcoming chapters, we will investigate how the relationships between f, g and τ determine the long-term behaviour of (0.1.6). Firstly, we determine the conditions under which the equation with unconstrained delay coefficient does indeed grow at a rate identical to that of the equivalent ODE.

The boundedness of g/f is far from being a necessary condition for an instantaneously–dominated growth rate of the equation. It is intuitive to conjecture that the size of the delay term τ is of critical importance. We may indeed have $g/f \to \infty$, but if the delay term is large enough the contribution of the delayed feedback could be insignificant in the determining the growth rate of the solution. In Section 5.2, we establish criteria on the size of g and τ relative to f under which the solution of the delay equation inherits the rate of growth of the equivalent ODE. We show that the general sufficient conditions we develop are quite sharp by showing that when they are relaxed, the growth rate changes. However in the chapter we restrict ourselves to equations for which the rate of growth is indeed determined by f, albeit perhaps not at the *exact* rate exhibited by the solution to the corresponding ordinary differential equation. We refer to such equations as being “instantaneously-dominated”. A treatment of the cases where the rate is no longer characterised by f is given in Chapter 7.

Examples are featured in Section 5.3, and proofs are for the most part deferred to Section 5.4.

5.2 Criteria for Inheriting the Growth Rate of ODE

Note that the solution of the ODE given by

$$y'(t) = f(y(t)), \quad t \geq 0; \quad y(0) = \xi > 0, \tag{5.2.1}$$

is given by $F(y(t)) = t$ for $t \geq 0$ so that

$$\lim_{t\to\infty} \frac{F(y(t))}{t} = 1. \tag{5.2.2}$$

We define for $\theta > 0$ the function

$$f_\theta(x) = f(F^{-1}(F(x) + \theta)), \quad x \geq 0. \tag{5.2.3}$$

In our first main result, we show that if the delayed term g is asymptotically dominated by the instantaneous term f, in the sense that $g = o(f_\tau)$, then the solution of (0.1.6) inherits the asymptotic behaviour of (5.2.1) characterised by the limit (5.2.2).

Theorem 5.2.1. *Suppose that f obeys* (0.1.7), (1.1.1) *and* (4.2.1)*. Let g obey* (0.1.8) *and let $\tau > 0$ and ψ obey* (0.1.9)*. Suppose that f_τ is defined by* (5.2.3)*, F is defined by* (1.2.4)*, and x is the unique continuous solution of* (0.1.6)*.*

(i) If

$$\lim_{x\to\infty} \frac{g(x)}{f_\tau(x)} = 0, \tag{5.2.4}$$

then

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1. \tag{5.2.5}$$

(ii) If

$$\textit{there exists } \Lambda > 0 \textit{ such that } \limsup_{x\to\infty} \frac{g(x)}{f_\tau(x)} = \Lambda, \tag{5.2.6}$$

then

$$1 \leq \liminf_{t\to\infty} \frac{F(x(t))}{t} \leq \limsup_{t\to\infty} \frac{F(x(t))}{t} \leq 1 + \Lambda. \tag{5.2.7}$$

The results of Theorem 5.2.1 hint that the rate of growth of $g = o(f_\tau)$ in (5.2.4) is close to being "critical", in the sense that if g grows more rapidly than (5.2.4), the rate of growth of solutions of (0.1.6) depart from those of (5.2.1). This is because the rate of growth of $g = O(f_\tau)$ allowed for in (5.2.6) leaves open the possibility that x does not obey (5.2.5), as suggested by the presence of a non-unit upper bound in the right-most member of the inequality (5.2.7). As we will see in a later *example*, the growth of g given by (5.2.6) can lead to the upper bound in (5.2.7) being sharp, so that x does not obey (5.2.5).

Roughly speaking, our next result shows in the *general* case that $g = O(f_\theta)$ for $\theta > \tau$, then x does not obey (5.2.5), justifying the notion that the hypothesis $g = O(f_\tau)$ is close to being sharp.

Theorem 5.2.2. *Suppose f obeys* (0.1.7), (1.1.1) *and* (4.2.1). *Let g be non-decreasing and obey* (0.1.8). *Let $\tau > 0$ and ψ obey* (0.1.9). *Suppose that f_θ is defined by* (5.2.3). *Let x be the unique continuous solution of* (0.1.6).

(i) If

$$\text{there exists } \tau_0 > \tau \text{ and } \Lambda_0 \in (0, \tau_0/\tau - 1] \text{ such that } \liminf_{x\to\infty} \frac{g(x)}{f_{\tau_0}(x)} = \Lambda_0, \tag{5.2.8}$$

then x obeys

$$\liminf_{t\to\infty} \frac{F(x(t))}{t} \geq 1 + \Lambda_0. \tag{5.2.9}$$

(ii) If

$$\text{there exists } \tau_0 > \tau \text{ and } \Lambda_0 > \tau_0/\tau - 1 \text{ such that } \liminf_{x\to\infty} \frac{g(x)}{f_{\tau_0}(x)} = \Lambda_0, \tag{5.2.10}$$

then x obeys

$$\liminf_{t\to\infty} \frac{F(x(t))}{t} \geq \frac{\tau_0}{\tau}. \tag{5.2.11}$$

(iii) If

$$\text{there exists } \tau_1 > \tau \text{ and } \Lambda_1 \geq \tau_1/\tau - 1 \text{ such that } \limsup_{x\to\infty} \frac{g(x)}{f_{\tau_1}(x)} = \Lambda_1, \tag{5.2.12}$$

then x obeys

$$\limsup_{t\to\infty} \frac{F(x(t))}{t} \leq 1 + \Lambda_1. \tag{5.2.13}$$

(iv) If

$$\text{there exists } \tau_1 > \tau \text{ and } \Lambda_1 < \tau_1/\tau - 1 \text{ such that } \limsup_{x\to\infty} \frac{g(x)}{f_{\tau_1}(x)} = \Lambda_1, \tag{5.2.14}$$

then x obeys

$$\limsup_{t\to\infty} \frac{F(x(t))}{t} \le \frac{\tau_1}{\tau}. \tag{5.2.15}$$

(v) If

$$\text{there exists } \Lambda > 0 \text{ such that } \lim_{x\to\infty} \frac{g(x)}{f_{\tau(1+\Lambda)}(x)} = \Lambda, \tag{5.2.16}$$

then

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1 + \Lambda. \tag{5.2.17}$$

The monotonicity assumption on g is a consequence of the method of proof of the Theorem. We arrive at our upper and lower estimates of the growth rate using a *constructive comparison principle* (see Appleby and Rodkina [2], and Appleby and Buckwar [7]). For example in the proof of part (iii) we construct a function which grows at the rate consistent with the right-hand side of the inequality (5.2.13). This function is constructed in order to satisfy a differential inequality closely related to the differential equation, and the monotonicity of g is sufficient to ensure that it does indeed give an upper bound on x.

The last two theorems highlight the importance of the question : what are the relative rates of growth of f_θ and f_τ for $\theta > \tau$? Our next result shows that if f is (roughly speaking) growing *sublinearly*, then the function f_θ has the same asymptotic behaviour for every $\theta > 0$, growing at a rate asymptotic to f. On the other hand, if f is (roughly speaking) growing *superlinearly*, the function f_θ (which appears as the denominator in each of (5.2.8), (5.2.10), (5.2.12), (5.2.14), and (5.2.16)) grows more rapidly than f_τ (which appears as the denominator in (5.2.4)) because in each of these hypotheses $\theta > \tau$.

Lemma 5.2.1. *Suppose that f obeys* (0.1.7)*,* (1.1.1) *and* (4.2.1) *and g obeys* (0.1.8)*.*

(i) If $f \in RV_\infty(1)$, $\lim_{x\to\infty} f(x)/x = \infty$, and $\theta > \tau$, then

$$\lim_{x\to\infty} f_\theta(x)/f_\tau(x) = \infty.$$

If moreover, $x \mapsto g(x)/f(x)$ is bounded, then

$$\lim_{x\to\infty} g(x)/f_\tau(x) = 0.$$

(ii) If $f \in RV_\infty(\alpha)$ for $\alpha \le 1$, $\lim_{x\to\infty} f(x)/x = 0$, and $\theta > 0$, then

$$\lim_{x\to\infty} f_\theta(x)/f(x) = 1.$$

(iii) If $\lim_{x\to\infty} f(x)/x = a > 0$, and $\theta > 0$, then $\lim_{x\to\infty} f_\theta(x)/x = ae^\theta$.

One consequence of part (i) is that Theorem 5.2.2 imposes hypotheses complementary to those of Theorem 5.2.1, and the asymptotic rates of growth of x given in (5.2.9), (5.2.11), and (5.2.17) differ from that given in (5.2.5) when f grows superlinearly.

Under the hypotheses that f obeys (0.1.7), (1.1.1) and (4.2.1),

$$f \in \mathrm{RV}_\infty(1), \quad \lim_{x\to\infty} f(x)/x = \infty, \quad \text{and } x \mapsto g(x)/f(x) \text{ is bounded,}$$

Theorem 1.2.4 showed that $F(x(t))/t \to 1$ as $t \to \infty$. By part (i) of Lemma 5.2.1 and part (i) of Theorem 5.2.1, we may draw independently the same conclusion.

We now use Lemma 5.2.1 and Theorems 5.2.1 and 5.2.2 to show that the critical rate of growth of $g(x)$ as $x \to \infty$ is $f_\tau(x)$.

Theorem 5.2.3. *Suppose f obeys* (0.1.7), (1.1.1) *and* (4.2.1)*. Let g be non-decreasing and obey* (0.1.8)*. Let $\tau > 0$ and ψ obey* (0.1.9)*. Suppose that f_θ is defined by* (5.2.3)*. Let x be the unique continuous solution of* (0.1.6)*.*

(i) If

$$\text{there exists } \theta > \tau \text{ such that } \liminf_{x\to\infty} \frac{g(x)}{f_\theta(x)} > 0, \tag{5.2.18}$$

then $\liminf_{t\to\infty} F(x(t))/t > 1$.

(ii) If $\lim_{x\to\infty} g(x)/f_\tau(x) = 0$, then $\lim_{t\to\infty} F(x(t))/t = 1$.

(iii) If $\lim_{t\to\infty} F(x(t))/t = 1$, then $\liminf_{x\to\infty} g(x)/f_\theta(x) = 0$ for all $\theta > \tau$.

Proof. When the limit inferior is less than or equal to $\theta/\tau - 1$, part (i) is a direct consequence of part (i) of Theorem 5.2.2. In the other case when $\liminf_{x\to\infty} g(x)/f_\theta(x) > \theta/\tau - 1$, part (ii) of Theorem 5.2.2 shows that $\liminf_{t\to\infty} F(x(t))/t \ge \theta/\tau > 1$. Part (iii) is a consequence of part (i). Part (ii) follows from Theorem 5.2.1, completing the proof. □

We remark that $g(x)/f_\tau(x) \to 0$ as $x \to \infty$ implies that $g(x)/f_\theta(x) \to 0$ as $x \to \infty$ for all $\theta > \tau$, which shows the consistency of the conclusion of part (iii) with the hypothesis of part (ii).

There are many possible corollaries to Theorem 5.2.2. It is sometimes difficult to apply (5.2.16) because of the presence of $\Lambda > 0$ on both the right and left hand sides. Here is one example in which the existence of a limit of the form (5.2.16) is posited without asserting the form of the dependence of the righthand side.

Theorem 5.2.4. *Suppose that f obeys* (0.1.7), (1.1.1) *and* (4.2.1). *Let g be non-decreasing and obey* (0.1.8). *Let $\tau > 0$ and ψ obey* (0.1.9). *Let x be the unique continuous solution of* (0.1.6). *If*

$$\text{there exists } \tau_0 > \tau \text{ such that } \lim_{x\to\infty} \frac{g(x)}{f_{\tau_0}(x)} = \lambda > 0, \tag{5.2.19}$$

then

$$1 < \min(1+\lambda, \tau_0/\tau) \le \liminf_{t\to\infty} \frac{F(x(t))}{t} \le \limsup_{t\to\infty} \frac{F(x(t))}{t} \le \max(1+\lambda, \tau_0/\tau).$$

Proof. In the case when $\lambda \le \tau_0/\tau - 1$, by part (i) of Theorem 5.2.2 we get

$$\liminf_{t\to\infty} \frac{F(x(t))}{t} \ge 1 + \lambda = \min(1+\lambda, \tau_0/\tau).$$

When $\lambda > \tau_0/\tau - 1$, by part (ii) of Theorem 5.2.2 we obtain

$$\liminf_{t\to\infty} \frac{F(x(t))}{t} \ge \tau_0/\tau = \min(1+\lambda, \tau_0/\tau).$$

The upper bound is found by applying parts (iii) and (iv) of Theorem 5.2.2. □

5.3 Examples

We now consider some examples which are motivated by the theorems of the previous section. Such examples are useful in illustrating the importance of the conditions required to give the different types of asymptotic behaviour described in the previous section.

Example 5.3.1. We use our first example to demonstrate the sharpness of the hypothesis $g = O(f_\tau)$. If f is regularly varying with index $\alpha \le 1$ with $\lim_{x\to\infty} f(x)/x = 0$, Lemma 5.2.1 implies for any $\theta > 0$ that

$$\lim_{x\to\infty} \frac{f_\theta(x)}{f(x)} = 1.$$

If $\lim_{x\to\infty} g(x)/f(x) = 0$, then $\lim_{x\to\infty} g(x)/f_\tau(x) = 0$ so by Theorem 5.2.1,

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1.$$

If $\lim_{x\to\infty} g(x)/f(x) = \Lambda > 0$, by Lemma 5.2.1 we have $\lim_{x\to\infty} g(x)/f_{\tau(1+\Lambda)}(x) = \Lambda$. Thus by Theorem 5.2.2,

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1 + \Lambda.$$

This recovers most of Theorem 1.2.1.

When $\lim_{x\to\infty} g(x)/f(x) = \Lambda > 0$, we actually have that $\lim_{x\to\infty} g(x)/f_\tau(x) = \Lambda$ by Lemma 5.2.1. This example demonstrates that given a condition of the form

$$\text{there exists } \Lambda > 0 \text{ such that } \lim_{x\to\infty} \frac{g(x)}{f_\tau(x)} = \Lambda,$$

we cannot in general conclude that $\lim_{t\to\infty} F(x(t))/t = 1$, more information is needed. Notice that this condition assumes a slower growth rate of g than (5.2.16).

Example 5.3.2. Next we recover a known result from linear equations using Theorem 5.2.2. Condition (5.2.16) in part (v) of Theorem 5.2.2 generalises the notion of a characteristic equation. We see this by showing for linear equations that it generates the classical characteristic equation associated with a linear delay differential equation. Suppose that $\lim_{x\to\infty} f(x)/x = a > 0$ and $\lim_{x\to\infty} g(x)/x = b \geq 0$. By Lemma 5.2.1 part (iii) we have for every $\tau_0 > 0$ that $f_{\tau_0}(x)/x \to ae^{a\tau_0}$ as $x \to \infty$. Therefore $\lim_{x\to\infty} g(x)/f_{\tau_0}(x) = b/(a\exp(a\tau_0))$.

If $b = 0$, by Theorem 5.2.1 we have $F(x(t))/t \to 1$ as $t \to \infty$, or $\log x(t)/t \to a$ as $t \to \infty$.

If $b > 0$, in order to apply Theorem 5.2.2 part (v), we seek $\Lambda > 0$ and $\tau_0 > 0$ such that $a\tau_0 = \tau(a + a\Lambda)$ and $b = a\Lambda e^{a\tau_0}$. If this can be done, then $\log x(t)/t \to a + a\Lambda$ as $t \to \infty$. The existence of such a $\Lambda > 0$ is equivalent to the existence of a $\lambda_0 > a$ for which $\lambda_0 = a + a\Lambda$, which in turn is equivalent to the existence of $\lambda_0 > a$ such that $b = (\lambda_0 - a)e^{\tau\lambda_0}$. In other words, if there exists $\lambda_0 > a$ such that $\lambda_0 = a + be^{-\tau\lambda_0}$, then $\log x(t)/t \to \lambda_0$ as $t \to \infty$. This recovers the asymptotic behaviour of the linear equation that can be inferred by standard theory.

5.3.1 An example generation algorithm

We now describe a method for generating test equations with explicit solutions. In practice checking the condition $g/f_\tau \to 0$ is difficult, as we cannot in general determine F and F^{-1} explicitly. However in our algorithm we introduce an auxiliary function f_* which is asymptotic to f, dominates f pointwise and for which the integral of $1/f_*$ can be determined explicitly. It can then be shown that a positive continuous (and if needed, non-decreasing) function g can be constructed so that there is a unique continuous solution of

$$x'(t) = f(x(t)) + g(x(t-\tau)), \quad t > 0$$

which is positive, obeys $x(t) \to \infty$ as $t \to \infty$, and which has the same growth rate as $t \to \infty$ as the ordinary differential equation $y'(t) = f(y(t))$. Moreover we can now calculate the asymptotic behaviour of g/f_τ and thus check the sharpness of our conditions. In particular, we will generate an example which illustrates the gap between the necessary and sufficient conditions for $F(x(t))/t \to 1$ (cf. Theorem 5.2.3).

The algorithm for generating test equations is now given along with some comments on its construction.

Construction of examples Let f obey (0.1.7) and (1.1.1). Suppose that f_* is a continuous function with the following properties

$$f_* \in C((0,\infty);(0,\infty)); \tag{5.3.1}$$

$$f(x) < f_*(x), \quad x > 0; \tag{5.3.2}$$

$$\lim_{x\to\infty} \frac{f_*(x)}{f(x)} = 1. \tag{5.3.3}$$

Let $c > 0$ and define

$$F_*(x) = \int_c^x \frac{1}{f_*(u)}\,du, \quad x \geq 0. \tag{5.3.4}$$

Under these assumptions, F_* has the following properties:

$$F_* \in C^1((0,\infty);(\mathbb{R})) \text{ is increasing}; \quad \lim_{x\to\infty} F_*(x) = +\infty; \quad \lim_{x\to\infty} \frac{F(x)}{F_*(x)} = 1.$$

Let $\tau > 0$. Since F_* is increasing, we have that $F_*(x) > F_*(0)$ for all $x > 0$. Therefore $F_*(x)+\tau > F_*(0)$ for $x \geq 0$. Since F_* is increasing, it is invertible and we have $F_*^{-1}(F_*(x)+$

$\tau) > 0$ for all $x \geq 0$. Therefore the function $u : [0, \infty) \to \mathbb{R}$ given by $u(x) := F_*^{-1}(F_*(x)+\tau)$ for $x \geq 0$ is well-defined and we have $u(x) > 0$ for all $x \geq 0$. Also $u \in C([0, \infty); (0, \infty))$. Note that the function $g : [0, \infty) \to \mathbb{R}$ given by $g(x) = f_*(u(x)) - f(u(x))$ for $x \geq 0$ is well-defined. Since $u(x) > 0$ for all $x \geq 0$ and u is continuous, by (0.1.7), (5.3.1) and (5.3.2), we have that $g \in C([0, \infty); (0, \infty))$, where g is given by

$$g(x) = f_*(F_*^{-1}(F_*(x) + \tau)) - f(F_*^{-1}(F_*(x) + \tau)), \quad x \geq 0. \tag{5.3.5}$$

Clearly, if $f_* - f$ is non-decreasing on $[0, \infty)$, and since u is increasing, it follows that $g = (f_* - f) \circ u$ is non-decreasing on $[0, \infty)$.

The following result shows that we can generate delay differential equations with the appropriate properties.

Theorem 5.3.1. *Suppose that f obeys* (0.1.7) *and* (1.1.1) *and that f_* is a function which obeys* (5.3.1), (5.3.2) *and* (5.3.3)*, and let F_* be the function defined by* (5.3.4)*. Suppose that $0 < \tau < -F_*(\epsilon)$ for some $\epsilon \in (0, c)$ where $c > 0$ is given in* (5.3.4)*. Let g be the function defined in* (5.3.5)*. Let ψ be the function defined by*

$$\psi(t) = F_*^{-1}(t), \quad t \in [-\tau, 0]. \tag{5.3.6}$$

(i) $g : [0, \infty) \to (0, \infty)$ and $\psi : [-\tau, 0] \to (0, \infty)$ are continuous and positive functions. If moreover, $f_* - f$ is non-decreasing, then g is non-decreasing.

(ii) The unique continuous solution of (0.1.6) *is $x(t) = F_*^{-1}(t)$ for $t \geq -\tau$.*

(iii) The solution x of (0.1.6) *obeys $x(t) \to \infty$ as $t \to \infty$ and moreover if F is given by* (1.2.4)*, then*

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1.$$

Proof. The properties of g were established in the paragraph preceding the statement of this theorem. Since $\epsilon \in (0, c)$ we have that $F_*(\epsilon) < 0$. Because $0 < \tau < -F_*(\epsilon)$, we have $0 > -\tau > F_*(\epsilon)$. Therefore $F_*^{-1}(-\tau) > \epsilon > 0$. Since F_*^{-1} is increasing, we have that $F_*^{-1}(t) \geq F_*^{-1}(-\tau) > 0$ for all $t \geq -\tau$. In particular, this means that ψ defined by (5.3.6) is a positive and continuous function. Also the function y defined by $y(t) := F^{-1}(t)$ for $t \geq -\tau$ is in $C^1((-\tau, \infty); (0, \infty))$.

Since ψ is continuous, g is continuous and f is locally Lipschitz continuous, it follows that there is a unique continuous solution of (0.1.6). We note that with ψ defined by (5.3.6), we have $y(t) = \psi(t)$ for all $t \in [-\tau, 0]$. For $t \geq 0$ we have that $t - \tau \geq -\tau$. Hence $y(t-\tau) = F_*^{-1}(t-\tau)$, so as g is defined by (5.3.5) we have

$$\begin{aligned} g(y(t-\tau)) &= g(F_*^{-1}(t-\tau)) \\ &= f_*(F_*^{-1}(F_*(F_*^{-1}(t-\tau)) + \tau)) - f(F_*^{-1}(F_*(F_*^{-1}(t-\tau)) + \tau)) \\ &= f_*(F_*^{-1}(t)) - f(F_*^{-1}(t)) \\ &= f_*(y(t)) - f(y(t)). \end{aligned}$$

Since $F_*(y(t)) = t$ for $t \geq 0$ we have $F_*'(y(t))y'(t) = 1$ for $t > 0$, or

$$y'(t) = \frac{1}{F_*'(y(t))} = f_*(y(t)) = f(y(t)) + g(y(t-\tau)).$$

Therefore y is a continuously differentiable solution of (0.1.6) on $(0, \infty)$. However, as there is a unique continuous solution of (0.1.6), we have that $x(t) = y(t) = F_*^{-1}(t)$ for $t \geq -\tau$, as claimed.

To show part (iii), we notice that $F_*(x) \to \infty$ as $x \to \infty$ by (1.1.1), (5.3.3) and (5.3.4), so $F_*^{-1}(x) \to \infty$ as $x \to \infty$. Therefore $x(t) \to \infty$ as $t \to \infty$. Moreover, as (5.3.3) implies that $F(x)/F_*(x) \to 1$ as $x \to \infty$, and we have that $x(t) \to \infty$ as $t \to \infty$ we get

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = \lim_{t\to\infty} \frac{F_*(x(t))}{t} \cdot \frac{F(x(t))}{F_*(x(t))} = 1,$$

because $F_*(x(t)) = t$ for all $t \geq 0$. $\square$

We now give examples of delay differential equations which have a known solution. In our first example, we verify that f and g obey all the properties that enabled us to use the theorems that determine the asymptotic behaviour of the equation.

Example 5.3.3. Let $\tau \in (0, \log(1/\log(2)))$. Let $\alpha \in (0,1)$ and suppose that

$$f(x) = (2+x)\log(2+x) - (2+x)^{\alpha}, \quad x \geq 0$$

and let g be given by

$$g(x) = (x+2)^{\alpha e^{\tau}}, \quad x \geq 0.$$

Then the unique continuous solution of

$$x'(t) = f(x(t)) + g(x(t-\tau)), \quad t > 0; \quad x(t) = \exp(e^t) - 2 \text{ for } t \in [-\tau, 0]$$

is given by

$$x(t) = \exp(e^t) - 2, \quad t \geq -\tau. \tag{5.3.7}$$

Furthermore, if F is given by (1.2.4), then

$$\lim_{x\to\infty} \frac{g(x)}{f(F^{-1}(F(x)+\tau))} = 0. \tag{5.3.8}$$

Also, there exists $\tau_1 < \tau$ such that

$$\lim_{x\to\infty} \frac{g(x)}{f(F^{-1}(F(x)+\tau_1))} = 0.$$

We will see in the next chapter that the condition $g/f_{\tau_1} \to 0$ for some $\tau_1 < \tau$ is important for replicating the growth rate in discrete–time. Moreover, if $\alpha > e^{-\tau}$, then

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = +\infty.$$

Proof. Let f_* be given by

$$f_*(x) = (2+x)\log(2+x), \quad x \geq 0.$$

Note for $x \geq 0$ that $f(x) > 0$ and $f_*(x) > 0$. Also note that f obeys (0.1.7) and (1.1.1), and that f_* satisfies (5.3.1), (5.3.2) and (5.3.3). Let $c := e - 2 > 0$. Define for $x \geq 0$

$$F_*(x) = \int_c^x \frac{1}{f_*(u)}\,du = \int_{e-2}^x \frac{1}{(2+u)\log(2+u)}\,du.$$

Then for $x \geq 0$ we have

$$F_*(x) = \int_e^{x+2} \frac{1}{v\log(v)}\,dv = \int_1^{\log(x+2)} \frac{1}{w}\,dw = \log(\log(x+2)).$$

Therefore, for $\eta := 2 + \epsilon \in (2, e)$ we have $-F_*(\epsilon) = -\log(\log(\epsilon+2)) = \log(1/\log\eta)$. Then $\theta := 1/\log\eta \in (1, 1/\log 2)$, so $-F_*(\epsilon) = \log\theta \in (0, \log(1/\log 2))$. Thus if $\tau \in (0, \log(1/\log 2)) = (0, 0.3665129\ldots)$, there exists $\epsilon \in (0, e-2)$ such that $\tau \in (0, -F_*(\epsilon))$. Note also that $x = \log\log(F_*^{-1}(x) + 2)$, so

$$F_*^{-1}(x) = \exp(e^x) - 2, \quad x > \log(1/\log 2).$$

Therefore we have that

$$F_*^{-1}(F_*(x)+\tau) = \exp(e^{F_*(x)}e^\tau) - 2 = \exp(e^\tau \log(x+2)) - 2 = (x+2)^{e^\tau} - 2.$$

Since $g(x) = (x+2)^{\alpha e^\tau}$, we have $g(x) = (2 + F_*^{-1}(F_*(x)+\tau))^\alpha$ for $x \geq 0$ and so g obeys (5.3.5). Therefore f, g and f_* satisfy all the properties of Theorem 5.3.1, and therefore it follows that $x(t) = F_*^{-1}(t) = \exp(e^t) - 2$ for $t \in [-\tau, \infty)$ is a solution of (0.1.6).

The proof that (5.3.8) holds involves using f_* to determine very precise asymptotic information about F (for which a closed form formula is not known) and therefore F^{-1}. The analysis is deferred to Appendix B. □

In our next example, we demonstrate the gap between the necessary and sufficient conditions for $F(x(t))/t \to 1$. This shows that condition (5.2.4) is not essential for this rate of growth, inferring that a condition of the form $\lim_{x\to\infty} g(x)/f_\tau(x) \in (0,\infty]$ does not enable us to conclude directly whether $\lim_{t\to\infty} F(x(t))/t = 1$ is true or false.

Example 5.3.4. Let $\tau \in (0, -\log(\log(\log(1/2)+e)))$. Suppose that $A = e^e/2$ and

$$f(x) = (A+x)\log(A+x)\log_2(A+x) - (A+x)\log_2(A+x), \quad x \geq 0,$$

and let g be given by

$$g(x) = \exp(\log^{e^\tau}(x+A))\log(\log^{e^\tau}(x+A)), \quad x \geq 0.$$

Then the unique continuous solution of

$$x'(t) = f(x(t)) + g(x(t-\tau)), \quad t > 0; \quad x(t) = \exp(\exp(e^t))) - A \text{ for } t \in [-\tau, 0]$$

is given by

$$x(t) = \exp(\exp(e^t))) - A, \quad t \geq -\tau. \tag{5.3.9}$$

Furthermore, if F is given by (1.2.4), then

$$\lim_{x\to\infty} \frac{g(x)}{f(F^{-1}(F(x)+\tau))} = \infty. \tag{5.3.10}$$

Again the proof of (5.3.10) is deferred to Appendix B.

5.4 Proofs

Proof of Theorem 5.2.1 Define $z(t) = F(x(t))$, $t \geq -\tau$. Then $z \in C^1((0,\infty),(0,\infty))$ and we have

$$z'(t) = F'(x(t))x'(t) = 1 + \frac{g(x(t-\tau))}{f(x(t))} = 1 + \frac{g(F^{-1}(z(t-\tau)))}{f(F^{-1}(z(t))} \tag{5.4.1}$$

for $t > 0$. Let $z_0(t) = z(t) - t$. Then

$$z_0'(t) = z'(t) - 1 = \frac{g(F^{-1}(z(t-\tau)))}{f(F^{-1}(z(t))} \geq 0, \quad t > 0.$$

Therefore for $t \geq \tau$ we have $z_0(t) \geq z_0(t-\tau)$, or $z(t) - t \geq z(t-\tau) - (t-\tau)$. Hence

$$z(t) \geq z(t-\tau) + \tau, \quad t \geq \tau. \tag{5.4.2}$$

We also have that $z'(t) \geq 1$ for all $t \geq 0$ so therefore

$$\liminf_{t\to\infty} \frac{F(x(t))}{t} = \liminf_{t\to\infty} \frac{z(t)}{t} \geq 1. \tag{5.4.3}$$

By (5.2.4), $g(x)/f_\tau(x) \to 0$ as $x \to \infty$, so it follows that for every $\varepsilon > 0$ there is $x_1(\varepsilon) > 0$ such that $g(x) < \varepsilon f_\tau(x)$ for all $x \geq x_1(\varepsilon)$. By Theorem 1.1.1, we have that $x(t) \to \infty$ as $t \to \infty$. Thus there exists $T(\varepsilon) > 0$ such that for $t > T(\varepsilon)$ we have $x(t) > x_1(\varepsilon)$. Hence for $t > T(\varepsilon) + \tau$ we have $x(t-\tau) > x_1(\varepsilon)$, so

$$g(x(t-\tau)) < \varepsilon f_\tau(x(t-\tau)) = \varepsilon f(F^{-1}(F(x(t-\tau)) + \tau)) = \varepsilon f(F^{-1}(z(t-\tau) + \tau)).$$

Since F^{-1} is increasing, by (5.4.2) we have $F^{-1}(z(t-\tau)) \leq F^{-1}(z(t) - \tau)$ for $t \geq \tau$. Therefore for $t > T(\varepsilon) + \tau$ we have $g(x(t-\tau)) < \varepsilon f(F^{-1}(z(t)))$, so for $t \geq T(\varepsilon) + \tau$ we use (5.4.1) to get the inequality

$$z'(t) = 1 + \frac{g(x(t-\tau))}{f(F^{-1}(z(t)))} \leq 1 + \epsilon.$$

Hence $z(t) \leq z(T(\varepsilon)+\tau) + (1+\varepsilon)(t - (T(\varepsilon)+\tau))$ for $t \geq T(\varepsilon)+\tau$. Hence $\limsup_{t\to\infty} z(t)/t \leq 1 + \varepsilon$. Letting $\varepsilon \to 0$ we have

$$\limsup_{t\to\infty} \frac{F(x(t))}{t} = \limsup_{t\to\infty} \frac{z(t)}{t} \leq 1. \tag{5.4.4}$$

Combining (5.4.3) and (5.4.4) yields (5.2.5), as required.

To prove part (ii), note that the first part of (5.2.7) is a consequence of (5.4.3). To prove the second part, by (5.2.6), $\limsup_{x\to\infty} g(x)/f_\tau(x) = \Lambda$, so it follows that for every $\varepsilon > 0$ there is $x_2(\varepsilon) > 0$ such that $g(x) < (\Lambda+\varepsilon)f_\tau(x)$ for all $x \geq x_2(\varepsilon)$. By Theorem 1.1.1, we have that $x(t) \to \infty$ as $t \to \infty$. Thus there exists $T_2(\varepsilon) > 0$ such that for $t > T_2(\varepsilon)$ we have $x(t) > x_2(\varepsilon)$. Hence for $t > T_2(\varepsilon) + \tau$ we have $x(t-\tau) > x_2(\varepsilon)$, so

$$\begin{aligned} g(x(t-\tau)) &< (\Lambda+\varepsilon)f_\tau(x(t-\tau)) = (\Lambda+\varepsilon)f(F^{-1}(F(x(t-\tau))+\tau)) \\ &= (\Lambda+\varepsilon)f(F^{-1}(z(t-\tau)+\tau)). \end{aligned}$$

Since F^{-1} is increasing, by (5.4.2) we have $F^{-1}(z(t-\tau)) \leq F^{-1}(z(t)-\tau)$ for $t \geq \tau$. Therefore for $t > T(\varepsilon)+\tau$ we have $g(x(t-\tau)) < (\Lambda+\varepsilon)f(F^{-1}(z(t)))$, so for $t \geq T_2(\varepsilon)+\tau$ we use (5.4.1) to get the inequality

$$z'(t) = 1 + \frac{g(x(t-\tau))}{f(F^{-1}(z(t)))} \leq 1 + \Lambda + \varepsilon.$$

Hence $z(t) \leq z(T_2(\varepsilon)+\tau) + (1+\Lambda+\varepsilon)(t-(T_2(\varepsilon)+\tau))$ for $t \geq T_2(\varepsilon)+\tau$, and so $\limsup_{t\to\infty} z(t)/t \leq 1+\Lambda+\varepsilon$. Letting $\varepsilon \to 0$ we have

$$\limsup_{t\to\infty} \frac{F(x(t))}{t} = \limsup_{t\to\infty} \frac{z(t)}{t} \leq 1+\Lambda,$$

proving the second part of (5.2.7).

Proof of Theorem 5.2.2 Clearly part (v) is a consequence of parts (i) and (iii) with $\tau_0 = \tau_1 = (1+\Lambda)\tau$ and $\Lambda_0 = \Lambda_1 = \Lambda$. We prove part (iii). By (5.2.12) for every $\epsilon > 0$ there exists $x_1(\epsilon) > 0$ such that

$$g(x) < \Lambda_1(1+\epsilon)f(F^{-1}(F(x)+\tau_1)), \quad x > x_1(\epsilon). \tag{5.4.5}$$

Since $\tau_1 \leq (1+\Lambda_1)\tau$, we have

$$\tau_1 \leq (1+\Lambda_1)\tau < (1+\Lambda_1(1+\epsilon))\tau. \tag{5.4.6}$$

Define $T_\epsilon > \tau$ so that

$$(1+\Lambda_1)(T_\epsilon - \tau) = F(\psi^* + x_1(\epsilon)). \tag{5.4.7}$$

and define the function x_ϵ so that

$$x_\epsilon(t) = F^{-1}((1+\Lambda_1(1+\epsilon))(t+T_\epsilon)), \quad t \geq -\tau. \tag{5.4.8}$$

Then for $t \in [-\tau, 0]$ by (5.4.7), we have

$$\begin{aligned} x_\epsilon(t) &= F^{-1}((1+\Lambda_1(1+\epsilon))(t+T_\epsilon)) \geq F^{-1}((1+\Lambda_1(1+\epsilon))(T_\epsilon - \tau)) \\ &> F^{-1}((1+\Lambda_1)(T_\epsilon - \tau)) = \psi^* + x_1(\epsilon) > \psi^* \geq \psi(t). \end{aligned}$$

We see for $t \geq 0$ that

$$\begin{aligned} x_\epsilon(t-\tau) &= F^{-1}((1+\Lambda_1(1+\epsilon))(t-\tau+T_\epsilon)) \geq F^{-1}((1+\Lambda_1(1+\epsilon))(T_\epsilon - \tau)) \\ &> F^{-1}((1+\Lambda_1)(T_\epsilon - \tau)) = \psi^* + x_1(\epsilon) > x_1(\epsilon). \end{aligned}$$

Therefore for $t \geq 0$, by (5.4.5) and (5.4.8) we have

$$\begin{aligned} g(x_\epsilon(t-\tau)) &< \Lambda_1(1+\epsilon) f(F^{-1}(F(x_\epsilon(t-\tau)) + \tau_1)) \\ &= \Lambda_1(1+\epsilon) f(F^{-1}((1+\Lambda_1(1+\epsilon))(t-\tau+T_\epsilon) + \tau_1) \\ &= \Lambda_1(1+\epsilon) f(F^{-1}((1+\Lambda_1(1+\epsilon))(t+T_\epsilon) - (1+\Lambda_1(1+\epsilon))\tau + \tau_1) \\ &\leq \Lambda_1(1+\epsilon) f(F^{-1}((1+\Lambda_1(1+\epsilon))(t+T_\epsilon))), \end{aligned}$$

where we have used (5.4.6) at the last step. Therefore for $t \geq 0$

$$g(x_\epsilon(t-\tau)) < \Lambda_1(1+\epsilon) f(F^{-1}((1+\Lambda_1(1+\epsilon))(t+T_\epsilon))) = \Lambda_1(1+\epsilon) f(x_\epsilon(t)).$$

Hence

$$f(x_\epsilon(t)) + g(x_\epsilon(t-\tau)) < (1+\Lambda_1(1+\epsilon)) f(x_\epsilon(t)), \quad t \geq 0. \tag{5.4.9}$$

For $t \geq 0$ we have $F(x_\epsilon(t)) = (1+\Lambda_1(1+\epsilon))(t+T_\epsilon)$. Hence for $t > 0$

$$x_\epsilon'(t)/f(x_\epsilon(t)) = F'(x_\epsilon(t)) x_\epsilon'(t) = 1 + \Lambda_1(1+\epsilon).$$

Therefore by (5.4.9), we have

$$x_\epsilon'(t) = (1+\Lambda_1(1+\epsilon)) f(x_\epsilon(t)) > f(x_\epsilon(t)) + g(x_\epsilon(t-\tau)), \quad t > 0.$$

Since we also have $x_\epsilon(t) > x(t)$ for $t \in [-\tau, 0]$, it follows that $x_\epsilon(t) > x(t)$ for all $t \geq -\tau$. Therefore

$$\begin{aligned} \limsup_{t\to\infty} \frac{F(x(t))}{t} &\leq \limsup_{t\to\infty} \frac{F(x_\epsilon(t))}{t} = \limsup_{t\to\infty} \frac{(1+\Lambda_1(1+\epsilon))(t+T_\epsilon)}{t} \\ &= 1 + \Lambda_1(1+\epsilon). \end{aligned}$$

Letting $\epsilon \to 0$, we obtain (5.2.13).

To prove part (iv), in the case when $\Lambda_1 < \tau_1/\tau - 1$, for every $\epsilon > 0$ there is an increasing, continuous and positive function γ_ε such that $\gamma_\varepsilon(x) > g(x)$ for all $x \geq 0$ and $\lim_{x\to\infty} \gamma_\varepsilon(x)/f_{\tau_1}(x) = (1+\varepsilon)(\tau_1/\tau - 1) > (\tau_1/\tau - 1)$. If we define by x_ε the solution of $x_\varepsilon'(t) = f(x_\varepsilon(t)) + \gamma_\varepsilon(x_\varepsilon(t-\tau))$ with $x_\varepsilon(t) = \psi(t) + 1$ for $t \in [-\tau, 0]$, then $x(t) < x_\varepsilon(t)$ for $t \geq 0$. Therefore we can apply the result of part (iii) above to get

$$\limsup_{t\to\infty} \frac{F(x(t))}{t} \leq \limsup_{t\to\infty} \frac{F(x_\varepsilon(t))}{t} \leq 1 + (1+\varepsilon)\left(\frac{\tau_1}{\tau} - 1\right).$$

Letting $\varepsilon \to 0$ we obtain $\limsup_{t\to\infty} F(x(t))/t \leq \tau_1/\tau$, as required.

To prove part (i), note that the condition (5.2.8) implies for every $\epsilon > 0$ that there exists $x_2(\epsilon) > 0$ such that

$$g(x) > \Lambda_0(1-\epsilon) f(F^{-1}(F(x) + \tau_0)), \quad x > x_2(\epsilon). \tag{5.4.10}$$

Since $x(t) \to \infty$ as $t \to \infty$, we have that there exists $T_0(\epsilon) > 0$ such that $x(t) > x_2(\epsilon)$ for all $t > T_0(\epsilon)$ and $x(T_0(\epsilon)) = x_2(\epsilon)$. Now since x is increasing, $T_1(\epsilon)$ given by

$$T_1(\epsilon) = \inf\{t > T_0(\epsilon) \,:\, F(x(t)) = F(x_2(\epsilon)) + (1 + \Lambda_0(1-\epsilon))\tau\} \tag{5.4.11}$$

is well-defined. In particular, we have

$$F(x(T_1(\epsilon))) = F(x_2(\epsilon)) + (1 + \Lambda_0(1-\epsilon))\tau. \tag{5.4.12}$$

Clearly as x is increasing $x(t) > x(T_1(\epsilon)) > x_2(\epsilon)$ for $t > T_1(\epsilon)$. Now define $x_\epsilon(t)$ by

$$x_\epsilon(t) = F^{-1}\left([1 + \Lambda_0(1-\epsilon)](t - T_1(\epsilon) - \tau) + F(x_2(\epsilon)) + [1 + \Lambda_0(1-\epsilon)]\tau\right), \quad t \geq T_1(\epsilon). \tag{5.4.13}$$

Thus for $t \in (T_1(\epsilon), T_1(\epsilon) + \tau]$, by (5.4.12) we have

$$\begin{aligned} x_\epsilon(t) \leq x_\epsilon(T_1(\epsilon) + \tau) &= F^{-1}((1 + \Lambda_0(1-\epsilon))\tau + F(x_2(\epsilon))) \\ &= F^{-1}(F(x(T_1(\epsilon)))) = x(T_1(\epsilon)) < x(t). \end{aligned}$$

Therefore

$$x_\epsilon(t) < x(t), \quad t \in (T_1(\epsilon), T_1(\epsilon) + \tau]. \tag{5.4.14}$$

Also

$$x_\epsilon(T_1(\epsilon)) < x_\epsilon(T_1(\epsilon)+\tau) = F^{-1}((1+\Lambda_0(1-\epsilon))\tau + F(x_2(\epsilon)))$$
$$= F^{-1}(F(x(T_1(\epsilon)))) = x(T_1(\epsilon)).$$

Therefore $x_\epsilon(T_1(\epsilon)) < x(T_1(\epsilon))$. Combining this and (5.4.14) we get

$$x_\epsilon(t) < x(t), \quad t \in [T_1(\epsilon), T_1(\epsilon)+\tau]. \tag{5.4.15}$$

Now by (5.4.13), $F(x_\epsilon(t)) = [1+\Lambda_0(1-\epsilon)](t-T_1(\epsilon)-\tau) + F(x_2(\epsilon)) + [1+\Lambda_0(1-\epsilon)]\tau$ for $t > T_1(\epsilon)+\tau$. Therefore for $t > T_1(\epsilon)+\tau$ we have $x_\epsilon'(t)/f(x_\epsilon(t)) = F'(x_\epsilon(t))x_\epsilon'(t) = 1+\Lambda_0(1-\epsilon)$. Hence

$$x_\epsilon'(t) = f(x_\epsilon(t)) + \Lambda_0(1-\epsilon)f(x_\epsilon(t)), \quad t > T_1(\epsilon)+\tau. \tag{5.4.16}$$

Now for $t > T_1(\epsilon)+\tau$ so $x_\epsilon(t-\tau) > x_\epsilon(T_1(\epsilon)) = x_2(\epsilon)$, by (5.4.13). Thus by (5.4.10) for $t > T_1(\epsilon)+\tau$ we have

$$g(x_\epsilon(t-\tau)) > \Lambda_0(1-\epsilon)f(F^{-1}(F(x_\epsilon(t-\tau))+\tau_0)).$$

Now $\tau_0 + F(x_\epsilon(t-\tau)) = \tau_0 + [1+\Lambda_0(1-\epsilon)](t-T_1(\epsilon)) - [1+\Lambda_0(1-\epsilon)]\tau + F(x_2(\epsilon))$. Since $\tau_0 \geq \tau(1+\Lambda_0)$, we have $\tau_0 - [1+\Lambda_0(1-\epsilon)]\tau \geq \tau(1+\Lambda_0) - [1+\Lambda_0(1-\epsilon)]\tau = \tau\left(1+\Lambda_0 - [1+\Lambda_0(1-\epsilon)]\right) = \tau\Lambda_0\epsilon > 0$. Therefore for $t > T_1(\epsilon)+\tau$ we have

$$\tau_0 + F(x_\epsilon(t-\tau)) = \tau_0 + [1+\Lambda_0(1-\epsilon)](t-T_1(\epsilon)) - [1+\Lambda_0(1-\epsilon)]\tau + F(x_2(\epsilon))$$
$$> [1+\Lambda_0(1-\epsilon)](t-T_1(\epsilon)) + F(x_2(\epsilon)).$$

Hence

$$F^{-1}(\tau_0 + F(x_\epsilon(t-\tau))) > F^{-1}([1+\Lambda_0(1-\epsilon)](t-T_1(\epsilon)) + F(x_2(\epsilon))), \quad t > T_1(\epsilon)+\tau.$$

Therefore by (5.4.13) we have $F^{-1}(\tau_0 + F_1(x_\epsilon(t-\tau))) > x_\epsilon(t)$ for $t > T_1(\epsilon)+\tau$. Since f is increasing we have $f(F^{-1}(\tau_0 + F(x_\epsilon(t-\tau)))) > f(x_\epsilon(t))$ for $t > T_1(\epsilon)+\tau$. Therefore for $t > T_1(\epsilon)+\tau$ we have

$$g(x_\epsilon(t-\tau)) > \Lambda_0(1-\epsilon)f(F^{-1}(F(x_\epsilon(t-\tau))+\tau_0)) > \Lambda_0(1-\epsilon)f(x_\epsilon(t)).$$

Hence by (5.4.16) for $t > T_1(\epsilon) + \tau$ we have $x_\epsilon'(t) = f(x_\epsilon(t)) + \Lambda_0(1-\epsilon)f(x_\epsilon(t)) < f(x_\epsilon(t)) + g(x_\epsilon(t-\tau))$. By this and (5.4.15) we have that $x(t) > x_\epsilon(t)$ for all $t \geq T_1(\epsilon)$. Therefore

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{F(x(t))}{t} \\
&\geq \liminf_{t\to\infty} \frac{F(x_\epsilon(t))}{t} \\
&= \liminf_{t\to\infty} \frac{[1+\Lambda_0(1-\epsilon)](t - T_1(\epsilon) - \tau) + F(x_2(\epsilon)) + [1+\Lambda_0(1-\epsilon)]\tau}{t} \\
&= 1 + \Lambda_0(1-\epsilon).
\end{aligned}$$

Letting $\epsilon \to 0$ we obtain (5.2.9).

To prove part (ii), define $\phi_0(x) = f(F^{-1}(F(x) + \tau_0))$. Then by (5.2.10), for every $\varepsilon \in (0,1)$ there exists $x_2(\varepsilon) > 0$ such that $g(x) \geq \Lambda_0(1-\varepsilon)\phi_0(x)$ for all $x \geq x_2(\varepsilon)$. Define $\gamma_1(x) = (\tau_0/\tau - 1)(1-\varepsilon)\phi_0(x)$ for $x \geq x_2(\varepsilon)$. Then $g(x) > \gamma_1(x)$ for $x \geq x_2(\varepsilon)$ and γ_1 is increasing on $[x_2, \infty)$. We extend γ_1 to $[0, x_2)$ so that it is continuous, positive and increasing on $[0, x_2]$, and obeys $g(x) > \gamma_1(x)$ for $x \in [0, x_2)$. If we define by y_ε the solution of $y_\varepsilon'(t) = f(y_\varepsilon(t)) + \gamma_1(y_\varepsilon(t-\tau))$ for $t > 0$ with $y_\varepsilon(t) = \psi(t)/2$ for $t \in [-\tau, 0]$, then $x(t) > y_\varepsilon(t)$ for $t \geq -\tau$. By applying part (i), we see that

$$\liminf_{t\to\infty} \frac{F(x(t))}{t} \geq \liminf_{t\to\infty} \frac{F(y_\varepsilon(t))}{t} \geq 1 + \left(\frac{\tau_0}{\tau} - 1\right)(1-\varepsilon).$$

Letting $\varepsilon \to 0$ gives the desired result (5.2.11).

Proof of Lemma 5.2.1 In each case, we note that the solution of $y'(t) = f(y(t))$ with $y(0) = \psi^*$ is $y(t) = F^{-1}(t)$. Clearly $y(t) \to \infty$ as $t \to \infty$.

To prove part (i), by (5.2.3), and the fact that $F(x) \to \infty$ as $x \to \infty$, we have

$$\lim_{x\to\infty} \frac{f_\theta(x)}{f_\tau(x)} = \lim_{x\to\infty} \frac{f(F^{-1}(F(x)+\theta))}{f(F^{-1}(F(x)+\tau))} = \lim_{y\to\infty} \frac{f(F^{-1}(F(y)-\tau+\theta))}{f(y)}.$$

Next, we have $\lim_{t\to\infty} y'(t)/y(t) = \lim_{t\to\infty} f(y(t))/y(t) = \infty$. Since $\tau < \theta$, it follows that $y(t+\theta-\tau)/y(t) \to \infty$ as $t \to \infty$. Thus we have

$$\lim_{y\to\infty} \frac{F^{-1}(F(y)-\tau+\theta)}{y} = \lim_{z\to\infty} \frac{F^{-1}(z+\theta-\tau)}{F^{-1}(z)} = \infty.$$

Hence $f_\theta(x)/f_\tau(x) \to \infty$ as $x \to \infty$, as required.

If $M = \limsup_{x\to\infty} g(x)/f(x)$, then

$$\limsup_{x\to\infty} \frac{g(x)}{f_\tau(x)} \le M \limsup_{x\to\infty} \frac{f(x)}{f_\tau(x)} = M \limsup_{x\to\infty} \frac{f(x)}{f(F^{-1}(F(x)+\tau))}.$$

Since f is in $\mathrm{RV}_\infty(1)$ and $F^{-1}(F(x)+\tau))/x \to \infty$, we have

$$\lim_{x\to\infty} f(F^{-1}(F(x)+\tau))/f(x) = +\infty.$$

Hence $g(x)/f_\tau(x) \to 0$ as $x \to \infty$, as required.

To prove part (ii), note that $\lim_{t\to\infty} y'(t)/y(t) = \lim_{t\to\infty} f(y(t))/y(t) = 0$. Thus $\lim_{t\to\infty} y(t+\theta)/y(t) = 1$ for any $\theta > 0$. Since $f \in \mathrm{RV}_\infty(\alpha)$, we have $\lim_{t\to\infty} f(y(t+\theta))/f(y(t)) = 1$ for any $\theta > 0$, or $\lim_{y\to\infty} f(F^{-1}(y+\theta))/f(F^{-1}(y)) = 1$. By (0.1.7) we have $F(x) \to \infty$ as $x \to \infty$, so $\lim_{x\to\infty} f(F^{-1}(F(x)+\theta))/f(x) = 1$, which yields $\lim_{x\to\infty} f_\theta(x)/f(x) = \lim_{x\to\infty} f(F^{-1}(F(x)+\theta))/f(x) = 1$ as required.

To prove part (iii), note that $\lim_{t\to\infty} y'(t)/y(t) = a$. Let $\theta > 0$. Therefore, as $t \to \infty$, we get

$$\log\left(\frac{y(t+\theta)}{y(t)}\right) = \int_t^{t+\theta} \frac{y'(s)}{y(s)}\,ds \to a\theta.$$

Thus $\lim_{t\to\infty} y(t+\theta)/y(t) = e^{a\theta}$. Hence $\lim_{t\to\infty} f(y(t+\theta))/f(y(t)) = e^{a\theta}$, which implies $\lim_{y\to\infty} f(F^{-1}(y+\theta))/f(F^{-1}(y)) = e^{a\theta}$. By (0.1.7) we have $F(x) \to \infty$ as $x \to \infty$, so $\lim_{x\to\infty} f(F^{-1}(F(x)+\theta))/f(x) = e^{a\theta}$, which yields

$$\lim_{x\to\infty} f_\theta(x)/ax = \lim_{x\to\infty} f(F^{-1}(F(x)+\theta))/f(x) = e^{a\theta}$$

as required.

Chapter 6

Numerical Approximation of Instantaneously-Dominated Equations

6.1 Introduction

In the case that g/f tends to a finite limit, we saw in Chapter 4 that the state–dependent discretisation introduced in Chapter 2 correctly replicated the growth rate of the solution to the instantaneously-dominated superlinear differential equation. This condition was critical for the proof of this result, as it was used to facilitate the use of a construction comparison argument, as we were able to consider an upper estimate on the discrete equation which inherited its asymptotic behaviour. For equations with unconstrained delay coefficients however, constructing the upper estimate is more problematic and we are unable to use a constructive comparison argument to replicate the exact growth rate. In the chapter, we adopt a different strategy.

Since $F(x(t)) \to 1$ as $x \to \infty$, it follows that the nonlinear transformation z given by $z(t) = F(x(t))$ for will obey the differential equation

$$z'(t) = 1 + \frac{g(F^{-1}(z(t-\tau)))}{f(F^{-1}(z(t)))}, \quad t > 0; \quad z(t) = F(\psi(t)), \quad t \in [-\tau, 0] \tag{6.1.1}$$

Moreover as we have seen in the proof of Theorem 5.2.1, z will grow linearly, so we will expect a *uniform* Euler discretisation to reproduce its growth rate. The problem however, is in constructing the discrete approximation of z. The function F^{-1} is not known a priori, so we cannot do this directly. Instead we replace F^{-1} by F_{Δ}^{-1}, an auxiliary function obtained from applying a state–dependent discretisation to the ODE given by $y'(t) = f(y(t))$. Once our numerical method is now behaving like that of a linear equation, constructing the comparison estimates is straightforward.

It is worth noting that the method of "prediscretisation" applies to all instantaneously-dominated equations. However if f is sublinear, we have seen in Theorem 1.2.1 that a uniform method will ascertain the correct growth rate, rendering the use of a more computationally demanding method somewhat unnecessary. However if f is superlinear, Theorem 1.2.4 showed that with g/f tending to a finite limit, constant step-sizes will return an exact rate of growth which will look entirely plausible, but will in fact underestimate the true rate. Indeed if g/f tends to infinity, uniform discretisations will still underestimate the growth rate (in that we can determine a lower bound which is incorrect), however in this thesis we have not attempted to determine the precise nature of this incorrect rate.

In Section 6.2 we construct this auxiliary function F_Δ^{-1} and investigate some of its useful properties. The transformation is detailed in Section 6.3, and the growth rates of both the transformed equation and the original equation are determined. Convergence of this transformed numerical method is featured in Section 6.4 and certain proofs are deferred to Section 6.5.

6.2 Constructing an Auxiliary Function

In this section we show that a non-uniform discretisation captures the dynamics of the ODE (5.2.1) as well as constructing auxiliary functions which enable us to develop upper and lower bounds for the solutions of the DDE (0.1.6).

Let F be given by (1.2.4). Then the solution y of the initial value problem (5.2.1) is given by

$$y(t) = F^{-1}(t), \quad t \geq 0.$$

Let $\Delta > 0$. Define now the sequence $(r_n)_{n=0}^{\infty}$ by

$$r_n = \sum_{j=0}^{n-1} \frac{\Delta}{f(\xi + j\Delta)}, \quad n \geq 1; \quad r_0 = 0. \tag{6.2.1}$$

Clearly $(r_n)_{n\geq 0}$ is an increasing sequence. Conditions (1.1.1) and (4.2.1) guarantee that $r_n \to \infty$ as $n \to \infty$. Define the function $H_\Delta : [0, \infty) \to \mathbb{R}$ as follows:

$$H_\Delta(r_n) = \xi + n\Delta, \quad n = 0, 1, \ldots,$$
$$H_\Delta(t) = \xi + n\Delta + f(\xi + n\Delta)(t - r_n), \quad t \in [r_n, r_{n+1}].$$

It is clear that H_Δ is increasing with $H_\Delta(0) = \xi$; therefore it has an inverse $H_\Delta^{-1} : [\xi, \infty) \to$

$[0,\infty)$. Define $F_\Delta := H_\Delta^{-1}$ and therefore $F_\Delta^{-1} = H_\Delta$, so that

$$F_\Delta^{-1}(r_n) = \xi + n\Delta, \quad n = 0, 1, \ldots, \tag{6.2.2a}$$

$$F_\Delta^{-1}(t) = \xi + n\Delta + f(\xi + n\Delta)(t - r_n), \quad t \in [r_n, r_{n+1}]. \tag{6.2.2b}$$

Notice that F_Δ^{-1} is differentiable on (r_n, r_{n+1}) for every $n \geq 0$ and indeed

$$\left(F_\Delta^{-1}\right)'(t) = f(\xi + n\Delta), \quad t \in (r_n, r_{n+1}). \tag{6.2.3}$$

Clearly from (6.2.2a) we have

$$r_n = F_\Delta(\xi + n\Delta), \quad n \geq 0.$$

We now record some properties of F_Δ and F_Δ^{-1} that will be of use not only in analysing the asymptotic behaviour of the solution y of (5.2.1) but also of the asymptotic behaviour of the solution of (0.1.6)

Lemma 6.2.1. *Suppose that F_Δ is defined by* (6.2.2) *and F by* (1.2.4)*. Then we have*

$$F_\Delta(x) \leq \frac{\Delta}{f(\xi)} + F(x), \quad x \geq \xi; \tag{6.2.4a}$$

$$F_\Delta(x) \geq F(x), \quad x \geq \xi; \tag{6.2.4b}$$

$$F_\Delta^{-1}(t) \leq F^{-1}(t), \quad t \geq 0; \tag{6.2.4c}$$

$$F^{-1}(t) \leq F_\Delta^{-1}\left(t + \frac{\Delta}{f(\xi)}\right), \quad t \geq 0. \tag{6.2.4d}$$

Theorem 6.2.1. *Suppose that f obeys* (0.1.7) *and $f \in RV_\infty(1)$. Let $\Delta > 0$ and let F be defined by* (1.2.4) *and F_Δ be the function defined by* (6.2.2)*. Let y_Δ be given by*

$$y_\Delta(t) = F_\Delta^{-1}(t), \quad t \geq 0. \tag{6.2.5}$$

If y is the solution of (5.2.1) *then*

$$y(t - \Delta/f(\xi)) \leq y_\Delta(t) \leq y(t), \quad t \geq \Delta/f(\xi). \tag{6.2.6}$$

and

$$\lim_{t\to\infty} \frac{F(y_\Delta(t))}{t} = 1. \tag{6.2.7}$$

Proof. By (6.2.4c) and the fact that y solves (5.2.1) we have for all $t \geq 0$.

$$y_\Delta(t) = F_\Delta^{-1}(t) \leq F^{-1}(t) = y(t).$$

Suppose that $t \geq \Delta/f(\xi)$. Then by (6.2.4d) we have

$$y_\Delta(t) = F_\Delta^{-1}(t - \Delta/f(\xi) + \Delta/f(\xi)) \geq F^{-1}(t - \Delta/f(\xi)),$$

completing the proof of (6.2.6). It is equivalent to

$$F^{-1}(t - \Delta/f(\xi)) \leq y_\Delta(t) \leq F^{-1}(t), \quad t \geq \Delta/f(\xi).$$

Since F is increasing, we have

$$t - \Delta/f(\xi) \leq F(y_\Delta(t)) \leq t, \quad t \geq \Delta/f(\xi),$$

from which we can immediately infer (6.2.7). □

6.3 Transformed Equation

Suppose that x is the solution of (0.1.6). Let $z(t) = F(x(t))$ for $t \geq -\tau$, where F is given by (1.2.4) with $\xi \in (0, \psi_*)$ where

$$\psi_* = \min_{t\in[-\tau,0]} \psi(t) > 0.$$

Then z obeys the differential equation

$$z'(t) = 1 + \frac{g(F^{-1}(z(t-\tau)))}{f(F^{-1}(z(t)))}, \quad t > 0; \quad z(t) = F(\psi(t)), \quad t \in [-\tau, 0]. \tag{6.3.1}$$

Our idea now is to discretise this differential equation: however, because F^{-1} is not data, we cannot do this directly. Instead, we replace F^{-1} with F_Δ^{-1} given by (6.2.2) for some suitably chosen $\Delta > 0$. Moreover, as the instantaneous part of the equation dominates, we have that $F(x(t))/t \to 1$ as $t \to \infty$, implying that

$$\lim_{t\to\infty} \frac{z(t)}{t} = 1.$$

Since z does not grow rapidly, we can expect that a uniform discretisation of (6.3.1) will recover the appropriate asymptotic behaviour. Also, we should expect that this discretisation will control the error on compact intervals.

Let

$$\Delta \in (0, \tau f(\xi)). \tag{6.3.2}$$

Suppose that F_Δ is given by (6.2.2). Let $N_\Delta = \lceil \tau/\Delta \rceil \in \mathbb{N}$ and define

$$h_\Delta := \frac{\tau}{N_\Delta}. \tag{6.3.3}$$

Note $h_\Delta \leq \Delta$. Define also

$$z_{n+1}(\Delta) = z_n(\Delta) + h_\Delta + h_\Delta \frac{g(F_\Delta^{-1}(z_{n-N_\Delta}(\Delta)))}{f(F_\Delta^{-1}(z_n(\Delta)))}, \quad n \geq 0 \tag{6.3.4a}$$

$$z_n(\Delta) = F_\Delta(\psi(nh_\Delta)), \quad n = -N_\Delta, \ldots, 0. \tag{6.3.4b}$$

Let $n(t) = \lfloor t/h_\Delta \rfloor$ for $t \geq 0$. Hence define the functions $\bar{Z}_\Delta, Z_\Delta$ by

$$\bar{Z}_\Delta(t) = z_{n(t)}(\Delta) + \frac{z_{n(t)+1}(\Delta) - z_{n(t)}(\Delta)}{h_\Delta}(t - n(t)h_\Delta), \quad n(t)h_\Delta \leq t < (n(t)+1)h_\Delta; \tag{6.3.5a}$$

$$Z_\Delta(t) = z_{n(t)}(\Delta), \quad n(t)h_\Delta \leq t < (n(t)+1)h_\Delta. \tag{6.3.5b}$$

$$\bar{Z}_\Delta(t) = F_\Delta(\psi(t)), \quad t \in [-\tau, 0];$$

$$Z_\Delta(t) = F_\Delta(\psi(nh_\Delta)), \quad nh_\Delta \leq t < (n+1)h_\Delta, \quad n = -N_\Delta, \ldots, 0.$$

and

$$\bar{x}_\Delta(t) = F_\Delta^{-1}(\bar{Z}_\Delta(t)), t \geq 0; \quad \bar{x}_\Delta(t) = \psi(t), t \in [-\tau, 0]. \tag{6.3.6}$$

Theorem 6.3.1. *Suppose that f obeys* (0.1.7), (1.1.1) *and* (4.2.1) *and g obeys* (0.1.8) *and is non-decreasing. Let $\tau > 0$ and ψ obey* (0.1.9)*. Let F be given by* (1.2.4) *where $\xi \in (0, \psi_*)$. Suppose also that f and g obey*

$$\text{There exists } \tau_1 < \tau \text{ such that } \lim_{x\to\infty} \frac{g(x)}{f_{\tau_1}(x)} = 0. \tag{6.3.7}$$

Suppose that Δ is so small that

$$\Delta < f(\xi)(\tau - \tau_1). \tag{6.3.8}$$

Let x be the unique continuous solution of (0.1.6)*. Let $N \in \mathbb{N}$ and $h = h_\Delta > 0$ be given by* (6.3.3)*. Suppose that $z_n(\Delta)$ is defined by* (6.3.4)*, $\bar{Z}_\Delta$ by* (6.3.5b) *and $\bar{x}_\Delta$ by* (6.3.6)*. Then $\bar{x}_\Delta$ obeys*

$$\lim_{t\to\infty} \frac{F(\bar{x}_\Delta(t))}{t} = 1. \tag{6.3.9}$$

(6.3.9) shows that the rate of growth of the approximation $\bar{x}_\Delta$ is the same as that of the true solution x under a hypothesis (6.3.7) similar to that required in an earlier theorem (namely condition (5.2.4)) about the asymptotic behaviour of the delay differential equation.

We notice that the condition (6.3.7) implies (5.2.4) because for $\tau_1 < \tau$

$$\frac{g(x)}{f(F^{-1}(F(x)+\tau_1))} > \frac{g(x)}{f(F^{-1}(F(x)+\tau))}.$$

Therefore (6.3.7) is strictly stronger than the hypothesis needed to recover the rate of growth of x. However, the gap between these hypotheses is very slight, and the asymptotic behaviour is recovered provided Δ is chosen sufficiently small. Notice that for any choice of $h > 0$ we recover the correct rate of growth of $\bar{x}_\Delta$. However, we can control the error on finite intervals only by adjusting h and Δ appropriately, i.e. by choosing $h = h_\Delta$. Note that (6.3.8) implies (6.3.2).

6.4 Control of the Error Estimate

We now show that $\bar{x}_\Delta$ does indeed approximate the true solution x on any compact interval. It it worth noting however that the primary use of the method of discretising the transformed differential equation is to replicate the growth rate of the true solution, as was demonstrated in Section 6.3. However from the point of view of error analysis this method is somewhat unnecessary, as we showed in Section 4.4 that the error associated with the numerical method described in Section 2.3 can be controlled on any compact interval when (1.1.1) holds. Thus, if we wanted to approximate the true solution with an arbitrary degree of accuracy, there is no need for pretransformation of the differential equation, we would simply use the state–dependent scheme. However for completeness the convergence of $\bar{x}_\Delta$ is now demonstrated.

Theorem 6.4.1. *Suppose that f obeys* (0.1.7), (1.1.1) *and* (4.2.1) *and g obeys* (0.1.8) *and is non-decreasing. Let $\tau > 0$ and ψ obey* (0.1.9)*. Suppose that f_τ is defined by* (5.2.3) *and* (5.2.4) *holds. Let F be given by* (1.2.4) *where $\xi \in (0, \underline{\psi})$. Let x be the unique continuous solution of* (0.1.6)*. Let $\Delta \in (0, \tau f(\xi))$ and let $N_\Delta \in \mathbb{N}$ and $h_\Delta > 0$ be given by* (6.3.3)*. Suppose that $z_n(\Delta)$ is defined by* (6.3.4)*, $\bar{Z}_\Delta$ by* (6.3.5b) *and $\bar{x}_\Delta$ by* (6.3.6)*. Then for any*

$T > 0$,

$$\lim_{\Delta\to 0}\sup_{0\le t\le T}|x(t)-\bar{x}_\Delta(t)| = 0. \tag{6.4.1}$$

6.5 Proofs

Proof of Lemma 6.2.1 Since f is non-decreasing and positive, we have

$$\frac{\Delta}{f(\xi+j\Delta)} \ge \int_{\xi+j\Delta}^{\xi+(j+1)\Delta}\frac{1}{f(x)}\,dx \ge \frac{\Delta}{f(\xi+(j+1)\Delta)}.$$

Therefore by (6.2.1) and the definition of F we have

$$r_n = \sum_{j=0}^{n-1}\frac{\Delta}{f(\xi+j\Delta)} \ge \sum_{j=0}^{n-1}\int_{\xi+j\Delta}^{\xi+(j+1)\Delta}\frac{1}{f(x)}\,dx = F(\xi+n\Delta).$$

Similarly

$$\begin{aligned} r_{n+1} &= \frac{\Delta}{f(\xi)} + \sum_{j=0}^{n-1}\frac{\Delta}{f(\xi+(j+1)\Delta)} \\ &\le \frac{\Delta}{f(\xi)} + \sum_{j=0}^{n-1}\int_{\xi+j\Delta}^{\xi+(j+1)\Delta}\frac{1}{f(x)}\,dx = \frac{\Delta}{f(\xi)} + F(\xi+n\Delta). \end{aligned}$$

Hence

$$r_n \ge F(\xi+n\Delta), \quad n\ge 0; \quad r_{n+1} \le \frac{\Delta}{f(\xi)} + F(\xi+n\Delta), \quad n \ge 0. \tag{6.5.1}$$

It is easy now to prove (6.2.4a). By (6.5.1) we have

$$F_\Delta(\xi+(n+1)\Delta) = r_{n+1} \le \frac{\Delta}{f(\xi)} + F(\xi+n\Delta).$$

Since F is increasing, we have $F(\xi+n\Delta) < F(\xi+(n+1)\Delta)$, so (6.2.4a) holds for $x = \xi+(n+1)\Delta$ and each $n\ge 0$. Now $F_\Delta(\xi) = t_0 = 0$, so

$$F_\Delta(\xi) = 0 \le \frac{\Delta}{f(\xi)} = \frac{\Delta}{f(\xi)} + F(\xi).$$

Hence (6.2.4a) holds for all $x = \xi+n\Delta$ for $n\ge 0$. Now consider $x\in[\xi+n\Delta,\xi+(n+1)\Delta)$. Since F_Δ and F are increasing we have

$$F_\Delta(x) < F_\Delta(\xi+(n+1)\Delta) \le \frac{\Delta}{f(\xi)} + F(\xi+n\Delta) \le \frac{\Delta}{f(\xi)} + F(x).$$

We now prove (6.2.4d). Let $x\ge\xi$. Then $y = F^{-1}(x)\ge 0$. Since F_Δ^{-1} is increasing and (6.2.4a) holds, we have

$$F^{-1}(x) = y = F_\Delta^{-1}(F_\Delta(y)) \le F_\Delta^{-1}(F(y)+\Delta/f(\xi)) = F_\Delta^{-1}(x+\Delta/f(\xi)).$$

Let y be the solution of (5.2.1). Then y is increasing, so because f is non-decreasing we have

$$y(r_{n+1}) = y(r_n) + \int_{r_n}^{r_{n+1}} f(y(s))\,ds \geq y(r_n) + (r_{n+1} - r_n) f(y(r_n)).$$

Note that $y(t_0) = \xi$. Let $y_n = \xi + n\Delta$. Then $y_0 = \xi$ and for $n \geq 0$ by (6.2.1) we have

$$y_{n+1} - y_n = \Delta = \frac{\Delta}{f(\xi + n\Delta)} f(y_n) = (r_{n+1} - r_n) f(y_n).$$

We now show $y(r_n) \geq y_n$ for $n \geq 0$. It is clearly true for $n = 0$. Suppose it is true for $n = k$. Then as f is non-decreasing and $y(r_k) \geq y_k$ we have

$$y(r_{k+1}) \geq y(r_k) + (r_{k+1} - r_k) f(y(r_k)) \geq y_k + (r_{k+1} - r_k) f(y_k) = y_{k+1},$$

so by induction we have $y(r_n) \geq y_n$ for $n \geq 0$, or

$$F^{-1}(r_n) \geq \xi + n\Delta = F_\Delta^{-1}(r_n), \quad n \geq 0.$$

Now, let $t > 0$ such that $t \neq r_n$. Since $r_n \to \infty$ as $n \to \infty$ there exists $n_t \in \mathbb{N}$ such that $t \in (r_{n_t}, r_{n_t+1})$. Then $(F_\Delta^{-1})'(t) = f(\xi + n_t\Delta)$. Now by (6.2.2a)

$$y(r_{n_t}) = F^{-1}(r_{n_t}) \geq F_\Delta^{-1}(r_{n_t}) = \xi + n_t\Delta.$$

Since f is non-decreasing and y solves (5.2.1) we have

$$(F_\Delta^{-1})'(t) = f(\xi + n_t\Delta) \leq f(y(r_{n_t})) \leq f(y(t)) = (F^{-1})'(t).$$

Next F_Δ^{-1} has right derivative $f(\xi + n_t\Delta)$ at r_{n_t}. Moreover

$$(F^{-1})'(r_{n_t}) = f(y(r_{n_t})) \geq f(\xi + n_t\Delta).$$

Therefore, as $F_\Delta^{-1}(r_n) \leq F^{-1}(r_n)$ we have $F^{-1}(t) \geq F_\Delta^{-1}(t)$ for all $t \geq 0$, proving (6.2.4c). Taking inverses gives (6.2.4b).

Proof of Theorem 6.3.1 We first prepare some estimates of us later in the proof. For any $\tau_2 > \tau_1 > 0$ we have

$$\frac{g(x)}{f(F^{-1}(F(x) + \tau_2))} < \frac{g(x)}{f(F^{-1}(F(x) + \tau_1))}.$$

By (6.3.8) we have $\tau - \Delta/f(\xi) > \tau_1$, so putting $\tau_2 := \tau - \Delta/f(\xi)$ we have

$$\limsup_{x\to\infty} \frac{g(x)}{f(F^{-1}(F(x) + \tau - \Delta/f(\xi)))} \leq \lim_{x\to\infty} \frac{g(x)}{f(F^{-1}(F(x) + \tau_1))} = 0,$$

so

$$\lim_{x\to\infty} \frac{g(x)}{f(F^{-1}(F(x)+\tau-\Delta/f(\xi)))} = 0. \tag{6.5.2}$$

By Lemma 6.2.1 we have

$$F^{-1}(y) \geq F_\Delta^{-1}(y), \quad y \geq 0; \quad F^{-1}(y) \leq F_\Delta^{-1}\left(y + \frac{\Delta}{f(\xi)}\right), \quad y \geq 0.$$

The second member implies

$$F^{-1}\left(z - \frac{\Delta}{f(\xi)}\right) \leq F_\Delta^{-1}(z), \quad z \geq \frac{\Delta}{f(\xi)}. \tag{6.5.3}$$

If z_Δ is given by (6.3.4), we rearrange (6.3.4a) to get

$$z_{n+1}(\Delta) - (n+1)h = z_n(\Delta) - nh + h\frac{g(F_\Delta^{-1}(z_{n-N_\Delta}(\Delta)))}{f(F_\Delta^{-1}(z_n(\Delta)))}, \quad n \geq 0.$$

Therefore $n \mapsto z_n(\Delta) - nh$ is increasing for $n \geq 0$. Let $n \geq N_\Delta$. Then by (6.3.3) we have

$$z_{n-N_\Delta}(\Delta) + \tau - nh = z_{n-N_\Delta}(\Delta) - (n - N_\Delta)h \leq z_n(\Delta) - nh.$$

Hence

$$z_{n-N_\Delta}(\Delta) + \tau \leq z_n(\Delta), \quad \text{for } n \geq N_\Delta. \tag{6.5.4}$$

Another consequence of the monotonicity of $n \mapsto z_n(\Delta) - nh$ is that

$$z_n(\Delta) \geq z_0(\Delta) + nh, \quad n \geq 0. \tag{6.5.5}$$

Note for $n \geq N_\Delta$ that by (6.5.5), (6.2.4b), (6.3.3) and (6.3.2) we have

$$\begin{aligned} z_n(\Delta) &\geq z_0(\Delta) + nh \geq z_0(\Delta) + N_\Delta h \\ &= z_0(\Delta) + \tau > F_\Delta(\psi(0)) + \frac{\Delta}{f(\xi)} \\ &\geq F(\psi(0)) + \frac{\Delta}{f(\xi)}, \end{aligned}$$

where we have used the fact that $\psi(0) > \xi$. Using this fact again, we have

$$z_n(\Delta) > \frac{\Delta}{f(\xi)}, \quad z_n(\Delta) \geq F(\psi(0)) + \tau > \tau, \quad \text{for } n \geq N_\Delta.$$

Hence by (6.5.3) we have

$$F^{-1}\left(z_n(\Delta) - \frac{\Delta}{f(\xi)}\right) \leq F_\Delta^{-1}(z_n(\Delta)), \quad n \geq N_\Delta. \tag{6.5.6}$$

For $n \geq N_\Delta$, by (6.5.4), (6.2.4c) we have

$$F_\Delta^{-1}(z_{n-N_\Delta}(\Delta)) \leq F_\Delta^{-1}(z_n(\Delta)-\tau) \leq F^{-1}(z_n(\Delta)-\tau), \quad n \geq N_\Delta. \tag{6.5.7}$$

Therefore, for $n \geq N_\Delta$ by the monotonicity of f and g and (6.5.6) and (6.5.7) we have

$$\frac{g(F_\Delta^{-1}(z_{n-N_\Delta}(\Delta)))}{f(F_\Delta^{-1}(z_n(\Delta)))} \leq \frac{g(F^{-1}(z_n(\Delta)-\tau))}{f\left(F^{-1}\left(z_n(\Delta)-\frac{\Delta}{f(\xi)}\right)\right)}.$$

Therefore for $n \geq N_\Delta$

$$z_{n+1}(\Delta) \leq z_n(\Delta) + h + h\frac{g(F^{-1}(z_n(\Delta)-\tau))}{f\left(F^{-1}\left(z_n(\Delta)-\frac{\Delta}{f(\xi)}\right)\right)}.$$

Define

$$\varphi(z) = \frac{g(F^{-1}(z-\tau))}{f\left(F^{-1}\left(z-\frac{\Delta}{f(\xi)}\right)\right)}, \quad z \geq \tau. \tag{6.5.8}$$

Then with $x(z) := F^{-1}(z-\tau)$, we have that $x(z) \to \infty$ as $z \to \infty$ and

$$\varphi(z) = \frac{g(x(z))}{f\left(F^{-1}\left(F(x(z))+\tau-\frac{\Delta}{f(\xi)}\right)\right)}, \quad z \geq \tau.$$

Therefore by (6.5.2), we have $\varphi(z) \to 0$ as $z \to \infty$. By the definition of φ we have

$$z_{n+1}(\Delta) - z_n(\Delta) \leq h + h\varphi(z_n(\Delta)), \quad n \geq N_\Delta. \tag{6.5.9}$$

Note that $z_n(\Delta) \to \infty$ as $n \to \infty$, so that $\varphi(z_n(\Delta)) \to 0$ as $n \to \infty$. Therefore we have

$$\lim_{n\to\infty} \frac{1}{n-N_\Delta+1}\sum_{j=N_\Delta}^{n} \varphi(z_j(\Delta)) = 0.$$

Summing both sides of (6.5.9) over $\{N_\Delta, \dots, n\}$ and using the last limit, we obtain

$$\limsup_{n\to\infty} \frac{z_n(\Delta)}{nh} \leq 1.$$

By (6.5.5) we have

$$\liminf_{n\to\infty} \frac{z_n(\Delta)}{nh} \geq 1,$$

and therefore we have

$$\lim_{n\to\infty} \frac{z_n(\Delta)}{nh} = 1. \tag{6.5.10}$$

By (6.3.4), $(z_\Delta(n))_{n=0}^\infty$ is increasing. Therefore by (6.3.5b) we get

$$z_{n(t)}(\Delta) \leq \bar{Z}_\Delta(t) \leq z_{n(t)+1}(\Delta), \quad t \geq 0.$$

Therefore by (6.5.10) we have

$$\lim_{t\to\infty} \frac{\bar{Z}_\Delta(t)}{n(t)h} = 1,$$

and since $n(t)/t \to 1/h$ as $t \to \infty$ we have

$$\lim_{t\to\infty} \frac{\bar{Z}_\Delta(t)}{t} = \lim_{t\to\infty} \frac{\bar{Z}_\Delta(t)}{n(t)h} \cdot \frac{n(t)h}{t} = 1.$$

By (6.3.6) we have

$$\lim_{t\to\infty} \frac{F_\Delta(\bar{x}_\Delta(t))}{t} = 1.$$

Finally by applying (6.2.4a) and (6.2.4b) we obtain (6.3.9) as required.

Proof of Theorem 6.4.1 Define for $v_1, v_2 \geq 0$,

$$\varrho(v_1, v_2) := 1 + \frac{g(F^{-1}(v_1))}{f(F^{-1}(v_2))}, \tag{6.5.11a}$$

$$\varrho_\Delta(v_1, v_2) := 1 + \frac{g(F_\Delta^{-1}(v_1))}{f(F_\Delta^{-1}(v_2))}. \tag{6.5.11b}$$

Then we can rewrite (6.3.1) and (6.3.5a) as

$$z(t) = z(0) + \int_0^t \varrho(z(s-\tau), z(s))\,ds$$
$$\bar{Z}_\Delta(t) = z_0(\Delta) + \int_0^t \varrho_\Delta(Z_\Delta(s-\tau), Z_\Delta(s))\,ds.$$

Thus

$$\begin{aligned}
&|z(t) - \bar{Z}_\Delta(t)| \\
&\quad = |F(\psi(0)) - F_\Delta(\psi(0)) + \int_0^t \varrho(z(s-\tau), z(s))\,ds - \int_0^t \varrho_\Delta(Z_\Delta(s-\tau), Z_\Delta(s))| \\
&\quad \leq |F(\psi(0)) - F_\Delta(\psi(0))| + |\int_0^t \varrho(z(s-\tau), z(s))\,ds - \int_0^t \varrho_\Delta(Z_\Delta(s-\tau), Z_\Delta(s))| \\
&\quad \leq \Delta/f(\xi) + \int_0^t |\varrho(z(s-\tau), z(s)) - \varrho_\Delta(Z_\Delta(s-\tau), Z_\Delta(s))|\,ds.
\end{aligned} \tag{6.5.12}$$

since $|F(\psi(0)) - F_\Delta(\psi(0))| \leq \Delta/f(\xi)$ using (6.2.4a) and (6.2.4b).

Now for $v_1, v_2, w_1, w_2 > 0$,

$$\begin{aligned}
&\varrho(v_1, v_2) - \varrho_\Delta(w_1, w_2) \\
&\quad = \frac{g(F^{-1}(v_1))}{f(F^{-1}(v_2))} - \frac{g(F_\Delta^{-1}(w_1))}{f(F_\Delta^{-1}(w_2))} \\
&\quad = \frac{g(F^{-1}(v_1))f(F_\Delta^{-1}(w_2)) - g(F_\Delta^{-1}(w_1))f(F^{-1}(v_2))}{f(F^{-1}(v_2))f(F_\Delta^{-1}(w_2))} \\
&\quad = \frac{1}{f(F^{-1}(v_2))f(F_\Delta^{-1}(w_2))}\Big(g(F^{-1}(v_1))\left[f(F_\Delta^{-1}(w_2)) - f(F_\Delta^{-1}(v_2))\right] \\
&\qquad + f(F_\Delta^{-1}(v_2))\left[g(F^{-1}(v_1)) - g(F^{-1}(w_1))\right] \\
&\qquad + g(F^{-1}(w_1))\left[f(F_\Delta^{-1}(v_2)) - f(F^{-1}(v_2))\right] \\
&\qquad + f(F^{-1}(v_2))\left[g(F^{-1}(w_1)) - g(F_\Delta^{-1}(w_1))\right]\Big).
\end{aligned} \tag{6.5.13}$$

We look to control each term inside the brackets in (6.5.13). To do this, we first need to determine the Lipschitz constants. If $t \leq \tau$,

$$g(F^{-1}(z(t-\tau))) = g(F^{-1}(F(\psi(t-\tau)))) = g(\psi(t-\tau)) \leq \max_{x\in(0,\psi^*]} g(x) =: \bar{g}_\psi.$$

Since f is monotone and z is non-decreasing on $[0,\infty)$, $f(F^{-1}(z(t))) \geq f(F^{-1}(z(0))) = f(F^{-1}(F(\psi(0)))) = f(\psi(0))$. Thus

$$z'(t) \leq 1 + \frac{\bar{g}_\psi}{f(\psi(0))} =: 1 + L_\psi, \quad t \leq \tau.$$

Since $g(x)/f_\tau(x) \to 0$ as $x \to \infty$ and is continuous and f and g obey (0.1.7) and (0.1.8),

$$\frac{g(x)}{f_\tau(x)} \leq \sup_{y>0} \frac{g(y)}{f_\tau(y)} =: S$$

Therefore $g(x) \leq S f_\tau(x)$ for all $x > 0$. Thus

$$g(x(t-\tau)) \leq S f_\tau(x(t-\tau)) = S f(F^{-1}(F(x(t-\tau)) + \tau)) = S f(F^{-1}(z(t-\tau) + \tau)).$$

Now since z obeys (5.4.2), $z(t-\tau) \leq z(t) - \tau$. Therefore

$$g(x(t-\tau)) \leq S f(F^{-1}(z(t) - \tau + \tau)) = S f(F^{-1}(z(t))).$$

Thus for $t > \tau$, $z'(t) \leq 1 + S$. Defining $S^* := \max(L_\psi, S)$ we have $z'(t) \leq 1 + S^*$ for $t \geq 0$. Integrating over $[0,t]$ yields

$$z(t) \leq z(0) + t(1 + S^*), \quad t \geq 0.$$

and specifically

$$z(T) \leq z(0) + T(1 + S^*) =: P(T), \quad t \geq 0. \tag{6.5.14}$$

Since z_n obeys (6.3.4), for $n \geq 0$

$$z_{n+1}(\Delta) = z_n(\Delta) + h_\Delta \left(1 + \frac{g(F_\Delta^{-1}(z_{n-N_\Delta}(\Delta)))}{f(F_\Delta^{-1}(z_n(\Delta)))}\right).$$

If $n - N_\Delta \leq 0$, $g(F_\Delta^{-1}(z_{n-N\Delta}(\Delta))) = g(\psi((n - N_\Delta)h_\Delta)) \leq \bar{g}_\psi$. Also since f is monotone, $f(F_\Delta^{-1}(z_n(\Delta))) \geq f(F_\Delta^{-1}(z_0(\Delta))) = f(\psi(0))$. Thus

$$z_{n+1}(\Delta) \leq z_n(\Delta) + h_\Delta(1 + L_\psi), \quad n \leq N_\Delta.$$

If $n - N_\Delta > 0$, first note that φ defined by (6.5.8) obeys $\varphi(z) \to 0$ as $z \to \infty$. Therefore since φ is continuous,

$$\varphi(z) \leq \sup_{y \in (0,\infty)} \varphi(y) =: \varphi^*, \quad z > 0.$$

Thus $\varphi(z_n(\Delta)) \leq \varphi^*$ for $n \geq 0$ and

$$z_{n+1}(\Delta) \leq z_n(\Delta) + h_\Delta(1 + \varphi^*), \quad n > N_\Delta.$$

With $R^* := \max(L_\psi, \varphi^*)$ we have $z_{n+1}(\Delta) \leq z_n(\Delta) + h_\Delta(1 + R^*)$ for $n \geq 0$ and so

$$z_n(\Delta) \leq z_0(\Delta) + nh_\Delta(1 + R^*), \quad n \geq 0. \tag{6.5.15}$$

For any $T > 0$ let $n(t)$ be such that $n(t)h_\Delta \geq T > (n(t) - 1)h_\Delta$. Then by (6.5.15)

$$\begin{aligned}
\bar{Z}_\Delta(T) \leq z_{n(t)}(\Delta) &\leq z_0(\Delta) + n(t)h_\Delta(1 + R^*) \\
&< z_0(\Delta) + (T + h_\Delta)(1 + R^*) \\
&= F_\Delta(\psi(0)) + (T + h_\Delta)(1 + R^*) \\
&\leq \Delta/f(\xi) + F(\psi(0)) + (T + \Delta)(1 + R^*) =: P_\Delta(T)
\end{aligned}$$

where we have used (6.2.4a) and $h_\Delta \leq \Delta$ at the last step. Thus for any $\Delta_0 \in (0, \tau f(\xi))$, $P_\Delta(T) < P_{\Delta_0}(T)$ for $\Delta < \Delta_0$. Thus

$$\bar{Z}_\Delta(T) < P_{\Delta_0}(T), \quad \Delta < \Delta_0. \tag{6.5.16}$$

Setting $P^*(T) = \max(\psi^*, P(T), P_{\Delta_0}(T))$ we have that $z(t), \bar{Z}_\Delta(t), Z_\Delta(t) \in [0, P^*(T)]$ for $t \in [-\tau, T]$ and $\Delta < \Delta_0$. Next define $M^*(T) := F^{-1}(P^*(T))$ and define the Lipschitz constants as $c^f_{M^*(T)}$ and $c^g_{M^*(T)}$.

We are now in a position to examine the terms inside the brackets in (6.5.13). For the first term, with $v_2, w_2 \leq P^*(T)$ and since F_Δ^{-1} is increasing, $F_\Delta^{-1}(v_2), F_\Delta^{-1}(w_2) \leq F_\Delta^{-1}(P^*(T)) \leq M^*(T)$ by (6.2.4c). Thus

$$|f(F_\Delta^{-1}(w_2)) - f(F_\Delta^{-1}(v_2))| \leq c^f_{M^*(T)}|F_\Delta^{-1}(w_2) - F_\Delta^{-1}(v_2)|.$$

Now for any $T > 0$ let n be such that $F_\Delta(\xi + n\Delta) \leq P^*(T) < F_\Delta(\xi + (n+1)\Delta)$. Using (6.2.3),

$$\left(F_\Delta^{-1}\right)'(t) \leq f(\xi + n\Delta), \quad t \leq P^*(T).$$

Therefore since f is non-decreasing,

$$\begin{aligned}
|F_\Delta^{-1}(w_2) - F_\Delta^{-1}(v_2)| &\leq f(\xi + n\Delta)|w_2 - v_2| \\
&\leq f(F_\Delta^{-1}(P^*(T)))|w_2 - v_2| \\
&\leq f(M^*(T))|w_2 - v_2|.
\end{aligned}$$

So

$$\begin{aligned}
|f(F_\Delta^{-1}(w_2)) - f(F_\Delta^{-1}(v_2))| &\leq c^f_{M^*(T)} f(M^*(T))|w_2 - v_2| \\
&=: \kappa_1(T)|w_2 - v_2|. \qquad (6.5.17)
\end{aligned}$$

For the second term in (6.5.13), $F^{-1}(v_1), F^{-1}(w_1) \leq M^*(T)$ so

$$|g(F^{-1}(v_1)) - g(F^{-1}(w_1))| \leq c^g_{M^*(T)}|F^{-1}(v_1) - F^{-1}(w_1)|.$$

Recall that y defined by (5.2.1) obeys $y(t) = F^{-1}(t)$ for $t \geq 0$. Thus

$$\begin{aligned}
|F^{-1}(v_1) - F^{-1}(w_1)| &= |y(v_1) - y(w_1)| \\
&= \left|\int_{w_1}^{v_1} f(y(s))\,ds\right| \\
&\leq \left|\int_{w_1}^{v_1} f(M^*(T))\,ds\right| \\
&\leq f(M^*(T))|v_1 - w_1|
\end{aligned}$$

as f is non-decreasing and so

$$\begin{aligned}
|g(F^{-1}(v_1)) - g(F^{-1}(w_1))| &\leq c^g_{M^*(T)} f(M^*(T))|v_1 - w_1| \\
&=: \kappa_2(T)|v_1 - w_1|. \qquad (6.5.18)
\end{aligned}$$

For the third term in (6.5.13), $F_\Delta^{-1}(v_2), F^{-1}(v_2) \le M^*(T)$ so

$$|f(F_\Delta^{-1}(v_2)) - f(F^{-1}(v_2))| \le c^f_{M^*(T)}|F_\Delta^{-1}(v_2) - F^{-1}(v_2)| = c^f_{M^*(T)}\left(F^{-1}(v_2) - F_\Delta^{-1}(v_2)\right)$$

since $F^{-1} - F_\Delta^{-1}$ is always positive. Now if $v_2 \le \Delta/f(\xi)$,

$$F^{-1}(v_2) - F_\Delta^{-1}(v_2) \le F^{-1}(v_2) - F_\Delta^{-1}(0) = F^{-1}(v_2) - F^{-1}(0) \le \frac{\Delta}{f(\xi)}f(M^*(T)).$$

If $v_2 > \Delta/f(\xi)$, by (6.2.4d)

$$F^{-1}(v_2) - F_\Delta^{-1}(v_2) \le F^{-1}(v_2) - F^{-1}(v_2 - \Delta/f(\xi)) \le \frac{\Delta}{f(\xi)}f(M^*(T)).$$

Combining both cases we have

$$|f(F_\Delta^{-1}(v_2)) - f(F^{-1}(v_2))| \le c^f_{M^*(T)}\frac{\Delta}{f(\xi)}f(M^*(T)) = \frac{\Delta}{f(\xi)}\kappa_1(T). \tag{6.5.19}$$

Analogously for the fourth term,

$$|g(F^{-1}(w_1)) - g(F_\Delta^{-1}(w_1))| \le c^g_{M^*(T)}\frac{\Delta}{f(\xi)}f(M^*(T)) = \frac{\Delta}{f(\xi)}\kappa_2(T). \tag{6.5.20}$$

Next noting that $f(F^{-1}(v_2)), f(F^{-1}(v_2)) \ge f(\xi)$, $1/[f(F^{-1}(v_2))f(F^{-1}(v_2))] \le 1/f^2(\xi)$. Therefore using this and inserting (6.5.17), (6.5.18), (6.5.19) and (6.5.20) into (6.5.13) we have

$$\begin{aligned}\varrho(v_1, v_2) - \varrho_\Delta(w_1, w_2) \le \frac{1}{f^2(\xi)}\Big(&g(M^*(T))\kappa_1(T)\left[\frac{\Delta}{f(\xi)}|w_2 - v_2|\right] \\ &+ f(M^*(T))\kappa_2(T)\left[\frac{\Delta}{f(\xi)}|v_1 - w_1|\right]\Big).\end{aligned} \tag{6.5.21}$$

Putting this into (6.5.12) yields

$$\begin{aligned}\sup_{t\in[0,T]}|z(t) - \bar{Z}_\Delta(t)| \le \Delta/f(\xi) + \int_0^T \frac{1}{f^2(\xi)}\Big(&g(M^*(T))\kappa_1(T)\left[\frac{\Delta}{f(\xi)}|Z_\Delta(s) - z(s)|\right] \\ &+ f(M^*(T))\kappa_2(T)\left[\frac{\Delta}{f(\xi)}|z(s-\tau) - Z_\Delta(s-\tau)|\right]\Big).\end{aligned}$$

Now $|z(t) - Z_\Delta(t)| \le |z(t) - \bar{Z}_\Delta(t)| + |\bar{Z}_\Delta(t) - Z_\Delta(t)|$ for all $t \ge \tau$, thus

$$\begin{aligned}\sup_{t\in[0,T]}|z(t) - \bar{Z}_\Delta(t)| &\le \Delta/f(\xi) + \kappa_3(T)\int_0^T \left(|z(s) - \bar{Z}_\Delta(s)| + |\bar{Z}_\Delta(s) - Z_\Delta(s)|\right)ds \\ &+ \kappa_4(T)\int_0^T \left(|z(s-\tau) - \bar{Z}_\Delta(s-\tau)| + |\bar{Z}_\Delta(s-\tau) - Z_\Delta(s-\tau)|\right)ds\end{aligned} \tag{6.5.22}$$

with $\kappa_3(T) := \frac{1}{f^3(\xi)} g(M^*(T))\kappa_1(T)$ and $\kappa_4(T) := \frac{1}{f^3(\xi)} f(M^*(T))\kappa_2(T)$.

Now given $s \in [0,T]$, note $n(s)h_\Delta \le s < (n(s)+1)h_\Delta$. Thus

$$\begin{aligned} |\bar{Z}_\Delta(s) - Z_\Delta(s)| &= \left| \int_{n(s)h_\Delta}^{s} \varrho_\Delta(Z_\Delta(u-\tau), Z_\Delta(u))\, du \right| \\ &\le (s - n(s)h_\Delta)|\varrho_\Delta(z_{n-N_\Delta}(\Delta), z_n(\Delta))| \\ &= (s - n(s)h_\Delta)\left(1 + \frac{g(F_\Delta^{-1}(z_{n-N_\Delta}(\Delta)))}{f(F_\Delta^{-1}(z_n(\Delta)))}\right) \\ &< h_\Delta \left(1 + \frac{g(M^*(T))}{f(\psi(0))}\right). \end{aligned}$$

and so

$$\int_0^T |\bar{Z}_\Delta(s) - Z_\Delta(s)|\, ds \le h_\Delta T \left(1 + \frac{g(M^*(T))}{f(\psi(0))}\right) =: h_\Delta \kappa_5(T). \tag{6.5.23}$$

Next,

$$\int_0^T |\bar{Z}_\Delta(s-\tau) - Z_\Delta(s-\tau)|\, ds \le \int_{-\tau}^{T-\tau} |\bar{Z}_\Delta(s) - Z_\Delta(s)|\, ds$$

For $T-\tau < 0$, there exists $t \in [-\tau, -h_\Delta]$ such that $n(t)h_\Delta \le T-\tau < (n(t)+1)h_\Delta$. Thus

$$\begin{aligned} \int_{-\tau}^{T-\tau} |\bar{Z}_\Delta(s) - Z_\Delta(s)|\, ds &= \sum_{j=-N_\Delta}^{n(t)-1} \int_{jh_\Delta}^{(j+1)h_\Delta} |\bar{Z}_\Delta(s) - Z_\Delta(s)|\, ds \\ &\quad + \int_{n(t)h_\Delta}^{T-\tau} |\bar{Z}_\Delta(s) - Z_\Delta(s)|\, ds \\ &= \sum_{j=-N_\Delta}^{n(t)-1} \int_{jh_\Delta}^{(j+1)h_\Delta} |F_\Delta(\psi(s)) - F_\Delta(\psi(jh_\Delta))|\, ds \\ &\quad + \int_{n(t)h_\Delta}^{T-\tau} |F_\Delta(\psi(s)) - F_\Delta(\psi(n(t)h_\Delta))|\, ds. \end{aligned}$$

Now using (6.2.3), $(F_\Delta)'(t) = 1/\left(F_\Delta^{-1}\right)'(F_\Delta(t)) \le 1/f(\xi)$, so

$$\begin{aligned} |F_\Delta(\psi(s)) - F_\Delta(\psi(jh_\Delta))| &\le 1/f(\xi)|\psi(s) - \psi(jh_\Delta)| \\ &\le 1/f(\xi) \sup_{0 \le s - jh_\Delta \le h_\Delta} |\psi(s) - \psi(jh_\Delta)| \\ &\le 1/f(\xi) \sup_{s,u \in [-\tau,0]: 0 \le s-u \le h_\Delta} |\psi(s) - \psi(jh_\Delta)| \\ &= 1/f(\xi)\omega_\psi(h_\Delta) \end{aligned}$$

where ω_ψ is a modulus of continuity of the continuous function ψ. Thus

$$\begin{aligned}\int_0^T |\bar{Z}_\Delta(s-\tau) - Z_\Delta(s-\tau)|\, ds &\le \sum_{j=-N_\Delta}^{n(t)-1} \frac{h_\Delta}{f(\xi)}\omega_\psi(h_\Delta) + \frac{h_\Delta}{f(\xi)}\omega_\psi(h_\Delta) \\ &\le \sum_{j=-N_\Delta}^{-1} \frac{h_\Delta}{f(\xi)}\omega_\psi(h_\Delta) = N_\Delta \frac{h_\Delta}{f(\xi)}\omega_\psi(h_\Delta) \\ &= \frac{\tau}{f(\xi)}\omega_\psi(h_\Delta), \end{aligned} \tag{6.5.24}$$

and this bound also holds for $T - \tau = 0$ since $\int_{-h_\Delta}^0 |F_\Delta(\psi(s)) - F_\Delta(\psi(n(t)h_\Delta))|\, ds \le \frac{h_\Delta}{f(\xi)}\omega_\psi(h_\Delta)$. Therefore for $T - \tau \ge 0$,

$$\begin{aligned}\int_0^T |\bar{Z}_\Delta(s-\tau) - Z_\Delta(s-\tau)| &= \int_{-\tau}^0 |\bar{Z}_\Delta(s) - Z_\Delta(s)|\, ds + \int_0^{T-\tau} |\bar{Z}_\Delta(s) - Z_\Delta(s)|\, ds \\ &\le \frac{\tau}{f(\xi)}\omega_\psi(h_\Delta) + h_\Delta(T-\tau)\kappa_5(T)\end{aligned}$$

where we have used (6.5.23) at the last step. Combining both cases for $T - \tau$ we have

$$\int_0^T |\bar{Z}_\Delta(s-\tau) - Z_\Delta(s-\tau)| \le \frac{\tau}{f(\xi)}\omega_\psi(h_\Delta) + h_\Delta(T-\tau)\kappa_5(T). \tag{6.5.25}$$

Thus inserting (6.5.23) and (6.5.25) into (6.5.22) yields

$$\begin{aligned}\sup_{t\in[0,T]} |z(t) - \bar{Z}_\Delta(t)| &\le \Delta/f(\xi) + h_\Delta\kappa_6(T) + \omega_\psi(h_\Delta)\kappa_7(T) \\ &\quad + \kappa_3(T)\int_0^T |z(s) - \bar{Z}_\Delta(s)| + \kappa_4(T)\int_0^T |z(s-\tau) - \bar{Z}_\Delta(s-\tau)|. \end{aligned} \tag{6.5.26}$$

with $\kappa_6(T) := T\kappa_3(T)\kappa_5(T) + (T-\tau)\kappa_5(T)$ and $\kappa_7(T) := \kappa_4(T)\frac{\tau}{f(\xi)}$. Now, let $e^*(s) := \sup_{-\tau\le t\le s} |z(t) - \bar{Z}_\Delta(t)|$ and note that

$$\begin{aligned} e^*(T) &= \sup_{-\tau\le t\le T} |z(t) - \bar{Z}_\Delta(t)| \\ &= \max\left(\sup_{-\tau\le t\le\ 0} |z(t) - \bar{Z}_\Delta(t)|,\ \sup_{0\le t\le\ T} |z(t) - \bar{Z}_\Delta(t)| \right) \\ &= \max\left(0,\ \sup_{0\le t\le\ T} |z(t) - \bar{Z}_\Delta(t)| \right) \\ &= \sup_{0\le t\le\ T} |z(t) - \bar{Z}_\Delta(t)|. \end{aligned}$$

Therefore

$$e^*(T) \le \Delta/f(\xi)h_\Delta\kappa_6(T) + \omega_\psi(h_\Delta)\kappa_7(T) + (\kappa_3(T) + \kappa_4(T))\int_0^T e^*(s)\, ds$$

and so by Gronwall's inequality

$$\sup_{0\leq t\leq T} |z(t) - \bar{Z}_\Delta(t)| \leq \Delta/f(\xi)\left(h_\Delta \kappa_6(T) + \omega_\psi(h_\Delta)\kappa_7(T)\right) e^{T(\kappa_3(T)+\kappa_4(T))}$$

Now since $h_\Delta \leq \Delta$ and ω_ψ is nondecreasing.

$$\sup_{0\leq t\leq T} |z(t) - \bar{Z}_\Delta(t)| \leq \Delta/f(\xi)\left(\Delta \kappa_6(T) + \omega_\psi(\Delta)\kappa_7(T)\right) e^{T(\kappa_3(T)+\kappa_4(T))}.$$

Now note that for all $t \geq 0$

$$\begin{aligned}|F(x(t)) - F(\bar{x}_\Delta(t))| &= F'(\zeta(t))|x(t) - \bar{x}_\Delta(t)|, \\ &= \frac{1}{f(\zeta(t))}|x(t) - \bar{x}_\Delta(t)|, \quad \zeta(t) \in [x(t), \bar{x}_\Delta(t)].\end{aligned}$$

Thus

$$\begin{aligned}|x(t) - \bar{x}_\Delta(t)| &= f(\zeta(t))|F(x(t)) - F(\bar{x}_\Delta(t))| \\ &\leq \sup_{0\leq x\leq F(M^*(T))} f(x)|F(x(t)) - F(\bar{x}_\Delta(t))|.\end{aligned}$$

Now since for $t \geq 0$

$$|F(x(t)) - F(\bar{x}_\Delta(t))| \leq |F(x(t)) - F_\Delta(\bar{x}_\Delta(t))| + |F_\Delta(\bar{x}_\Delta(t)) - F(\bar{x}_\Delta(t))|,$$

and using (6.2.4a),

$$\begin{aligned}\sup_{0\leq t\leq T} |x(t) - \bar{x}_\Delta(t)| &\leq \sup_{0\leq t\leq T} |z(t) - \bar{Z}_\Delta(t)| + \Delta/f(\xi) \\ &\leq \Delta/f(\xi)\left(\Delta \kappa_6(T) + \omega_\psi(\Delta)\kappa_7(T)\right) e^{T(\kappa_3(T)+\kappa_4(T))} + \Delta/f(\xi)\end{aligned}$$

Since ω_ψ is a modulus of continuity of ψ, $\lim_{\delta\to 0}\omega_\psi(\delta) = 0$. Taking limits as $\Delta \to 0$ yields (6.4.1).

Chapter 7

Delay-Dominated Equations

7.1 Introduction

In Chapter 5, we explored the necessary and sufficient conditions for the unique solution x to the delay differential equation (0.1.6) to grow at a rate characterised by the function F given by (1.2.4). Roughly speaking, we were able to ascertain that as g increased in relative asymptotic terms compared to f_τ, the rate of growth of x departed from that of (5.2.1). However because those results considered cases where g grew at a rate asymptotically equivalent to some f_θ where $\theta > \tau$, the rate of growth was still determined by $\bar{F}$, albeit with a non-unit normalising constant. It is therefore natural to consider the critical relative nonlinearity in g at which the solution to (0.1.6) no longer grows at a rate characterised by the instantaneous component of the equation. We will refer to such equations as being "delay-dominated".

Our results recover and extend Chapter 1 in a number of directions. In Section 7.2, we give general theorems on the growth rate of x in which the delay term in some sense asymptotically dominates the instantaneous term. In these general theorems the sufficient conditions which describe this dominance, as well as the rate of growth of solutions, depend on the existence of an auxiliary function ϕ obeying certain asymptotic properties: we do not attempt, in our general results, to demonstrate that such a function exists, nor do we indicate how it might be constructed. However, in Section 7.2.1, we give some representative examples for which the auxiliary function can be found, and the exact asymptotic behaviour determined. The general theorems are obtained by employing a *constructive comparison principle* (see [2, 7], for example).

Statements and discussion of the main results, as well as examples, are given in Section 7.2, while proofs are deferred to Section 7.3. Specific examples are featured in Section 7.2.2.

7.2 General Comparison Results

Before we state our main results, we first introduce some auxiliary functions. Suppose that $\phi : (\psi^*, \infty) \to (0, \infty)$ is continuous, and define

$$\Gamma(x) = \int_{\psi^*}^{x} \frac{1}{\phi(u)}\, du, \quad x > \psi^*. \tag{7.2.1}$$

Suppose that

$$\lim_{x\to\infty} \Gamma(x) = +\infty. \tag{7.2.2}$$

Define also for $c > 0$ the function Γ_c given by

$$\Gamma_c(x) = \frac{1}{c}\Gamma(x), \quad x > \psi^*. \tag{7.2.3}$$

In our first main result, we claim that if f is asymptotically dominated by the delayed term, then the solution of (0.1.6) behaves according to the ordinary differential equation $z'(t) = \phi(z(t))$.

Theorem 7.2.1. *Suppose that f obeys* (0.1.7) *and* (1.1.1)*. Let g be non–decreasing and obey* (0.1.8) *and let $\tau > 0$ and ψ obey* (0.1.9)*. Suppose that there exists a continuous function ϕ such that Γ, Γ_c are defined by* (7.2.1) *and* (7.2.3) *respectively, and that Γ obeys* (7.2.2)*. Suppose also that*

$$\lim_{\epsilon\to 0^+} \eta(\varepsilon) = \eta, \tag{7.2.4}$$

and suppose that

$$\lim_{x\to\infty} \frac{f(x)}{\phi(x)} = 0, \tag{7.2.5}$$

$$\limsup_{x\to\infty} \frac{g(x)}{\phi(\Gamma^{-1}_{\eta(\varepsilon)}(\Gamma_{\eta(\varepsilon)}(x) + \tau))} = \bar{\eta}_\varepsilon \in [0, \infty) \quad \textit{for every } \varepsilon \in (0, 1), \tag{7.2.6}$$

where

$$\sup_{\epsilon\in(0,1)} \bar{\eta}_\epsilon =: \bar{\eta} < \eta. \tag{7.2.7}$$

If x is the unique continuous solution of (0.1.6)*, then*

$$\limsup_{t\to\infty} \frac{\Gamma(x(t))}{t} \le \eta. \tag{7.2.8}$$

This offers an improvement on Theorem 2.2 of Appleby, McCarthy and Rodkina [4]. In that theorem the condition

$$\lim_{x\to\infty} \frac{f(x)}{g(\Gamma^{-1}_{\eta(\varepsilon)}(\Gamma_{\eta(\varepsilon)}(x) - \tau))} = 0, \quad \text{for every } \varepsilon \in (0, 1) \tag{7.2.9}$$

is relaxed. In later examples we show that this enables asymptotic estimates to be extended to a wider class of problems.

We comment briefly on Theorem 7.2.1 and its hypotheses. First, the existence of a function ϕ obeying (7.2.6) and (7.2.9) is not assured by the theorem; the existence or construction of such a function must be achieved independently. However, (7.2.6) describes an asymptotic relationship between ϕ and g only, and this is what identifies candidates for ϕ. In Section 7.2.1, we give examples of the function g for which a suitable ϕ can be chosen. The condition (7.2.9) characterises the fact that the instantaneous term f is dominated by the delayed term.

We now state a corresponding result which enables us to determine a lower bound on the rate of growth of solutions. It appeared as Theorem 2.3 in [4].

Theorem 7.2.2. *Suppose that f obeys* (0.1.7) *and* (1.1.1)*. Let g be non–decreasing and obey* (0.1.8) *and let $\tau > 0$ and ψ obey* (0.1.9)*. Suppose that there exists a continuous function ϕ such that Γ, Γ_c are defined by* (7.2.1) *and* (7.2.3) *respectively, and Γ obeys* (7.2.2)*. Suppose also that*

$$\lim_{\epsilon\to 0^+} \mu(\varepsilon) = \mu, \tag{7.2.10}$$

and that g and ϕ obey

$$\liminf_{x\to\infty} \frac{g(x)}{\phi(\Gamma^{-1}_{\mu(\varepsilon)}(\Gamma_{\mu(\varepsilon)}(x) + \tau(1-\epsilon)))} = \bar{\mu}_\varepsilon \in (0,\infty] \quad \textit{for every } \varepsilon \in (0,1), \tag{7.2.11}$$

where

$$\inf_{\epsilon\in(0,1)} \bar{\mu}_\epsilon =: \bar{\mu} > \mu. \tag{7.2.12}$$

If x is the unique continuous solution of (0.1.6)*, then*

$$\liminf_{t\to\infty} \frac{\Gamma(x(t))}{t} \geq \mu. \tag{7.2.13}$$

As in Theorem 7.2.1, in which the condition (7.2.6) determines a relationship between ϕ and g, in Theorem 7.2.2 there is a corresponding and closely related condition (7.2.11) which describes the relationship between g and ϕ.

Contingent on other hypotheses being satisfied, we notice that the lower bound (7.2.13) and the upper bound (7.2.8) incorporate the same function Γ. Therefore, under certain conditions we may combine Theorems 7.2.1 and 7.2.2 to arrive at the exact asymptotic behaviour of x. This is the subject of the next result, which improves on a result in [4].

Theorem 7.2.3. *Suppose f obeys* (0.1.7)*) and* (1.1.1)*. Let g be non–decreasing and obey* (0.1.8)*) and let $\tau > 0$ and ψ obey* (0.1.9)*. Suppose that there exists a continuous function ϕ such that Γ, Γ_c are defined by* (7.2.1) *and* (7.2.3)*, and that Γ obeys* (7.2.2)*. Suppose also that there is $\eta > 0$ such that $\mu(\epsilon) \to \eta$ and $\eta(\epsilon) \to \eta$ as $\epsilon \to 0$ and that f, g, and ϕ obey* (7.2.5)*,* (7.2.6) *and* (7.2.11)*, where*

$$\sup_{\epsilon \in (0,1)} \bar{\eta}_\epsilon =: \bar{\eta} < \eta, \qquad \inf_{\epsilon \in (0,1)} \bar{\mu}_\epsilon =: \bar{\mu} > \eta. \tag{7.2.14}$$

If x is the unique continuous solution of (0.1.6)*, then*

$$\lim_{t \to \infty} \frac{\Gamma(x(t))}{t} = \eta. \tag{7.2.15}$$

Provided that a function ϕ can be found so that all the hypotheses of Theorem 7.2.3 are satisfied, the conclusion of Theorem 7.2.3 (viz., (7.2.15)) which describes an exact rate of growth, is sharp.

7.2.1 Application to regularly varying equations

We consider some cases in which the unknown auxiliary function ϕ (and therefore Γ) in Theorems 7.2.1–7.2.3 can be constructed explicitly in terms of g. First we consider the case where g is in $\text{RV}_\infty(\beta)$ for $\beta \leq 1$ and $g(x)/x \to 0$ as $x \to \infty$.

Theorem 7.2.4. *Let f obey* (0.1.7)*,* (1.1.1)*. Let g obey* (0.1.8) *be non-decreasing and let $\tau > 0$ and $\psi \in C([-\tau, 0]; (0, \infty))$. Suppose $g \in RV_\infty(\beta)$ for some $\beta \leq 1$, $\lim_{x\to\infty} g(x)/x = 0$, and $\lim_{x\to\infty} f(x)/g(x) = 0$. If x is the unique continuous solution of* (0.1.6)*, then*

$$\lim_{t \to \infty} \frac{G(x(t))}{t} = 1 \tag{7.2.16}$$

where G is defined by (1.2.7). This result is proven using Theorems 7.2.1 and 7.2.2; it recovers part (ii) of Theorem 1.2.1. Next we consider the case where g is in $\text{RV}_\infty(1)$ but in which $g(x)/x \to \infty$ as $x \to \infty$, and use Theorem 7.2.3 to determine the growth rate.

Theorem 7.2.5. *Let f obey* (0.1.7)*,* (1.1.1)*. Let g obey* (0.1.8) *and be non-decreasing. Let $\tau > 0$ and ψ obey* (0.1.9)*. Suppose $g \in RV_\infty(1)$, $x \mapsto g(x)/x$ is asymptotic to a non-decreasing function, $\lim_{x\to\infty} g(x)/x = \infty$, and*

$$\lim_{x \to \infty} \frac{f(x)}{x \log(g(x)/x)} = 0. \tag{7.2.17}$$

Define

$$L(x) = \int_1^x \frac{1}{u\log(1+g(u)/u)}\,du, \quad x > 1. \tag{7.2.18}$$

Then the unique continuous solution x of (0.1.6) *obeys*

$$\lim_{t\to\infty} \frac{L(x(t))}{t} = \frac{1}{\tau}. \tag{7.2.19}$$

With a slightly stronger hypothesis on f we can obtain the same conclusion on the growth rate, but by an alternative proof.

Theorem 7.2.6. *Let f obey* (0.1.7), (1.1.1). *Let g obey* (0.1.8) *and be non-decreasing. Let $\tau > 0$ and ψ obey* (0.1.9). *Suppose $g \in RV_\infty(1)$, $x \mapsto g(x)/x$ is asymptotic to a non-decreasing function,* $\lim_{x\to\infty} g(x)/x = \infty$, *and* $\lim_{x\to\infty} f(x)/x = 0$. *Then the unique continuous solution x of* (0.1.6) *obeys*

$$\lim_{t\to\infty} \frac{L(x(t))}{t} = \frac{1}{\tau}. \tag{7.2.20}$$

The case where g grows according to $g \in \text{RV}_\infty(\beta)$ for some $\beta \le 1$ with $g(x)/x$ tending to a zero limit is covered by Theorem 7.2.4. The proof of Theorem 7.2.6 is facilitated by Lemma 1.2.1, which also motivates the choice of ϕ in Theorem 7.2.5. If $g(x)/x$ tends to a finite non-zero limit, we are in the standard linear case, but even this is recovered independently of the standard linear theory by applying Theorems 7.2.1 and 7.2.2.

Theorem 7.2.7. *Let $C > 0$, $\tau > 0$ and suppose that ψ obeys* (0.1.9). *Let x be the unique continuous solution of* (0.1.6) *with $f(x)/x \to 0$ and $g(x)/x \to C$ as $x \to \infty$. Then there is a unique $\lambda > 0$ such that $\lambda = Ce^{-\lambda\tau}$ and x obeys* $\lim_{t\to\infty} \log x(t)/t = \lambda$.

In the case when g has a power-like growth faster which is faster than linear, the rate of growth can be determined by means of Theorem 7.2.3.

Theorem 7.2.8. *Suppose that f obeys* (0.1.7) *and* (1.1.1). *Let g obey* (0.1.8) *be non-decreasing and let $\tau > 0$ and ψ obey* (0.1.9). *Suppose also that there exists $\beta > 1$ such that* $\lim_{x\to\infty} \log g(x)/\log x = \beta$ *and*

$$\lim_{x\to\infty} \frac{f(x)}{x\log x} = 0.$$

Then the unique continuous solution x of (0.1.6) *obeys*

$$\lim_{t\to\infty} \frac{\log\log x(t)}{t} = \frac{\log(\beta)}{\tau}. \tag{7.2.21}$$

7.2.2 Examples

We consider representatives example to which Theorem 7.2.3 can be applied. For simplicity, we set f to be identically zero.

Example 7.2.1. Suppose g obeys (0.1.8)) and is non–decreasing, and there exists $C_1 > 0$ and $\alpha \in (0,1)$ such that $\lim_{x\to\infty} g(x)/(x\exp((\log x)^\alpha)) = C_1$, and $f(x) = 0$ for all $x \geq 0$. Suppose $\tau > 0$ and ψ obeys (0.1.9)). Then the unique continuous solution x of (0.1.6) obeys

$$\lim_{t\to\infty} \frac{\log x(t)}{t^{1/(1-\alpha)}} = \left(\frac{\eta(1-\alpha)}{\tau}\right)^{1/(1-\alpha)} \tag{7.2.22}$$

To see this, we note that g obeys all the properties of Theorem 7.2.5. For $x > e$ let $\phi(x) = x(\log x)^\alpha$. Then $\Gamma(x) = (\log(x)^{1-\alpha} - 1)/(1-\alpha)$. By Theorem 7.2.5 we have $\lim_{t\to\infty} \Gamma(x(t))/t = 1/\tau$, which rearranges to give (7.2.22)

We remark that the results can be applied to equations in which g grows more rapidly than a polynomial function; here again is a representative example, which was considered without supporting calculations in [4].

Example 7.2.2. Suppose g obeys (0.1.8)) and is non–decreasing, and there exists $C_1 > 0$ and $\alpha > 1$ such that $\lim_{x\to\infty} g(x)/\exp((\log x)^\alpha) = C_1$, and $f(x) = 0$ for all $x \geq 0$. Suppose $\tau > 0$ and ψ obeys (0.1.9)). Then the unique continuous solution x of (0.1.6) obeys $\lim_{t\to\infty} \log_3 x(t)/t = \log\alpha/\tau$.

To justify Example 7.2.2, set $\phi(x) = (1+x)\log(1+x)\log_2(1+x)$ for $x > e^e - 1$. With $\psi^* := e^e - 1$, we have $\Gamma_\eta(x) = \log_3(1+x)/\eta$ and with $\lambda = e^{\eta\theta}$,

$$\Gamma_\eta^{-1}(\Gamma_\eta(x) + \theta) = \exp((\log(1+x))^\lambda) - 1.$$

Therefore we have

$$\lim_{x\to\infty} \frac{\phi(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta))}{\exp([\log(1+x)]^\lambda)(\log x)^\lambda \log_2 x} = \lambda.$$

Define $\eta(\epsilon) = (1+\epsilon)\log\alpha/\tau$ and $\mu(\epsilon) = \log\alpha/(\tau(1-\epsilon)^2)$. Then

$$\lim_{x\to\infty} \frac{g(x)}{\phi(\Gamma_{\eta(\epsilon)}^{-1}(\Gamma_{\eta(\epsilon)}(x)+\tau))} = 0$$

and

$$\lim_{x\to\infty} \frac{g(x)}{\phi(\Gamma_{\mu(\epsilon)}^{-1}(\Gamma_{\mu(\epsilon)}(x)+\tau(1-\epsilon)))} = \infty.$$

Since $\eta(\epsilon), \mu(\epsilon) \to \log \alpha/\tau$ as $\epsilon \to \infty$, from Theorem 7.2.3 we have $\lim_{t\to\infty} \Gamma(x(t))/t = \log \alpha/\tau$, from which the result follows.

7.3 Proofs

In this section, we give the proofs of the main results from Section 7.2, with the exception of Theorem 7.2.6, whose proof is strongly based on that of Theorem 8.2.3. The proofs of these two results, along with Theorem 8.2.4, are given in Section 8.3 of the following chapter.

Proof of Theorem 7.2.1 By (7.2.6) for every $\epsilon \in (0,1)$ there exists $x_2(\epsilon) > 0$ such that for $x > x_2(\epsilon)$ we have

$$g(x) < (\bar{\eta}_\epsilon + \epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(x) + \tau)) \leq (\bar{\eta} + \epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(x) + \tau)),$$

where the last inequality is a consequence of (7.2.7). Since $\bar{\eta} < \eta = \lim_{\epsilon\to 0^+} \eta(\epsilon)$, there exists $\epsilon' \in (0,1)$ such that for $\epsilon < \epsilon'$, we have $\eta(\epsilon) > \bar{\eta} + \epsilon$. Thus for all $\epsilon < \epsilon' < 1$ we have

$$g(x) < \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(x) + \tau)), \quad x > x_2(\epsilon). \tag{7.3.1}$$

By (7.2.5) for every $\epsilon \in (0,1)$ there exists an $x_1(\epsilon) > 0$ such that

$$f(x) \leq \epsilon\eta(\epsilon)\phi(x), \quad x > x_1(\epsilon). \tag{7.3.2}$$

Define

$$c(\epsilon) = \Gamma_{\eta(\epsilon)}(\psi^* + x_1(\epsilon) + x_2(\epsilon)) + (1+\epsilon)\tau, \tag{7.3.3}$$

and define also

$$x_\epsilon(t) = \Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)t + c(\epsilon)), \quad t \geq -\tau. \tag{7.3.4}$$

This function is well–defined since $c(\epsilon) > \Gamma_{\eta(\epsilon)}(\psi^*) + (1+\epsilon)\tau$, so $c(\epsilon) - (1+\epsilon)\tau > \Gamma_{\eta(\epsilon)}(\psi^*)$, or $x_\epsilon(t) > \psi^*$ for all $t \in [-\tau, 0]$. Since $c(\epsilon) > \Gamma_{\eta(\epsilon)}(x_1(\epsilon)) + (1+\epsilon)\tau$ and $\Gamma_{\eta(\epsilon)}$ is increasing, $\Gamma^{-1}_{\eta(\epsilon)}(c(\epsilon) - (1+\epsilon)\tau) > x_1(\epsilon)$, so $x_\epsilon(t) > x_1(\epsilon)$ for all $t \geq -\tau$. Therefore by (7.3.2), $f(x_\epsilon(t)) \leq \epsilon\eta(\epsilon)\phi(x_\epsilon(t))$. Also for $t \geq 0$, we have

$$\begin{aligned} g(x_\epsilon(t-\tau)) &= g(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)(t-\tau) + c(\epsilon)) = g(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)t - \tau - \epsilon\tau + c(\epsilon))) \\ &< g(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)t - \tau + c(\epsilon))). \end{aligned}$$

Now, because $c(\epsilon) > \Gamma_{\eta(\epsilon)}(x_2(\epsilon)) + \tau$, we have that the argument of g on the right-hand side exceeds $x_2(\epsilon)$ for all $t \geq 0$. Therefore by (7.3.1), we have

$$\begin{aligned}
g(x_\epsilon(t-\tau)) &< g(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)t - \tau + c(\epsilon))) \\
&< \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)t - \tau + c(\epsilon))) + \tau)) \\
&= \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)t - \tau + c(\epsilon)) + \tau) \\
&= \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)t + c(\epsilon))) \\
&= \eta(\epsilon)\phi(x_\epsilon(t)).
\end{aligned}$$

Hence for $t \geq 0$

$$f(x_\epsilon(t)) + g(x_\epsilon(t-\tau)) < (1+\epsilon)\eta(\epsilon)\phi(x_\epsilon(t)). \tag{7.3.5}$$

Now for $t > 0$, $\Gamma_{\eta(\epsilon)}(x_\epsilon(t)) = (1+\epsilon)t + c(\epsilon)$, so $\Gamma'_{\eta(\epsilon)}(x_\epsilon(t))x'_\epsilon(t) = (1+\epsilon)$, or $x'_\epsilon(t) = (1+\epsilon)\eta(\epsilon)\phi(x_\epsilon(t))$. Hence

$$x'_\epsilon(t) = (1+\epsilon)\eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)t + c(\epsilon))), \quad t > 0. \tag{7.3.6}$$

Thus by (7.3.5) and (7.3.6) for $t > 0$ we have $x'_\epsilon(t) > f(x_\epsilon(t)) + g(x_\epsilon(t-\tau))$.

Now as $x_\epsilon(t) > \psi^*$, we have $x_\epsilon(t) > x(t)$ for $t \in [-\tau, 0]$ and $x'_\epsilon(t) > f(x_\epsilon(t)) + g(x_\epsilon(t-\tau))$ for $t \geq 0$. Suppose that there is a $t_0 > 0$ such that $x_\epsilon(t) > x(t)$ for $t \in [-\tau, t_0)$ $x_\epsilon(t_0) = x(t_0)$. Therefore $x'_\epsilon(t_0) \leq x'(t_0)$. Then as g is non–decreasing,

$$\begin{aligned}
x'_\epsilon(t_0) &\leq x'(t_0) = f(x(t_0)) + g(x(t_0 - \tau)) \\
&= f(x_\epsilon(t_0)) + g(x(t_0-\tau)) \leq f(x_\epsilon(t_0)) + g(x_\epsilon(t_0 - \tau)) \\
&< x'_\epsilon(t_0),
\end{aligned}$$

a contradiction. Thus $x_\epsilon(t) > x(t)$ for all $t \geq -\tau$. Hence $\Gamma_{\eta(\epsilon)}(x(t)) < \Gamma_{\eta(\epsilon)}(x_\epsilon(t))$ for all $t \geq -\tau$. Hence

$$\Gamma_{\eta(\epsilon)}(x(t)) < \Gamma_{\eta(\epsilon)}(x_\epsilon(t)) = (1+\epsilon)t + c(\epsilon), \quad t \geq -\tau.$$

But $\Gamma(x(t)) = \eta(\epsilon)\Gamma_{\eta(\epsilon)}(x(t)) < (1+\epsilon)\eta(\epsilon)t + \eta(\epsilon)c(\epsilon)$. Therefore

$$\limsup_{t\to\infty} \frac{\Gamma(x(t))}{t} \leq (1+\epsilon)\eta(\epsilon).$$

Since $\epsilon > 0$ is arbitrary, and $\eta(\epsilon) \to \eta$ as $\epsilon \to 0$, we have (7.2.8).

Proof of Theorem 7.2.2 Suppose first that $\bar{\mu}_\epsilon$ is finite. Then by (7.2.11) for every $\epsilon \in (0,1)$ there exists $x_3(\epsilon) > 0$ such that for $x > x_3(\epsilon)$

$$\begin{aligned} g(x) &> \bar{\mu}_\epsilon(1-\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x)+\tau(1-\epsilon))) \geq \bar{\mu}(1-\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x)+\tau(1-\epsilon))) \\ &> \mu(\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x)+\tau(1-\epsilon))), \end{aligned}$$

where the penultimate inequality is a consequence of (7.2.12), and the last inequality holds for all $\epsilon < \epsilon'$, because for such ϵ we have $\mu(\epsilon) < (1-\epsilon)\bar{\mu}$. This holds for the following reason.

By (7.2.10), there exists $\epsilon_1 \in (0,1)$ such that $\epsilon \in (0,\epsilon_1)$ implies $-\epsilon < \mu(\epsilon)-\mu < \mu\epsilon$. Since $\mu < \bar{\mu}$, it follows that there exists $\epsilon_2 \in (0,1)$ such that $\epsilon < \epsilon_2$ implies $\bar{\mu} > (1+\epsilon)\mu/(1-\epsilon)$. Hence for all $\epsilon < \epsilon' := \epsilon_1 \wedge \epsilon_2$, we have $\mu(\epsilon) < \mu(1+\epsilon) < (1-\epsilon)\bar{\mu}$.

Thus for all $0 < \epsilon < \epsilon' < 1$, and $x > x_3(\epsilon)$ we have

$$g(x) > \mu(\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x)+\tau(1-\epsilon))), \quad x > x_3(\epsilon). \tag{7.3.7}$$

When $\bar{\mu}_\epsilon = +\infty$, because $\mu(\epsilon)$ is finite, (7.3.7) is trivial.

Define $y_3(\epsilon) = \Gamma_{\mu(\epsilon)}(x_3(\epsilon)) + \tau(1-\epsilon)$. Then for $y > y_3(\epsilon)$, if we define $x = \Gamma^{-1}_{\mu(\epsilon)}(y - \tau(1-\epsilon))$, for $x > x_3(\epsilon)$ we have that $y > y_3(\epsilon)$. Thus by (7.3.7)

$$g(\Gamma^{-1}_{\mu(\epsilon)}(y-\tau(1-\epsilon))) > \mu(\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}(y)), \quad y > y_3(\epsilon). \tag{7.3.8}$$

Next let $T_0(\epsilon) = \inf\{t > 0 : x(t) = x_3(\epsilon)\}$ and define $T_1 > T_0$ such that $(1-\epsilon)\tau + \Gamma_{\mu(\epsilon)}(x(T_0(\epsilon))) = \Gamma_{\mu(\epsilon)}(x(T_1(\epsilon)))$, or $(1-\epsilon)\tau + \Gamma_{\mu(\epsilon)}(x_3(\epsilon))) = \Gamma_{\mu(\epsilon)}(x(T_1(\epsilon)))$. Define

$$x_\epsilon(t) = \Gamma^{-1}_{\mu(\epsilon)}((1-\epsilon)(t-T_1(\epsilon)) + \Gamma_{\mu(\epsilon)}(x(T_0(\epsilon))), \quad t \geq -\tau. \tag{7.3.9}$$

Therefore for $t \geq T_1(\epsilon)+\tau$ we have

$$\begin{aligned} (1-\epsilon)(t-T_1(\epsilon)) + \Gamma_{\mu(\epsilon)}(x(T_0(\epsilon))) &\geq (1-\epsilon)\tau + \Gamma_{\mu(\epsilon)}(x(T_0(\epsilon))) = \Gamma_{\mu(\epsilon)}(x(T_1(\epsilon))) \\ &= (1-\epsilon)\tau + \Gamma_{\mu(\epsilon)}(x_3(\epsilon))) = y_3(\epsilon). \end{aligned}$$

Setting $y = (1-\epsilon)(t-T_1(\epsilon)) + \Gamma_{\mu(\epsilon)}(x(T_0(\epsilon)))$ in (7.3.8) yields

$$\begin{aligned} & g(\Gamma^{-1}_{\mu(\epsilon)}((1-\epsilon)(t-T_1-\tau) + \Gamma_{\mu(\epsilon)}(x(T_0(\epsilon))))) \\ & \qquad > \mu(\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}((1-\epsilon)(t-T_1(\epsilon)) + \Gamma_{\mu(\epsilon)}(x(T_0(\epsilon))))) \end{aligned}$$

for $t \geq T_1(\epsilon) + \tau$. By (7.3.9) we have

$$g(x_\epsilon(t-\tau)) > \mu(\epsilon)\phi(x_\epsilon(t)), \quad t \geq T_1(\epsilon) + \tau. \tag{7.3.10}$$

Therefore by (7.3.10) for $t > T_1(\epsilon) + \tau$, $\Gamma_{\mu(\epsilon)}(x_\epsilon(t)) = (1-\epsilon)(t - T_1(\epsilon)) + \Gamma_{\mu(\epsilon)}(x(T_0(\epsilon)))$, we have

$$\begin{aligned} x_\epsilon'(t) = (1-\epsilon)\frac{1}{\Gamma'_{\mu(\epsilon)}(x_\epsilon(t))} &= (1-\epsilon)\mu(\epsilon)\phi(x_\epsilon(t)) \\ &< \mu(\epsilon)\phi(x_\epsilon(t)) < g(x_\epsilon(t-\tau)) \leq f(x_\epsilon(t)) + g(x_\epsilon(t-\tau)). \end{aligned}$$

Now for $t \in [T_1, T_1 + \tau]$ we have

$$\begin{aligned} x_\epsilon(t) \leq x_\epsilon(T_1 + \tau) &= \Gamma^{-1}_{\mu(\epsilon)}((1-\epsilon)\tau + \Gamma_{\mu(\epsilon)}(x(T_0(\epsilon)))) \\ &= \Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x(T_1(\epsilon)))) = x(T_1(\epsilon)) < x(t), \end{aligned}$$

where we used at the last step the fact that x is increasing on $[T_1(\epsilon), T_1(\epsilon)+\tau] \subset [\tau, \infty)$. Finally $x_\epsilon(T_1(\epsilon)) < x_\epsilon(T_1(\epsilon)+\tau) = x(T_1(\epsilon))$. Therefore we have $x_\epsilon(t) < x(t)$ for $t \in [T_1(\epsilon), T_1(\epsilon)+\tau]$, and also $x_\epsilon'(t) < f(x_\epsilon(t)) + g(x_\epsilon(t-\tau))$ for $t \geq T_1 + \tau$.

Suppose that there is a $t_1 > T_1(\epsilon) + \tau$ such that $x_\epsilon(t) < x(t)$ for $t \in [T_1(\epsilon), t_1)$ and $x_\epsilon(t_1) = x(t_1)$. Therefore $x_\epsilon'(t_1) \geq x'(t_1)$. Then as g is non–decreasing,

$$\begin{aligned} x_\epsilon'(t_1) \geq x'(t_1) = f(x(t_1)) + g(x(t_1-\tau)) &= f(x_\epsilon(t_1)) + g(x(t_1-\tau)) \\ &\geq f(x_\epsilon(t_1)) + g(x_\epsilon(t_1-\tau)) > x_\epsilon'(t_1), \end{aligned}$$

a contradiction. Thus $x_\epsilon(t) < x(t)$ for all $t \geq T_1$. Hence $\Gamma_{\mu(\epsilon)}(x(t)) > \Gamma_{\mu(\epsilon)}(x_\epsilon(t))$ for all $t \geq T_1(\epsilon)$. Hence

$$\Gamma_{\mu(\epsilon)}(x(t)) > \Gamma_{\mu(\epsilon)}(x_\epsilon(t)) = (1-\epsilon)(t - T_1(\epsilon)) + \Gamma_{\mu(\epsilon)}(x(T_0(\epsilon))), \quad t \geq T_1(\epsilon).$$

But $\Gamma(x(t)) = \mu(\epsilon)\Gamma_{\mu(\epsilon)}(x(t)) > (1-\epsilon)\mu(\epsilon)t + \mu(\epsilon)\Gamma_{\mu(\epsilon)}(x(T_0(\epsilon)))$. Therefore

$$\limsup_{t\to\infty} \frac{\Gamma(x(t))}{t} \geq (1-\epsilon)\mu(\epsilon).$$

Since $\epsilon > 0$ is arbitrary, and $\mu(\epsilon) \to \mu$ as $\epsilon \to 0$, we have (7.2.13).

Proof of Theorem 7.2.4 Suppose that $\phi(x) = g(x)$ for $x > 0$. Thus $\Gamma_\eta(x) = \eta^{-1}\int_{\psi^*}^x du/g(u)$. Let $z(t) = \Gamma_\eta^{-1}(t)$ for $t \geq 0$. Then $z'(t) = \eta g(z(t))$ for $t > 0$ with $z(0) = \psi^*$. Thus $z'(t)/z(t) \to 0$ as $t \to \infty$. Therefore

$$\log\left(\frac{z(t)}{z(t-\theta)}\right) = \int_{t-\theta}^t \frac{z'(s)}{z(s)}\,ds \to 0 \quad \text{as } t \to \infty,$$

so $\lim_{t\to\infty} z(t-\theta)/z(t) = 1$ for any $\theta \in \mathbb{R}$. Since $g \in \mathrm{RV}_\infty(\beta)$, we have

$$\lim_{t\to\infty} g(z(t-\theta))/g(z(t)) = 1.$$

Hence $\lim_{t\to\infty} g(\Gamma_\eta^{-1}(t-\theta))/g(\Gamma_\eta^{-1}(t)) = 1$. Since $\Gamma_\eta^{-1}(t) \to \infty$ as $t \to \infty$, we have

$$\lim_{x\to\infty} \frac{g(x)}{\phi(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta))} = \lim_{x\to\infty} \frac{g(x)}{g(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta))} = 1. \tag{7.3.11}$$

Since this holds for every $\eta > 0$ and $\theta \in \mathbb{R}$ it follows that (7.2.6) and (7.2.11) hold with $\bar{\eta}_\epsilon = \bar{\mu}_\epsilon = 1$. Let $\rho \in (0,1)$. Define $\mu(\epsilon) = 1-\rho$ and $\eta(\epsilon) = 1+\rho$. Then with $\eta = 1+\rho$ and $\mu = 1-\rho$, (7.2.4), (7.2.10), (7.2.7) and (7.2.12) hold. To prove (7.2.5), we note that

$$\lim_{x\to\infty} \frac{f(x)}{\phi(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Since all the hypotheses of Theorems 7.2.1 and 7.2.2 hold, we have

$$\limsup_{t\to\infty} \frac{\Gamma(x(t))}{t} \leq 1+\rho, \quad \text{and} \quad \liminf_{t\to\infty} \frac{\Gamma(x(t))}{t} \geq 1-\rho.$$

Letting $\rho \to 0$, we have $\lim_{t\to\infty} \Gamma(x(t))/t = 1$, whence the result.

Proof of Theorem 7.2.5 Before starting, we obtain a preparatory result for use in the proof.

Lemma 7.3.1. *Let $c > 0$ and suppose $z \in C^1((0,\infty);(0,\infty))$ obeys*

$$\lim_{t\to\infty} \frac{z'(t)}{z(t)} = 0.$$

Then

$$\max_{r\in[0,c]} \left|\frac{z(t-r)}{z(t)} - 1\right| = 0. \tag{7.3.12}$$

Proof. Since $z'(t)/z(t) \to 0$ as $t \to \infty$, for every $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that

$$-\epsilon < \frac{z'(t)}{z(t)} < \epsilon, \quad t > T(\epsilon).$$

Let $r \in [0,c]$ and suppose $t \geq T(\epsilon) + c$. Then

$$-\epsilon r < \int_{t-r}^{r} \frac{z'(s)}{z(s)}\,ds < \epsilon r,$$

and so

$$e^{-\epsilon r} \leq \frac{z(t-r)}{z(t)} \leq e^{\epsilon r}.$$

Therefore $-(1-e^{-\epsilon r}) \leq z(t-r)/z(t) - 1 \leq e^{\epsilon r} - 1$ for $t \geq T(\epsilon) + c$. Thus

$$\begin{aligned}\max_{r\in[0,c]} \left| \frac{z(t-r)}{z(t)} - 1 \right| &\leq \max_{r\in[0,c]} (e^{\epsilon r} - 1, 1 - e^{-\epsilon r}) \\ &= \max(e^{\epsilon c} - 1, 1 - e^{-\epsilon c}).\end{aligned}$$

Letting $\epsilon \to 0$ obtains (7.3.12). □

Now to prove (7.2.19). Since $g \in \mathrm{RV}_\infty(1)$, it follows that there exists an increasing and continuously differentiable function $\delta : [\psi^*, \infty) \to (0,\infty)$ with $\delta(\psi^*) > e\psi^*$ such that

$$\lim_{x\to\infty} \frac{\delta(x)}{g(x)} = 1, \quad \text{and} \quad \lim_{x\to\infty} \frac{x\delta'(x)}{\delta(x)} = 1.$$

Define $\phi(x) = x \log(\delta(x)/x)$ for $x \geq \psi^*$. Recall $\Gamma(x) = \int_{\psi^*}^{x} du/\phi(u)$ for $x \geq \psi^*$. Since $(g(x)/x)/(\delta(x)/x) \to 1$ as $x \to \infty$, we have $\log(g(x)/x)/\log(\delta(x)/x) \to 1$ as $x \to \infty$. Therefore by L'Hôpital's rule, we have

$$\lim_{x\to\infty} \frac{\Gamma(x)}{L(x)} = 1.$$

Define $\Gamma_\eta(x) = \Gamma(x)/\eta$ and $\delta_1(x) = \delta(x)/x$ for $x \geq \psi^*$. Since $x\delta'(x)/\delta(x) \to 1$ as $x \to \infty$, we have that δ_1 is continuously differentiable and $x\delta_1'(x)/\delta_1(x) \to 0$ as $x \to \infty$. Define $y(t) = \Gamma_\eta^{-1}(t)$ for $t \geq 0$ and $u(t) = \log \delta_1(y(t))$. Then

$$y'(t) = \eta\phi(y(t)) = \eta y(t) \log \delta_1(y(t)) = \eta y(t) u(t).$$

Moreover since $\Gamma_\eta(x) \to \infty$ as $x \to \infty$, we have that $y(t) \to \infty$ as $t \to \infty$. Thus

$$\begin{aligned}\lim_{x\to\infty} \frac{g(x)}{\phi(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta))} &= \lim_{x\to\infty} \frac{\delta(x)}{\phi(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta))} \\ &= \lim_{t\to\infty} \frac{\delta(\Gamma_\eta^{-1}(t-\theta))}{\Gamma_\eta^{-1}(t)\log(\delta(\Gamma_\eta^{-1}(t))/\Gamma_\eta^{-1}(t))},\end{aligned}$$

and therefore we have

$$\lim_{x\to\infty} \frac{g(x)}{\phi(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta))} = \lim_{t\to\infty} \frac{y(t-\theta)\delta_1(y(t-\theta))}{y(t)\log\delta_1(y(t))}.$$

Since $\log(y(t)/y(t-\theta)) = \int_{t-\theta}^t y'(s)/y(s)\,ds = \int_{t-\theta}^t \eta u(s)\,ds$. Hence

$$\begin{aligned}&\log\left(\lim_{x\to\infty}\frac{g(x)}{\phi(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta))}\right)\\ &= \lim_{t\to\infty}\{-\log(y(t)/y(t-\theta))+u(t-\theta)-\log u(t)\}\\ &= \lim_{t\to\infty}u(t)\left\{-\eta\frac{1}{u(t)}\int_{t-\theta}^t u(s)\,ds+\frac{u(t-\theta)}{u(t)}-\frac{\log u(t)}{u(t)}\right\}.\end{aligned}$$

Since δ_1, y are continuously differentiable, so is u, and we have

$$u'(t) = \delta_1'(y(t))y'(t)/\delta_1(y(t)) = \eta u(t)\cdot y(t)\delta_1'(y(t))/\delta_1(y(t)).$$

Since $x\delta_1'(x)/\delta_1(x) \to 0$ as $x\to\infty$ and $y(t)\to\infty$ as $t\to\infty$, we have $u'(t)/u(t)\to 0$ as $t\to\infty$. Now note

$$\int_{t-\theta}^t u(s)\,ds/u(t)-\theta = \int_0^\theta u(t-r)/u(t)\,dr-\theta = \int_0^\theta (u(t-r)/u(t)-1)\;dr.$$

By Lemma 7.3.1,

$$\limsup_{t\to\infty}\left|\int_{t-\theta}^t u(s)\,ds/u(t)-\theta\right| \le \limsup_{t\to\infty}\max_{r\in[0,\theta]}|u(t-r)/u(t)-1| = 0.$$

Thus $\int_{t-\theta}^t u(s)\,ds/u(t)\to\theta$ and moreover $u(t-\theta)/u(t)\to 1$ as $t\to\infty$. Also we have $u(t)\to\infty$ as $t\to\infty$ and so

$$\lim_{t\to\infty}\left\{-\eta\frac{1}{u(t)}\int_{t-\theta}^t u(s)\,ds+\frac{u(t-\theta)}{u(t)}-\frac{\log u(t)}{u(t)}\right\} = 1-\eta\theta.$$

Therefore we have

$$\log\left(\lim_{x\to\infty}\frac{g(x)}{\phi(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta))}\right) = \begin{cases}-\infty & \text{if } 1-\eta\theta<0\\ +\infty & \text{if } 1-\eta\theta>0.\end{cases}$$

Therefore, with $\eta(\epsilon) = (1+\epsilon)/\tau$ and $\mu(\epsilon) = (1-\epsilon)/\tau$, we have

$$\lim_{x\to\infty}\frac{g(x)}{\phi(\Gamma_{\eta(\epsilon)}^{-1}(\Gamma_{\eta(\epsilon)}(x)+\tau))} = 0, \quad \lim_{x\to\infty}\frac{g(x)}{\phi(\Gamma_{\mu(\epsilon)}^{-1}(\Gamma_{\mu(\epsilon)}(x)+\tau(1-\epsilon)))} = \infty.$$

Since $\mu(\epsilon),\eta(\epsilon)\to 1/\tau$ as $\epsilon\to 0$, and we have $\bar\eta_\epsilon = 0 =: \bar\eta < 1/\tau$ and $\bar\mu_\epsilon = +\infty =: \bar\mu > 1/\tau$.

Next, note that (7.2.17) implies

$$\lim_{x\to\infty}\frac{f(x)}{\phi(x)} = \lim_{x\to\infty}\frac{f(x)}{x\log(\delta(x)/x)} = \lim_{x\to\infty}\frac{f(x)}{x\log(g(x)/x)} = 0.$$

Therefore by Theorem 7.2.3, we have $\lim_{t\to\infty}\Gamma(x(t))/t = 1/\tau$, and due to the fact that $\lim_{x\to\infty}L(x)/\Gamma(x) = 1$, we get $\lim_{t\to\infty}L(x(t))/t = 1/\tau$, as required.

Proof of Theorem 7.2.7 Set $\phi(x) = x$ for $x \geq \psi^*$. Then $\Gamma_\eta(x) = \eta^{-1}\log(x/\psi^*)$, $\Gamma_\eta^{-1}(x) = \psi^* e^{\eta x}$, and $\phi(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta)) = xe^{\eta\theta}$. Thus

$$\lim_{x\to\infty} \frac{g(x)}{\phi(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta))} = Ce^{-\eta\theta}.$$

Define $c(\nu) := \nu - Ce^{-\nu\tau}$. Then c is increasing on $[0,\infty)$ and there is a unique $\lambda > 0$ such that $c(\lambda) = 0$, or $\lambda = Ce^{-\lambda\tau}$. Let $\sigma \in \mathbb{R}$ and $\lambda_\sigma := \lambda(1+\sigma)$. For $\sigma > 0$, $c(\lambda_\sigma) > 0$ or $\lambda_\sigma > Ce^{-\lambda_\sigma\tau}$. Similarly, $\lambda_{-\sigma} < Ce^{-\lambda_{-\sigma}\tau}$. Define $\eta(\epsilon) = \lambda_\sigma(1+\epsilon)$. Then $\eta(\epsilon) \to \lambda_\sigma =: \eta$ as $\epsilon \to 0$. Also

$$\lim_{x\to\infty} \frac{g(x)}{\phi(\Gamma_{\eta(\epsilon)}^{-1}(\Gamma_{\eta(\epsilon)}(x)+\tau))} = Ce^{-\lambda_\sigma(1+\epsilon)\tau} =: \bar{\eta}_\epsilon.$$

Then $\sup_{\epsilon\in(0,1)} \bar{\eta}_\epsilon = Ce^{-\lambda_\sigma\tau} =: \bar{\eta}$. But $\bar{\eta} = Ce^{-\lambda_\sigma\tau} < \lambda_\sigma = \eta$. Finally, $f(x)/\phi(x) = f(x)/x \to 0$ as $x \to \infty$, and so by Theorem 7.2.1,

$$\limsup_{t\to\infty} \frac{\Gamma(x(t))}{t} \leq \lambda_\sigma,$$

or

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} \leq \lambda(1+\sigma).$$

Letting $\sigma \downarrow 0$ yields $\limsup_{t\to\infty} \log x(t)/t \leq \lambda$.

Define $\mu(\epsilon) = \lambda_{-\sigma}(1-\epsilon)$. Then $\lim_{\epsilon\to 0}\mu(\epsilon) = \lambda_{-\sigma} =: \mu$. Also

$$\lim_{x\to\infty} \frac{g(x)}{\phi(\Gamma_{\mu(\epsilon)}^{-1}(\Gamma_{\mu(\epsilon)}(x)+\tau(1-\epsilon)))} = Ce^{-\lambda_{-\sigma}(1-\epsilon)\tau} =: \bar{\mu}_\epsilon.$$

Then $\inf_{\epsilon\in(0,1)} \bar{\mu}_\epsilon = Ce^{-\lambda_{-\sigma}\tau} =: \bar{\mu}$. But $\bar{\mu} = Ce^{-\lambda_{-\sigma}\tau} > \lambda_{-\sigma} = \mu$. Thus by Theorem 7.2.2,

$$\liminf_{t\to\infty} \frac{\Gamma(x(t))}{t} \geq \lambda_{-\sigma},$$

or

$$\liminf_{t\to\infty} \frac{\log x(t)}{t} \geq \lambda(1-\sigma).$$

Letting $\sigma \downarrow 0$ yields $\liminf_{t\to\infty} \log x(t)/t \geq \lambda$, whence the result.

Proof of Theorem 7.2.8 Define $\phi(x) = (1+x)\log(1+x)$ for $x \geq \psi^*$. Hence for $\eta > 0$ we have

$$\Gamma_\eta(x) = \frac{1}{\eta}\log\left(\frac{\log(1+x)}{\log(1+\psi^*)}\right), \quad \Gamma_\eta^{-1}(x) = \exp\left(\log(1+\psi^*)e^{\eta x}\right) - 1.$$

Thus $\phi(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta)) = e^{\eta\theta}(1+x)^{e^{\eta\theta}}\log(1+x)$. Also $\Gamma_\eta^{-1}(\Gamma_\eta(x)-\tau) = (1+x)^{e^{-\eta\tau}}-1$. Therefore

$$\begin{aligned}
\lim_{x\to\infty}\frac{g(x)}{\phi(\Gamma_\eta^{-1}(\Gamma_\eta(x)+\theta))} &= e^{-\eta\theta}\lim_{x\to\infty}\frac{g(x)}{(1+x)^{e^{\eta\theta}}\log(1+x)},\\
\lim_{x\to\infty}\frac{f(x)}{g(\Gamma_\eta^{-1}(\Gamma_\eta(x)-\tau))} &= \lim_{x\to\infty}\frac{f(x)}{g((1+x)^{e^{-\eta\tau}}-1)}.
\end{aligned}$$

Next, $\eta(\epsilon) := \epsilon + \log(\beta)/\tau$. Then $\lim_{\epsilon\to 0}\eta(\epsilon) = \log(\beta)/\tau =: \eta$, and so

$$\lim_{x\to\infty}\frac{g(x)}{\phi(\Gamma_{\eta(\epsilon)}^{-1}(\Gamma_{\eta(\epsilon)}(x)+\tau))} = 0.$$

Therefore $\bar{\eta}_\epsilon = 0$, so $\bar{\eta} = 0 < \log(\beta)/\tau = \eta$. Next, as $f(x)/(x\log x) \to 0$ as $x\to\infty$, we have

$$\lim_{x\to\infty}\frac{f(x)}{\phi(x)} = \frac{f(x)}{(1+x)\log(1+x)} = 0.$$

By Theorem 7.2.1,

$$\limsup_{t\to\infty}\Gamma(x(t))/t \le \eta,$$

or equivalently $\limsup_{t\to\infty}\log\log x(t)/t \le \log(\beta)/\tau$. We now obtain a lower bound. Define $\mu(\epsilon) = \log(\beta)/\tau$ for $\epsilon > 0$. Then

$$\lim_{x\to\infty}\frac{g(x)}{\phi(\Gamma_{\mu(\epsilon)}^{-1}(\Gamma_{\mu(\epsilon)}(x)+\tau(1-\epsilon)))} = \beta^{-(1-\epsilon)}\lim_{x\to\infty}\frac{g(x)}{(1+x)^{\beta^{1-\epsilon}}\log(1+x)}.$$

Therefore

$$\lim_{x\to\infty}\frac{g(x)}{\phi(\Gamma_{\mu(\epsilon)}^{-1}(\Gamma_{\mu(\epsilon)}(x)+\tau(1-\epsilon)))} = \infty,$$

so $\bar{\mu}_\epsilon = +\infty = \bar{\mu} > \mu = \log(\beta)/\tau$. By Theorem 7.2.2, $\liminf_{t\to\infty}\Gamma(x(t))/t \ge \mu$, or $\liminf_{t\to\infty}\log\log x(t)/t \ge \log\beta/\tau$, which proves (7.2.21).

Chapter 8

Uniform Numerics and Delay-Dominated Equations

8.1 Introduction

In this chapter, we look to obtain discrete–time results analogous to those that were demonstrated in Chapter 7 for the delay-dominated continuous–time equation (0.1.6). We showed in Chapters 1, 4 and 6 that for instantaneously-dominated equations, it is the nonlinearity of f that determines a suitable numerical method for replicating the growth rate. If f grows sublinearly, a uniform Euler scheme will recover the correct rate, whereas if f grows superlinearly we must apply a state–dependent discretisation to recover the instantaneously-dominated rate of growth.

For delay-dominated equations however, a uniform method will determine the correct rate irrespective of the degree of nonlinearity in the delay coefficient. This is very much in contrast to the instantaneously-dominated case. As a consequence of this, two different equations which have the same rates of growth may require different numerical methods to replicate this rate. We illustrate this in Section 8.2.2 using a representative example.

Section 8.2 shows that the rate of growth of the delay-dominated equation (0.1.6) is preserved under a uniform discretisation and the application of these results for regularly varying equations are demonstrated in Section 8.2.1.

8.2 Preservation of Growth Rates under a Uniform Discretisation

Let $N \in \mathbb{N}$, and suppose that $h = \tau/N$. Consider the discretisation of (0.1.6) according to (1.1.2) and its related continuous–time extension given by (1.2.13).

Theorem 8.2.1. *Suppose that f obeys* (0.1.7) *and* (1.1.1)*. Let g be non-decreasing and obey* (0.1.8) *and let $\tau > 0$ and ψ obey* (0.1.9)*. Suppose that there exists a continuous* *function ϕ such that* Γ, Γ_c *are defined by* (7.2.1) *and* (7.2.3) *respectively, and that* Γ *obeys* (7.2.2). *Suppose also that* (7.2.4) *holds and* f *obeys* (7.2.5), *and that* g *and* ϕ *obey* (7.2.6) *where* $\bar{\eta}_\varepsilon$ *obeys* (7.2.7). *Suppose finally that* ϕ *and* f *are non-decreasing. If* $x_n(h)$ *is the unique solution of* (1.1.2), *then it obeys*

$$\limsup_{n\to\infty} \frac{\Gamma(x_n(h))}{nh} \le \eta. \tag{8.2.1}$$

Theorem 8.2.2. *Suppose that* f *obeys* (0.1.7) *and* (1.1.1). *Let* g *be non-decreasing and obey* (0.1.8) *and let* $\tau > 0$ *and* $\psi \in C([-\tau,0];(0,\infty))$. *Suppose that there exists a continuous function* ϕ *such that* Γ, Γ_c *are defined by* (7.2.1) *and* (7.2.3) *respectively, and* Γ *obeys* (7.2.2). *Suppose also that* (7.2.10) *holds and that* g *and* ϕ *obey*

$$\liminf_{x\to\infty} \frac{g(x)}{\phi(\Gamma^{-1}_{\mu(\varepsilon)}(\Gamma_{\mu(\varepsilon)}(x)+(\tau+h)(1-\epsilon)))} = \bar{\mu}_\varepsilon \in (0,\infty] \quad \text{for every } \varepsilon \in (0,1), \tag{8.2.2}$$

where (7.2.12) *also holds. If* $x_n(h)$ *is the unique solution of* (1.1.2), *then*

$$\liminf_{n\to\infty} \frac{\Gamma(x_n(h))}{nh} \ge \mu. \tag{8.2.3}$$

8.2.1 Preservation of growth rate for regularly varying equations

In Chapter 1, it was shown that the uniform Euler scheme (1.1.2) and the continuous time extension (1.2.13) preserve the rate of growth of the underlying continuous equation (0.1.6) in the case when g is in $\text{RV}_\infty(\beta)$ for $\beta \le 1$, and g is sublinear (cf. Theorems 1.2.2 and 1.2.3)

We now demonstrate that for superlinear equations the essential growth rate is preserved for all $h > 0$, and that the exact rate of growth is recovered in the limit as $h \to 0^+$, in a sense now made precise. We first consider the discrete analogue of Theorem 7.2.6.

Theorem 8.2.3. *Let* f *obey* (0.1.7), (1.1.1). *Let* g *obey* (0.1.8). *Let* $\tau > 0$ *and* ψ *obey* (0.1.9). *Suppose* $g \in RV_\infty(1)$, $x \mapsto g(x)/x$ *is asymptotic to a non-decreasing function,* $\lim_{x\to\infty} g(x)/x = \infty$, *and* $\lim_{x\to\infty} f(x)/x = 0$. *If* L *is defined by* (7.2.18), *then the unique solution* $x_n(h)$ *of* (1.1.2) *obeys*

$$\lim_{n\to\infty} \frac{L(x_n(h))}{nh} = \frac{1}{\tau+h}. \tag{8.2.4}$$

Moreover, if $\bar{x}_h$ *is the linear interpolant given by* (1.2.13), *then*

$$\lim_{t\to\infty} \frac{L(\bar{x}_h(t))}{t} = \frac{1}{\tau+h}. \tag{8.2.5}$$

By comparing (7.2.19) and (8.2.5), it can be seen that the essential growth rate is recovered by the linear interpolant for all $h > 0$, and the exact rate is recovered in the limit as $h \to 0^+$.

The rate of growth is also recovered in the same manner in the case when g grows polynomially at a superlinear rate, as confirmed by the following discrete analogue of Theorem 7.2.8.

Theorem 8.2.4. *Let f obey* (0.1.7), (1.1.1)*. Let g obey* (0.1.8)*. Let $\tau > 0$ and ψ obey* (0.1.9)*. Suppose that there exists $\beta > 1$ such that g obeys*

$$\lim_{x\to\infty} \frac{\log g(x)}{\log x} = \beta,$$

and $\lim_{x\to\infty} f(x)/x = 0$. Then the unique solution $x_n(h)$ of (1.1.2) *obeys*

$$\lim_{n\to\infty} \frac{\log_2 x_n(h)}{nh} = \frac{\log\beta}{\tau+h}. \tag{8.2.6}$$

Moreover, if $\bar{x}_h$ is the linear interpolant given by (1.2.13)*, then*

$$\lim_{t\to\infty} \frac{\log_2 \bar{x}_h(t)}{t} = \frac{\log\beta}{\tau+h}. \tag{8.2.7}$$

Once again, by comparing (7.2.21) and (8.2.7), we see that the essential growth rate is recovered by the linear interpolant for all $h > 0$, and the exact rate is recovered in the limit as $h \to 0^+$.

8.2.2 Delay-dominant numerics: a comparative example

Firstly, consider Example 5.3.3, where it was shown that if $\tau \in (0, \log(1/\log(2)))$, $\alpha \in (0,1)$,

$$f(x) = (2+x)\log(2+x) - (2+x)^\alpha, \quad x \geq 0,$$

and

$$g(x) = (x+2)^{\alpha e^\tau}, \quad x \geq 0,$$

then the unique continuous solution of

$$x'(t) = f(x(t)) + g(x(t-\tau)), \quad t > 0; \quad x(t) = \exp(e^t) - 2 \text{ for } t \in [-\tau, 0]$$

is given by

$$x(t) = \exp(e^t) - 2, \quad t \geq -\tau.$$

This obeys

$$\lim_{t\to\infty}\frac{F(x(t))}{t}=\frac{\log_2 x(t)}{t}=1,$$

that is, x is instantaneously-dominated. We can see that f and g satisfy the conditions of Theorem 6.3.1, which demonstrated that a state–dependent discretisation was required to replicate this growth rate, namely

$$\lim_{t\to\infty}\frac{F(\bar{x}_\Delta(t))}{t}=1,$$

where $\bar{x}_\Delta$ is given by (6.3.6). But now suppose instead that

$$\lim_{x\to\infty}\frac{f(x)}{x\log x}=0; \qquad g(x)=(x+2)^{e^\tau}, \quad x\geq 0.$$

Note that f is now growing slower than before and g growing faster. Then in accordance with Theorem 7.2.8 we have

$$\lim_{t\to\infty}\frac{\log_2 x(t)}{t}=\frac{\log e^\tau}{\tau}=1,$$

and using Theorem 8.2.4 we have

$$\lim_{t\to\infty}\frac{\log_2 \bar{x}_h(t)}{t}=\frac{\tau}{\tau+h},$$

where $\bar{x}_h$ is given by (1.2.13). Once again we see that the essential growth rate is recovered for all $h>0$, and since $\tau/(\tau+h)\to 1$ as $h\to 0^+$ the exact rate is recovered in the limit.

This demonstrates an interesting property of delay differential equations. We see that the discretisation method required to recover the growth rate of the solution is independent of the growth rate itself. Even an equation which grows at a very rapid rate, for example $x(t)\sim\exp(\exp(\exp\ldots\exp(t)))$, will not require state–dependent numerics to replicate this rate provided it is the delayed component of the equation which is generating this growth.

8.3 Proofs

In this section, we give the proofs of results from Section 8.2. We also give the proof of Theorem 7.2.6, which is greatly facilitated by the proof of Theorem 8.2.3.

Proof of Theorem 8.2.1 By (7.2.6) for every $\epsilon \in (0,1)$ there exists $x_2(\epsilon) > 0$ such that for $x > x_2(\epsilon)$ we have

$$g(x) < (\bar{\eta}_\epsilon + \epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(x) + \tau)) \leq (\bar{\eta} + \epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(x) + \tau)),$$

where the last inequality is a consequence of (7.2.7). Since $\bar{\eta} < \eta = \lim_{\epsilon \to 0^+} \eta(\epsilon)$, there exists $\epsilon' \in (0,1)$ such that for $\epsilon < \epsilon'$, we have $\eta(\epsilon) > \bar{\eta} + \epsilon$. Thus for all $\epsilon < \epsilon' < 1$ we have

$$g(x) < \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(x) + \tau)), \quad x > x_2(\epsilon). \tag{8.3.1}$$

By (7.2.5) for every $\epsilon \in (0,1)$ there exists an $x_1(\epsilon) > 0$ such that

$$f(x) \leq \epsilon\eta(\epsilon)\phi(x), \quad x > x_1(\epsilon). \tag{8.3.2}$$

Define

$$c(\epsilon) = \Gamma_{\eta(\epsilon)}(\psi^* + x_1(\epsilon) + x_2(\epsilon)) + (1+\epsilon)\tau, \tag{8.3.3}$$

and define also

$$x_\epsilon(n) = \Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)nh + c(\epsilon)), \quad n \geq -N. \tag{8.3.4}$$

This function is well-defined since $c(\epsilon) > \Gamma_{\eta(\epsilon)}(\psi^*) + (1+\epsilon)\tau$, so $c(\epsilon) - (1+\epsilon)\tau > \Gamma_{\eta(\epsilon)}(\psi^*)$, or $x_\epsilon(n) > \psi^*$ for all $n \in \{-N, \ldots, 0\}$. Since $c(\epsilon) > \Gamma_{\eta(\epsilon)}(x_1(\epsilon)) + (1+\epsilon)\tau$ and $\Gamma_{\eta(\epsilon)}$ is increasing, $\Gamma^{-1}_{\eta(\epsilon)}(c(\epsilon) - (1+\epsilon)\tau) > x_1(\epsilon)$, so $x_\epsilon(n) > x_1(\epsilon)$ for all $n \geq -N$. Therefore by (8.3.2), $f(x_\epsilon(n)) \leq \epsilon\eta(\epsilon)\phi(x_\epsilon(n))$ for $n \geq 0$. Also for $n \geq 0$, we have

$$\begin{aligned} g(x_\epsilon(n-N)) &= g(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)(n-N)h + c(\epsilon))) \\ &= g(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)nh - \tau - \epsilon\tau + c(\epsilon))) \\ &< g(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)nh - \tau + c(\epsilon))). \end{aligned}$$

Now, because $c(\epsilon) > \Gamma_{\eta(\epsilon)}(x_2(\epsilon)) + \tau$, we have that the argument of g on the right-hand side exceeds $x_2(\epsilon)$ for all $t \geq 0$. Therefore by (8.3.1), we have

$$\begin{aligned} g(x_\epsilon(n-N)) &< g(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)nh - \tau + c(\epsilon))) \\ &< \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)nh - \tau + c(\epsilon))) + \tau)) \\ &= \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)nh - \tau + c(\epsilon)) + \tau) \\ &= \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}((1+\epsilon)nh + c(\epsilon))) \\ &= \eta(\epsilon)\phi(x_\epsilon(n)). \end{aligned}$$

Hence

$$f(x_\epsilon(n)) + g(x_\epsilon(n-N)) < (1+\epsilon)\eta(\epsilon)\phi(x_\epsilon(n)), \quad n \geq 0. \tag{8.3.5}$$

Now for $n \geq 0$, $\Gamma_{\eta(\epsilon)}(x_\epsilon(n)) = (1+\epsilon)nh + c(\epsilon)$, so

$$\Gamma_{\eta(\epsilon)}(x_\epsilon(n+1)) - \Gamma_{\eta(\epsilon)}(x_\epsilon(n)) = (1+\epsilon)h.$$

Since Γ_η is in C^1 and $(x_\epsilon(n))_{n\geq 0}$ is an increasing sequence, there exists $\xi(n) \in [x_\epsilon(n), x_\epsilon(n+1)]$ such that

$$\Gamma_{\eta(\epsilon)}(x_\epsilon(n+1)) = \Gamma_{\eta(\epsilon)}(x_\epsilon(n)) + \Gamma'_{\eta(\epsilon)}(\xi(n))(x_\epsilon(n+1) - x_\epsilon(n)).$$

Therefore we have

$$(1+\epsilon)h = \Gamma'_{\eta(\epsilon)}(\xi(n))(x_\epsilon(n+1) - x_\epsilon(n)) = \frac{1}{\eta(\epsilon)}\frac{1}{\phi(\xi(n))}(x_\epsilon(n+1) - x_\epsilon(n)).$$

Thus as ϕ is non-decreasing, as $\xi(n) \geq x_\epsilon(n)$, we have

$$\begin{aligned} x_\epsilon(n+1) &= x_\epsilon(n) + (1+\epsilon)\eta(\epsilon)h\phi(\xi(n)) \\ &\geq x_\epsilon(n) + (1+\epsilon)\eta(\epsilon)h\phi(x_\epsilon(n)), \quad n \geq 0. \end{aligned} \tag{8.3.6}$$

Thus by (8.3.5) and (8.3.6) for $n \geq 0$ we have

$$x_\epsilon(n+1) \geq x_\epsilon(n) + (1+\epsilon)\eta(\epsilon)h\phi(x_\epsilon(n)) > x_\epsilon(n) + hf(x_\epsilon(n)) + hg(x_\epsilon(n-N)).$$

Now as $x_\epsilon(n) > \max_{n\in\{-N,\dots,0\}} \psi(nh)$, we have $x_\epsilon(n) > x_h(n)$ for $n \in \{N, \dots, 0\}$.

Suppose that there is a $n_0 \geq 1$ such that $x_\epsilon(n) > x_h(n)$ for $t \in \{-N, \dots, n_0 - 1\}$ $x_\epsilon(n_0) \leq x_h(n_0)$. Therefore $x_\epsilon(n_0) - x_\epsilon(n_0 - 1) \leq x_h(n_0) - x_h(n_0 - 1)$. Since f and g are non-decreasing,

$$\begin{aligned} x_\epsilon(n_0) - x_\epsilon(n_0 - 1) &\leq x_h(n_0) - x_h(n_0 - 1) \\ &= hf(x_h(n_0 - 1)) + hg(x_h(n_0 - 1 - N)) \\ &\leq hf(x_\epsilon(n_0 - 1)) + hg(x_h(n_0 - N)) \\ &\leq hf(x_\epsilon(n_0 - 1)) + hg(x_\epsilon(n_0 - 1 - N)) \\ &< x_\epsilon(n_0) - x_\epsilon(n_0 - 1), \end{aligned}$$

a contradiction.

Thus $x_\epsilon(n) > x_h(n)$ for all $n \geq -N$. Hence $\Gamma_{\eta(\epsilon)}(x_h(n)) < \Gamma_{\eta(\epsilon)}(x_\epsilon(n))$ for all $n \geq -N$. Hence

$$\Gamma_{\eta(\epsilon)}(x_h(n)) < \Gamma_{\eta(\epsilon)}(x_\epsilon(n)) = (1+\epsilon)nh + c(\epsilon), \quad n \geq -N.$$

But $\Gamma(x_h(n)) = \eta(\epsilon)\Gamma_{\eta(\epsilon)}(x_h(n)) < (1+\epsilon)\eta(\epsilon)nh + \eta(\epsilon)c(\epsilon)$. Therefore

$$\limsup_{n\to\infty} \frac{\Gamma(x_h(n))}{nh} \leq (1+\epsilon)\eta(\epsilon).$$

Since $\epsilon > 0$ is arbitrary, and $\eta(\epsilon) \to \eta$ as $\epsilon \to 0$, we have (8.2.1).

Proof of Theorem 8.2.2 Suppose first that $\bar{\mu}_\epsilon$ is finite. Then by (7.2.11) for every $\epsilon \in (0,1)$ there exists $x_3(\epsilon) > 0$ such that for $x > x_3(\epsilon)$

$$\begin{aligned} g(x) &> \bar{\mu}_\epsilon(1-\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x) + (\tau+h)(1-\epsilon))) \\ &\geq \bar{\mu}(1-\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x) + (\tau+h)(1-\epsilon))) \\ &> \mu(\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x) + (\tau+h)(1-\epsilon))), \end{aligned}$$

where the penultimate inequality is a consequence of (7.2.12), and the last inequality holds for all $\epsilon < \epsilon'$, because for such ϵ we have $\mu(\epsilon) < (1-\epsilon)\bar{\mu}$. This holds for the following reason.

By (7.2.10), there exists $\epsilon_1 \in (0,1)$ such that $\epsilon \in (0,\epsilon_1)$ implies $-\epsilon < \mu(\epsilon) - \mu < \mu\epsilon$. Since $\mu < \bar{\mu}$, it follows that there exists $\epsilon_2 \in (0,1)$ such that $\epsilon < \epsilon_2$ implies $\bar{\mu} > (1+\epsilon)\mu/(1-\epsilon)$. Hence for all $\epsilon < \epsilon' := \epsilon_1 \wedge \epsilon_2$, we have $\mu(\epsilon) < \mu(1+\epsilon) < (1-\epsilon)\bar{\mu}$.

Thus for all $0 < \epsilon < \epsilon' < 1$, and $x > x_3(\epsilon)$ we have

$$g(x) > \mu(\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x) + (\tau+h)(1-\epsilon))), \quad x > x_3(\epsilon). \tag{8.3.7}$$

When $\bar{\mu}_\epsilon = +\infty$, because $\mu(\epsilon)$ is finite, (8.3.7) is trivial.

Define $y_3(\epsilon) = \Gamma_{\mu(\epsilon)}(x_3(\epsilon)) + (\tau+h)(1-\epsilon)$. Then for $y > y_3(\epsilon)$, if we define $x = \Gamma^{-1}_{\mu(\epsilon)}(y - (\tau+h)(1-\epsilon))$, for $x > x_3(\epsilon)$ we have that $y > y_3(\epsilon)$. Thus by (8.3.7)

$$g(\Gamma^{-1}_{\mu(\epsilon)}(y - (\tau+h)(1-\epsilon))) > \mu(\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}(y)), \quad y > y_3(\epsilon). \tag{8.3.8}$$

Next let $N_0(\epsilon) = \inf\{n > 0 : x_h(n) \geq x_3(\epsilon)\}$ and define $N_1 > N_0$ such that

$$(1-\epsilon)(\tau+h)\Gamma_{\mu(\epsilon)}(x_h(N_0)) \leq \Gamma_{\mu(\epsilon)}(x_h(N_1)).$$

Define

$$x_\epsilon(n) = \Gamma^{-1}_{\mu(\epsilon)}((1-\epsilon)(n-N_1)h + \Gamma_{\mu(\epsilon)}(x_h(N_0))), \quad n \geq N_1. \tag{8.3.9}$$

Therefore for $n \geq N_1 + N$ we have

$$\begin{aligned}(1-\epsilon)(n+1-N_1)h + \Gamma_{\mu(\epsilon)}(x_h(N_0)) &\geq (1-\epsilon)(\tau+h) + \Gamma_{\mu(\epsilon)}(x_h(N_0)) \\ &\geq (1-\epsilon)(\tau+h) + \Gamma_{\mu(\epsilon)}(x_3(\epsilon))) = y_3(\epsilon).\end{aligned}$$

Setting $y = (1-\epsilon)(n+1-N_1)h + \Gamma_{\mu(\epsilon)}(x_h(N_0))$ in (8.3.8) yields

$$\begin{aligned}&g(\Gamma^{-1}_{\mu(\epsilon)}((1-\epsilon)(n-N_1-N)h + \Gamma_{\mu(\epsilon)}(x_h(N_0)))) \\ &\qquad > \mu(\epsilon)\phi(\Gamma^{-1}_{\mu(\epsilon)}((1-\epsilon)(n+1-N_1)h + \Gamma_{\mu(\epsilon)}(x_h(N_0)))), \quad n \geq N_1 + N.\end{aligned}$$

By (8.3.9) we have

$$g(x_\epsilon(n-N)) > \mu(\epsilon)\phi(x_\epsilon(n+1)), \quad n \geq N_1 + N. \tag{8.3.10}$$

Therefore by (8.3.10) for $n \geq N_1 + N$, and the fact that

$$\Gamma_{\mu(\epsilon)}(x_\epsilon(n)) = (1-\epsilon)(n-N_1)h + \Gamma_{\mu(\epsilon)}(x_h(N_0)),$$

we have

$$\Gamma_{\mu(\epsilon)}(x_\epsilon(n+1)) - \Gamma_{\mu(\epsilon)}(x_\epsilon(n)) = (1-\epsilon)h.$$

Hence there is $\xi(n) \in [x_\epsilon(n), x_\epsilon(n+1)]$ such that

$$x_\epsilon(n+1) - x_\epsilon(n) = (1-\epsilon)h\mu(\epsilon)\phi(\xi(n)).$$

Since ϕ is non-decreasing and $\xi(n) \leq x_\epsilon(n+1)$, we have

$$\begin{aligned}x_\epsilon(n+1) &= x_\epsilon(n) + (1-\epsilon)h\mu(\epsilon)\phi(\xi(n)) \\ &\leq x_\epsilon(n) + (1-\epsilon)h\mu(\epsilon)\phi(x_\epsilon(n+1)).\end{aligned}$$

Therefore by (8.3.10), we get for $n \geq N_1 + N$

$$\begin{aligned}x_\epsilon(n+1) &\leq x_\epsilon(n) + (1-\epsilon)h\mu(\epsilon)\phi(x_\epsilon(n+1)) \\ &< x_\epsilon(n) + h(1-\epsilon)g(x_\epsilon(n-N)) \\ &\leq x_\epsilon(n) + hf(x_\epsilon(n)) + h(1-\epsilon)g(x_\epsilon(n-N)) \\ &< x_\epsilon(n) + hf(x_\epsilon(n)) + hg(x_\epsilon(n-N)).\end{aligned}$$

Now for $n \in \{N_1, \dots, N_1 + N\}$ we have

$$\begin{aligned} x_\epsilon(n) \le x_\epsilon(N_1 + N) &= \Gamma^{-1}_{\mu(\epsilon)}((1-\epsilon)\tau + \Gamma_{\mu(\epsilon)}(x_h(N_0))) \\ &< \Gamma^{-1}_{\mu(\epsilon)}((1-\epsilon)(\tau + h) + \Gamma_{\mu(\epsilon)}(x_h(N_0))) \\ &\le \Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x_h(N_1))) = x_h(N_1) \le x_h(n), \end{aligned}$$

where we used at the last step the fact that x_h is increasing on $\{N_1, \dots, N_1 + N\} \subset \{N, N+1, \dots\}$. Therefore we have $x_\epsilon(n) < x_h(n)$ for $n \in \{N_1(\epsilon), \dots, N_1(\epsilon) + N\}$, and also $x_\epsilon(n+1) < x_\epsilon(n) + hf(x_\epsilon(n)) + hg(x_\epsilon(n-N))$ for $n \ge N_1 + N$.

Suppose there is a $n_1 \ge N_1(\epsilon) + N + 1$ such that $x_\epsilon(n) < x_h(n)$ for $n \in \{N_1(\epsilon), \dots, n_1\}$ and $x_\epsilon(n_1) \ge x_h(n_1)$. Therefore $x_\epsilon(n_1) - x_\epsilon(n_1 - 1) \ge x_h(n_1) - x_h(n_1 - 1)$. Then as f and g are non-decreasing,

$$\begin{aligned} x_\epsilon(n_1) - x_\epsilon(n_1 - 1) &\ge x_h(n_1) - x_h(n_1 - 1) \\ &= hf(x_h(n_1 - 1)) + hg(x_h(n_1 - 1 - N)) \\ &\ge hf(x_\epsilon(n_1 - 1)) + hg(x_\epsilon(n_1 - 1 - N)) \\ &> x_\epsilon(n_1) - x_\epsilon(n_1 - 1), \end{aligned}$$

a contradiction. Thus $x_\epsilon(n) < x_h(n)$ for all $n \ge N_1$. Hence $\Gamma_{\mu(\epsilon)}(x_h(n)) > \Gamma_{\mu(\epsilon)}(x_\epsilon(n))$ for all $n \ge N_1(\epsilon)$. Hence

$$\Gamma_{\mu(\epsilon)}(x_h(n)) > \Gamma_{\mu(\epsilon)}(x_\epsilon(n)) = (1-\epsilon)(n - N_1) + \Gamma_{\mu(\epsilon)}(x_h(N_0)), \quad n \ge N_1(\epsilon).$$

But $\Gamma(x_h(n)) = \mu(\epsilon)\Gamma_{\mu(\epsilon)}(x_h(n)) > (1-\epsilon)\mu(\epsilon)n + \mu(\epsilon)\Gamma_{\mu(\epsilon)}(x_h(N_0))$. Therefore

$$\liminf_{n\to\infty} \frac{\Gamma(x_h(n))}{nh} \ge (1-\epsilon)\mu(\epsilon).$$

Since $\epsilon > 0$ is arbitrary, and $\mu(\epsilon) \to \mu$ as $\epsilon \to 0$, we have (8.2.3).

Proof of Theorem 8.2.3 Let $j \ge N$. Summing across both sides of (1.1.2a) yields

$$x_h(j+1) = x_h(j-N) + h\sum_{n=j-N}^{j} f(x_h(n)) + h\sum_{n=j-N}^{j} g(x_h(n-N)).$$

Let $\epsilon(\tau+h)<1/2$. Since $x_h(n)\to\infty$ as $n\to\infty$ and $f(x)/x\to 0$ as $x\to\infty$, there exists $N_1(\epsilon)$ such that $f(x_h(n))\le \epsilon x_h(n)$ for all $n\ge N_1(\epsilon)$. Hence for $j\ge N_1(\epsilon)$ we have

$$\begin{aligned} x_h(j+1) &\le x_h(j-N) + h\sum_{n=j-N}^{j}\epsilon x_h(n) + h\sum_{n=j-N}^{j} g(x_h(n-N)) \\ &\le x_h(j-N) + h(N+1)\epsilon x_h(j) + h\sum_{n=j-N}^{j} g(x_h(n-N)) \\ &\le x_h(j-N) + h(N+1)\epsilon x_h(j+1) + h\sum_{n=j-N}^{j} g(x_h(n-N)). \end{aligned}$$

Hence for $j\ge N_1(\epsilon)$ we have

$$x_h(j+1)\le \frac{1}{1-(\tau+h)\epsilon}x_h(j-N) + \frac{1}{1-(\tau+h)\epsilon}h\sum_{n=j-N}^{j} g(x_h(n-N)).$$

Since g is in $\mathrm{RV}_\infty(1)$, $x\mapsto g(x)/x$ is asymptotic to a non-decreasing function, there exists g_0 such that g_0 is non-decreasing, $g_0(x)\to\infty$ as $x\to\infty$ and $g_0(x)/g(x)/x\to 1$ as $x\to\infty$. Therefore g_1 defined by $g_1(x):=xg_0(x)$ is increasing and is in $\mathrm{RV}_\infty(1)$. Since $x_h(n)\to\infty$ as $n\to\infty$, for every $\epsilon>0$ there exists $N_2(\epsilon)\ge N$ such that $g(x_h(n-N))<(1+\epsilon)g_1(x_h(n-N))$. Thus for $j\ge N_2(\epsilon)$ we have

$$\begin{aligned} h\sum_{n=j-N}^{j} g(x_h(n-N)) &\le h(1+\epsilon)\sum_{n=j-N}^{j} g_1(x_h(n-N)) \\ &\le h(N+1)(1+\epsilon)g_1(x_h(j-N)). \end{aligned}$$

Hence

$$h\sum_{n=j-N}^{j} g(x_h(n-N))\le (\tau+h)(1+\epsilon)g_1(x_h(j-N)),\quad j\ge N_2(\epsilon).$$

Let $N_3=\max(N_1,N_2)$. Then for $j\ge N_3$ we have

$$x_h(j+1)\le x_h(j-N)+\left(\frac{1}{1-(\tau+h)\epsilon}-1\right)x_h(j-N)+\frac{(\tau+h)(1+\epsilon)}{1-(\tau+h)\epsilon}g_1(x_h(j-N)).$$

Define $x_h^*(n)=x_h(n(N+1))$ for $n\ge 0$. Therefore for $n\ge N_3$ we have

$$x_h^*(j+1)\le x_h^*(j)+\left(\frac{1}{1-(\tau+h)\epsilon}-1\right)x_h^*(j)+\frac{(\tau+h)(1+\epsilon)}{1-(\tau+h)\epsilon}g_1(x_h^*(j)).$$

Define

$$g_\epsilon(x)=\left(\frac{1}{1-(\tau+h)\epsilon}-1\right)x+\frac{(\tau+h)(1+\epsilon)}{1-(\tau+h)\epsilon}g_1(x),\quad x>0. \tag{8.3.11}$$

Then g_ϵ is in $\mathrm{RV}_\infty(1)$, $x \mapsto g_\epsilon(x)/x$ is positive and non-decreasing, and $g_\epsilon(x)/x \to \infty$ as $x \to \infty$. Moreover

$$x_h^*(n+1) \le x_h^*(n) + g_\epsilon(x_h^*(n)), \quad n \ge N_3(\epsilon).$$

Next, define

$$y_\epsilon(n+1) = y_\epsilon(n) + g_\epsilon(y_\epsilon(n)), \quad n \ge N_3(\epsilon); \quad y_\epsilon(N_3) = 2x_h^*(N_3(\epsilon)).$$

Since g_ϵ is increasing, it follows that $x_h^*(n) \le y_\epsilon(n)$ for all $n \ge N_3(\epsilon)$. Define

$$J_\epsilon(x) = \int_1^x \frac{1}{u\log(1+g_\epsilon(u)/u)}\,du, \quad x \ge 0.$$

Then by applying Lemma 1.2.1 to (y_ϵ), we have that

$$\lim_{n\to\infty} \frac{J_\epsilon(y_\epsilon(n))}{n} = 1.$$

Since J_ϵ is increasing, and $x_h^*(n) \le y_\epsilon(n)$ for all $n \ge N_3(\epsilon)$, we have by the definition of x_h^* that

$$\limsup_{n\to\infty} \frac{J_\epsilon(x_h(n(N+1)))}{n} = \limsup_{n\to\infty} \frac{J_\epsilon(x_h^*(n))}{n} \le \lim_{n\to\infty} \frac{J_\epsilon(y_\epsilon(n))}{n} = 1.$$

Now by L'Hôpital's rule and (8.3.11)

$$\lim_{x\to\infty} \frac{J_\epsilon(x)}{L(x)} = \lim_{x\to\infty} \frac{\log(1+g(x)/x)}{\log(1+g_\epsilon(x)/x)} = \lim_{x\to\infty} \frac{\log(1+g(x)/x)}{\log\left(\frac{1}{1-(\tau+h)\epsilon} + \frac{(\tau+h)(1+\epsilon)}{1-(\tau+h)\epsilon}\frac{g_1(x)}{x}\right)}.$$

Since $g(x)/g_1(x) \to 1$ as $x \to \infty$, we have that

$$\lim_{x\to\infty} \frac{\log(1+g(x)/x)}{\log(1+g_1(x)/x)} = 1.$$

Therefore $\lim_{x\to\infty} J_\epsilon(x)/L(x) = 1$. Hence

$$\limsup_{n\to\infty} \frac{L(x_h(n(N+1)))}{n} \le 1. \tag{8.3.12}$$

Suppose $j \ge N+1$. Then there exists an integer $n = n(j) \ge 1$ such that $n(N+1) \le j < (n+1)(N+1)$. Since L is increasing, and $(x_h(n))_{n\ge 0}$ is increasing, we have

$$\begin{aligned}\frac{L(x_h(j))}{jh} &\le \frac{L(x_h((n+1)(N+1)))}{jh}\\ &\le \frac{L(x_h((n+1)(N+1)))}{n(N+1)h}\\ &= \frac{1}{\tau+h}\frac{L(x_h((n+1)(N+1)))}{n+1}\cdot\frac{n+1}{n}.\end{aligned}$$

By (8.3.12), we have

$$\limsup_{j\to\infty} \frac{L(x_h(j))}{jh} \le \frac{1}{\tau+h},$$

which gives the desired upper limit in (8.2.4).

To get a lower bound, since $f(x) \ge 0$, we have $x_h(n+1) \ge x_h(n) + hg(x_h(n-N))$ for $n \ge 0$. Since $x_h(n) \to \infty$ as $n \to \infty$, for every $\epsilon \in (0,1)$ there exists $N_4(\epsilon) \ge N$ such that $g(x_h(n-N)) > (1-\epsilon)g_1(x_h(n-N))$. Let $N_5(\epsilon) = \max(N_4(\epsilon), N_1(\epsilon))$. Let $y_h^{(1)}$ be defined by

$$y_h^{(1)}(n+1) = y_h^{(1)}(n) + h(1-\epsilon)g_1(y_h^{(1)}(n-N)), \quad n \ge N_5(\epsilon);$$
$$y_h^{(1)}(n) = x_h(n)/2, \quad n = N_5(\epsilon)-N, \dots, N_5(\epsilon).$$

Then we have for $n \ge N_5(\epsilon)$ the inequality $x_h(n+1) \ge x_h(n) + h(1-\epsilon)g_1(x_h(n-N))$. Hence $y_h^{(1)}(n) \le x_h(n)$ for $n \ge N_5(\epsilon) - N$. Clearly $(y_h^{(1)}(n))_{n\ge N_5(\epsilon)}$ is increasing and $y_h^{(1)}(n) \to \infty$ as $n \to \infty$.

Let $n \ge N_5(\epsilon) + N$. Then as $y_h^{(1)}$ is increasing, we have

$$y_h^{(1)}(n+1) = y_h^{(1)}(n) + h(1-\epsilon)g_1(y_h^{(1)}(n-N)) \ge y_h^{(1)}(n-N) + h(1-\epsilon)g_1(y_h^{(1)}(n-N)).$$

Therefore for $n \ge N_5(\epsilon) + N$ we have

$$\log y_h^{(1)}(n+1) \ge \log\left(\frac{g_1(y_h^{(1)}(n-N))}{y_h^{(1)}(n-N)}\right) + \log y_h^{(1)}(n-N) + \log\left(h(1-\epsilon) + \frac{y_h^{(1)}(n-N)}{g_1(y_h^{(1)}(n-N))}\right),$$

and so

$$\log y_h^{(1)}(n+1) \ge \log y_h^{(1)}(n-N) + \log(h(1-\epsilon)) + \log g_0(y_h^{(1)}(n-N)).$$

Define $u(n) := \log y_h^{(1)}(n)$ for $n \ge N_5(\epsilon)$. Then $(u(n))_{n\ge N_5}$ is increasing and tends to infinity as $n \to \infty$, and with $\gamma_0(x) := \log(h(1-\epsilon)) + \log g_0(e^x)$, we have

$$u(n+1) \ge u(n-N) + \gamma_0(u(n-N)), \quad n \ge N_5(\epsilon) + N.$$

Since g_0 is non-decreasing, so is γ_0, and moreover $\gamma_0(x) \to \infty$ as $x \to \infty$. Since g_0 is in $\mathrm{RV}_\infty(0)$, there is g_3 in $\mathrm{RV}_\infty(0)$ which is also in C^1 such that $g(x)/g_3(x) \to 1$ as $x \to \infty$,

$xg_3'(x)/g_3(x) \to 0$ as $x \to \infty$. Clearly for x^* sufficiently large we have $g_3(e^x) > e$ for all $x > x^*$, and so we may define

$$G_3(x) = \int_{x^*}^{x} \frac{1}{\log g_3(e^u)}\,du.$$

Then $G_3'(x) = 1/\log g_3(e^x) > 0$ for $x > x^*$ and since g_3 is in C^1 we have

$$G_3''(x) = -\frac{d}{dx}\log g_3(e^x)\cdot\frac{1}{(\log g_3(e^x))^2} = -\frac{1}{g_3(e^x))}g_3'(e^x)e^x\cdot\frac{1}{(\log g_3(e^x))^2}.$$

Since there $u(n) \to \infty$, there is N_6 is such that $u(n) > x^*$ for $n \geq N_6$. Let $N_7(\epsilon) = \max(N_5(\epsilon), N_6) + N$. Then for $n \geq N_7(\epsilon)$ we have $G_3(u(n+1)) \geq G_3(u(n-N) + \gamma_0(u(n-N)))$ and so by Taylor's theorem, there exists $\xi_n \in [u(n-N), u(n-N) + \gamma_0(u(n-N))]$ such that

$$\begin{aligned}
&G_3(u(n+1))\\
&\geq G_3(u(n-N) + \gamma_0(u(n-N)))\\
&= G_3(u(n-N)) + G_3'(u(n-N))\gamma_0(u(n-N)) + \frac{1}{2}G_3''(\xi_n)\gamma_0^2(u(n-N)),
\end{aligned}$$

for $n \geq N_7(\epsilon)$. Next, with $\eta_n := g_3'(e_n^{\xi})e^{\xi_n}/g_3(e^{\xi_n}))$ and using the fact that $xg_3'(x)/g_3(x) \to 0$ as $x \to \infty$, we have that $\eta_n \to 0$ as $n \to \infty$. Define for $n \geq N_7(\epsilon)$ the sequence

$$\delta(n) := \frac{\log(h(1-\epsilon)) + \log g_0(e^{u(n-N)})}{\log g_3(e^{u(n-N)})} - 1 - \frac{1}{2}\eta_n\frac{\left(\log(h(1-\epsilon)) + \log g_0(e^{u(n-N)})\right)^2}{(\log g_3(e^{\xi_n}))^2}.$$

so that

$$G_3(u(n+1)) \geq G_3(u(n-N)) + 1 + \delta(n), \quad n \geq N_7(\epsilon).$$

Since $\xi_n \to \infty$ as $n \to \infty$ and $g_3(x)/g_0(x) \to 1$ as $x \to \infty$ we have that for every $\epsilon \in (0,1)$ that there exists $N_8(\epsilon)$ such that $\log g_3(e^{\xi_n}) > \log(1-\epsilon) + \log g_0(e^{\xi_n})$ for all $n \geq N_8(\epsilon)$ and so for $n \geq N_9(\epsilon) = \max(N_8(\epsilon), N_7(\epsilon)) + N$ and so

$$\frac{\left(\log(h(1-\epsilon)) + \log g_0(e^{u(n-N)})\right)^2}{(\log g_3(e^{\xi_n}))^2} \leq \frac{\left(\log(h(1-\epsilon)) + \log g_0(e^{u(n-N)})\right)^2}{(\log(1-\epsilon) + \log g_0(e^{\xi_n}))^2}.$$

Since g_0 is increasing and $\xi_n \geq u(n-N)$ we have $\log g_0(e^{\xi_n}) \geq \log g_0(e^{u(n-N)})$. Hence

$$\frac{\left(\log(h(1-\epsilon)) + \log g_0(e^{u(n-N)})\right)^2}{(\log g_3(e^{\xi_n}))^2} \leq \frac{\left(\log(h(1-\epsilon)) + \log g_0(e^{\xi_n})\right)^2}{(\log(1-\epsilon) + \log g_0(e^{\xi_n}))^2}.$$

Therefore

$$\limsup_{n\to\infty}\frac{\left(\log(h(1-\epsilon)) + \log g_0(e^{u(n-N)})\right)^2}{(\log g_3(e^{\xi_n}))^2} \leq 1,$$

and so $\delta(n) \to 0$ as $n \to \infty$. Let $z(n) = G_3(u(n))$. Note that z is increasing and $z(n) \to \infty$ as $n \to \infty$. Then we have $z(n+1) \geq z(n-N) + 1 + \delta(n)$. Let $j \in \{0, \dots, N\}$. Define $z_j^*(n) = z((N+1)n + j)$. Then

$$\begin{aligned} z_j^*(n) &= z(Nn + n + j - 1 + 1) \\ &\geq z(Nn + n + j - 1 - N) + 1 + \delta(Nn + n + j - 1) \\ &= z_j^*(n-1) + 1 + \delta(Nn + n + j - 1). \end{aligned}$$

Now for $n \geq n'$ we have

$$\sum_{m=n'}^{n} z_j^*(m) \geq \sum_{m=n'}^{n} z_j^*(m-1) + n - n' + 1 + \sum_{m=n'}^{n} \delta(Nm + m + j - 1),$$

so

$$\frac{z_j^*(n)}{n} \geq \frac{z_j^*(n'-1)}{n} + 1 + \frac{-n'+1}{n} + \frac{1}{n}\sum_{m=n'}^{n} \delta(Nm + m + j - 1).$$

Since $\delta(n) \to 0$ as $n \to \infty$, we have $\liminf_{n\to\infty} z_j^*(n)/n \geq 1$. Therefore

$$\liminf_{n\to\infty} \frac{z((N+1)n + j)}{n(N+1)} \geq \frac{1}{N+1}, \quad \text{for each } j = 0, \dots, N.$$

Hence

$$\liminf_{n\to\infty} \frac{G_3(\log y_h^{(1)}(n))}{n} = \liminf_{n\to\infty} \frac{G_3(u(n))}{n} = \liminf_{n\to\infty} \frac{z(n)}{n} \geq \frac{1}{N+1}.$$

Since $x_h(n) \geq y_h^{(1)}(n)$ for $n \geq N_5(\epsilon) - N$, and G_3 is increasing, we have

$$\liminf_{n\to\infty} \frac{G_3(\log x_h(n))}{nh} \geq \liminf_{n\to\infty} \frac{G_3(\log y_h^{(1)}(n))}{nh} \geq \frac{1}{Nh+h} = \frac{1}{\tau+h}. \tag{8.3.13}$$

Now

$$G_3(\log x)) = \int_{x^*}^{\log x} \frac{1}{\log g_3(e^v)}\,dv = \int_{e^{x^*}}^{x} \frac{1}{u\log g_3(u)}\,du =: G_4(x). \tag{8.3.14}$$

Since $g_3(x)/g_0(x) \to 1$ as $x \to \infty$ and each belongs to $\text{RV}_\infty(0)$, we have that

$$\lim_{x\to\infty} \frac{\log g_0(x)}{\log g_3(x)} = 1.$$

Similarly, as $(1 + g(x)/x)/g_0(x) \to 1$ as $x \to \infty$ and g_0 is in $\text{RV}_\infty(0)$,

$$\lim_{x\to\infty} \frac{\log(1 + g(x)/x)}{\log g_0(x)} = 1.$$

Using these limits and L'Hôpital's rule, we arrive at

$$\lim_{x\to\infty} \frac{G_4(x)}{L(x)} = \lim_{x\to\infty} \frac{\log(1+g(x)/x)}{\log g_3(x)} = \lim_{x\to\infty} \frac{\log(1+g(x)/x)}{\log g_0(x)} \cdot \frac{\log g_0(x)}{\log g_3(x)} = 1.$$

Since $x_h(n) \to \infty$ as $n \to \infty$ and (8.3.13) and G_4 is defined by (8.3.14), by using the last limit, we get

$$\liminf_{n\to\infty} \frac{L(x_h(n))}{nh} = \liminf_{n\to\infty} \frac{L(x_h(n))}{G_4(x_h(n))} \frac{G_4(x_h(n))}{nh} = \liminf_{n\to\infty} \frac{G_3(\log x_h(n))}{nh} \geq \frac{1}{\tau + h},$$

which is the lower limit in (8.2.4).

In order to prove (8.2.5), notice for any $t > 0$ that there exists $n \geq 0$ such that $nh \leq t < (n+1)h$. Also as the linear interpolant $\bar{x}_h$ defined by (1.2.13), we have $x_h(n) \leq \bar{x}_h(t) \leq x_h(n+1)$. Therefore

$$\frac{L(\bar{x}_h(t))}{t} \leq \frac{L(x_h(n+1))}{nh} = \frac{L(x_h(n+1))}{(n+1)h} \cdot \frac{n+1}{n}.$$

Therefore by (8.2.4), we have

$$\limsup_{t\to\infty} \frac{L(\bar{x}_h(t))}{t} \leq \frac{1}{\tau + h}. \tag{8.3.15}$$

To get the lower bound, we observe that for $nh \leq t < (n+1)h$, we have

$$\frac{L(\bar{x}_h(t))}{t} \geq \frac{L(x_h(n))}{(n+1)h} = \frac{L(x_h(n))}{nh} \cdot \frac{n}{n+1}.$$

Therefore by (8.2.4), we have

$$\liminf_{t\to\infty} \frac{L(\bar{x}_h(t))}{t} \geq \frac{1}{\tau + h}.$$

Combining this limit with (8.3.15) yields (8.2.5).

Proof of Theorem 7.2.6 Let $N \in \mathbb{N}$ and set $h = \tau/N$. Let $j \geq N$. Integrating over $[(j-N)h, (j+1)h]$ yields

$$x((j+1)h) = x((j-N)h) + \int_{(j-N)h}^{(j+1)h} f(x(s))\,ds + \int_{(j-N)h}^{(j+1)h} g(x(s-Nh))\,ds.$$

Let $\epsilon(\tau + h) < 1/2$. Since $x(t) \to \infty$ as $t \to \infty$ and $f(x)/x \to 0$ as $x \to \infty$, there exists $T_1(\epsilon) > \tau$ such that $f(x(s)) \leq \epsilon x(s)$ for all $s \geq T_1(\epsilon)$. Let $N_1(\epsilon)$ be an integer such that $N_1(\epsilon)h > T_1(\epsilon)$. Then for $j \geq N_1(\epsilon)$, and using the fact that x is increasing, we have

$$\begin{aligned} x((j+1)h) &\leq x((j-N)h) + \int_{(j-N)h}^{(j+1)h} \epsilon x(s)\,ds + \int_{(j-N)h}^{(j+1)h} g(x(s-Nh))\,ds \\ &\leq x((j-N)h) + h(N+1)\epsilon x((j+1)h) + \int_{(j-N)h}^{(j+1)h} g(x(s-Nh))\,ds \\ &\leq x((j-N)h) + (\tau+h)\epsilon x((j+1)h) + \int_{(j-N)h}^{(j+1)h} g(x(s-Nh))\,ds. \end{aligned}$$

Hence for $j \geq N_1(\epsilon)$ we have

$$x((j+1)h) \leq x((j-N)h) + \left(\frac{1}{1-(\tau+h)\epsilon} - 1\right) x((j-N)h) + \frac{1}{1-(h+\tau)\epsilon} \int_{(j-N)h}^{(j+1)h} g(x(s-Nh))\, ds.$$

Since g is in $\mathrm{RV}_\infty(1)$, $x \mapsto g(x)/x$ is asymptotic to a non-decreasing function, there exists g_0 such that g_0 is non-decreasing, $g_0(x) \to \infty$ as $x \to \infty$ and $g_0(x)/g(x)/x \to 1$ as $x \to \infty$. Therefore g_1 defined by $g_1(x) := xg_0(x)$ is increasing and is in $\mathrm{RV}_\infty(1)$. Since $x(t) \to \infty$ as $t \to \infty$, for every $\epsilon > 0$ there exists $T_2(\epsilon) \geq \tau$ such that $g(x(t-\tau)) < (1+\epsilon)g_1(x(t-\tau))$ for all $t \geq T_2(\epsilon)$. Let $N_2(\epsilon)$ be an integer such that $N_2(\epsilon)h > T_2(\epsilon)$. Thus for $j \geq N_2(\epsilon) + N$ we have $jh \geq N_2(\epsilon)h + Nh > T_2 + \tau \geq 2\tau = 2Nh$, so as x is increasing on $[0,\infty)$ we have

$$\begin{aligned}\int_{(j-N)h}^{(j+1)h} g(x(s-Nh))\, ds &\leq (1+\epsilon) \int_{(j-N)h}^{(j+1)h} g_1(x(s-Nh))\, ds \\ &\leq h(N+1)(1+\epsilon)g_1(x((j+1-N)h)).\end{aligned}$$

Let $N_3(\epsilon) = \max(N_1(\epsilon), N_2(\epsilon)+N)$. Then for $j \geq N_3(\epsilon)$ we have

$$x((j+1)h) \leq x((j-N)h) + \left(\frac{1}{1-(h+\tau)\epsilon} - 1\right) x((j-N)h) + \frac{(h+\tau)(1+\epsilon)}{1-(h+\tau)\epsilon} g_1(x(j+1-N)h)),$$

which, as x is increasing, implies

$$x((j+1)h) \leq x((j+1-N)h) + \left(\frac{1}{1-(h+\tau)\epsilon} - 1\right) x((j+1-N)h) + \frac{(h+\tau)(1+\epsilon)}{1-(h+\tau)\epsilon} g_1(x(j+1-N)h)), \quad j \geq N_3(\epsilon).$$

Define $x_h^*(n) = x(nNh)$ for $n \geq -1$. Therefore for $n \geq N_3$, and since $N \geq 1$ we have

$$x_h^*(j+1) \leq x_h^*(j) + \left(\frac{1}{1-(h+\tau)\epsilon} - 1\right) x_h^*(j) + \frac{(h+\tau)(1+\epsilon)}{1-(h+\tau)\epsilon} g_1(x_h^*(j)).$$

The proof now continues as in the proof of Theorem 8.2.3, where τ is replaced by $\tau + h$. Proceeding in this manner we arrive at

$$\limsup_{n\to\infty} \frac{L(x(nNh))}{n} \leq 1. \tag{8.3.16}$$

For every $t > 0$ there exists $n \in \mathbb{N}$ such that $nNh \le t < (n+1)Nh$. Since L is increasing, and x is increasing, we have

$$\frac{L(x(t))}{t} \le \frac{L(x((n+1)Nh))}{t} \le \frac{L(x((n+1)Nh))}{nNh} = \frac{1}{\tau}\frac{L(x((n+1)Nh))}{n+1} \cdot \frac{n+1}{n}.$$

By (8.3.16), we have

$$\limsup_{t\to\infty} \frac{L(x(t))}{t} \le \frac{1}{\tau},$$

and therefore the desired upper limit in (7.2.20).

To get a lower bound, since $f(x) \ge 0$, we have

$$x((n+1)h) \ge x(nh) + \int_{nh}^{(n+1)h} g(x(s-Nh))\,ds, \quad n \ge 0.$$

Since $x(t) \to \infty$ as $t \to \infty$, for every $\epsilon \in (0,1)$ there exists $T_4(\epsilon) \ge \tau$ such that $g(x(t-\tau)) > (1-\epsilon)g_1(x(t-\tau))$. Let $N_4(\epsilon)$ be an integer such that $N_4(\epsilon)h > T_4(\epsilon)$. Let $N_5(\epsilon) = \max(N_4(\epsilon), N_1(\epsilon))$. Thus for $n \ge N_5(\epsilon)$ we have $nh \ge N_5(\epsilon))h \ge \max(T_4(\epsilon), \tau)$, so as x is increasing on $[0,\infty)$ we have

$$\begin{aligned} x((n+1)h) &\ge x(nh) + (1-\epsilon)\int_{nh}^{(n+1)h} g_1(x(s-Nh))\,ds \\ &\ge x(nh) + (1-\epsilon)hg_1(x(nh-Nh)). \end{aligned}$$

Then with $x_h(n) := x(nh)$, we have the inequality

$$x_h(n+1) \ge x_h(n) + (1-\epsilon)hg_1(x_h(n-N)), \quad n \ge N_5(\epsilon).$$

Let $y_h^{(1)}$ be defined by

$$\begin{aligned} y_h^{(1)}(n+1) &= y_h^{(1)}(n) + h(1-\epsilon)g_1(y_h^{(1)}(n-N)), \quad n \ge N_5(\epsilon); \\ y_h^{(1)}(n) &= x(nh)/2, \quad n = N_5(\epsilon)-N, \dots, N_5(\epsilon). \end{aligned}$$

Hence $y_h^{(1)}(n) \le x(nh)$ for $n \ge N_5(\epsilon) - N$. The proof now proceeds exactly as in Theorem 8.2.3, and we arrive at the analogue of (8.3.13), namely

$$\liminf_{n\to\infty} \frac{G_3(\log x(nh))}{nh} \ge \liminf_{n\to\infty} \frac{G_3(\log y_h^{(1)}(n))}{nh} \ge \frac{1}{Nh+h} = \frac{1}{\tau+h}, \tag{8.3.17}$$

where we have used the fact that $x(nh) = x_h(n)$. By (8.3.14), we have $G_3(\log x) = G_4(x)$, so once again we have that $\lim_{x\to\infty} G_4(x)/L(x) = 1$. Since $x(nh) \to \infty$ as $n \to \infty$, (8.3.17) holds, and G_4 is defined by (8.3.14), by using the last limit, we get

$$\liminf_{n\to\infty} \frac{L(x(nh))}{nh} = \liminf_{n\to\infty} \frac{L(x(nh))}{G_4(x(nh))}\frac{G_4(x(nh))}{nh} = \liminf_{n\to\infty} \frac{G_3(\log x(nh))}{nh} \ge \frac{1}{\tau+h}.$$

Now, for every $t > 0$ there exists n such that $nh \le t < (n+1)h$. Since x is increasing and L is increasing, we have

$$\frac{L(x(t))}{t} \ge \frac{L(x(nh))}{t} \ge \frac{L(x(nh))}{(n+1)h} = \frac{L(x(nh))}{nh}\frac{n}{n+1}.$$

Therefore

$$\liminf_{t\to\infty} \frac{L(x(t))}{t} \ge \liminf_{n\to\infty} \frac{L(x(nh))}{nh} \ge \frac{1}{\tau+h}.$$

Letting $h \to 0$ yields

$$\liminf_{t\to\infty} \frac{L(x(t))}{t} \ge \frac{1}{\tau},$$

which is the lower limit in (7.2.20).

Proof of Theorem 8.2.4 Let $j \ge N$. Summing across both sides of (1.1.2a) yields

$$x_h(j+1) = x_h(j-N) + h\sum_{n=j-N}^{j} f(x_h(n)) + h\sum_{n=j-N}^{j} g(x_h(n-N)).$$

Let $\epsilon(\tau+h) < 1/2$. Since $x_h(n) \to \infty$ as $n \to \infty$ and $f(x)/x \to 0$ as $x \to \infty$, there exists $N_1(\epsilon)$ such that $f(x_h(n)) \le \epsilon x_h(n)$ for all $n \ge N_1(\epsilon)$. Hence for $j \ge N_1(\epsilon)$ we have

$$\begin{aligned} x_h(j+1) &\le x_h(j-N) + h\sum_{n=j-N}^{j} \epsilon x_h(n) + h\sum_{n=j-N}^{j} g(x_h(n-N)) \\ &\le x_h(j-N) + h(N+1)\epsilon x_h(j) + h\sum_{n=j-N}^{j} g(x_h(n-N)) \\ &\le x_h(j-N) + h(N+1)\epsilon x_h(j+1) + h\sum_{n=j-N}^{j} g(x_h(n-N)). \end{aligned}$$

Hence for $j \ge N_1(\epsilon)$ we have

$$x_h(j+1) \le \frac{1}{1-(\tau+h)\epsilon} x_h(j-N) + \frac{1}{1-(\tau+h)\epsilon} h\sum_{n=j-N}^{j} g(x_h(n-N)).$$

Since $\log g(x)/\log x \to \beta$ as $x \to \infty$, and $x_h(n) \to \infty$ as $n \to \infty$, for every $\epsilon > 0$ there exists $N_2(\epsilon) \ge N$ such that $g(x_h(n-N)) < x_h(n-N)^{\beta+\epsilon}$. Thus for $j \ge N_2(\epsilon) + N$ we have

$$\begin{aligned} h\sum_{n=j-N}^{j} g(x_h(n-N)) &\le h\sum_{n=j-N}^{j} x_h(n-N)^{\beta+\epsilon} \\ &\le h(N+1)x_h(j-N)^{\beta+\epsilon}. \end{aligned}$$

Hence

$$h\sum_{n=j-N}^{j} g(x_h(n-N)) \le (\tau+h)x_h(j-N)^{\beta+\epsilon}, \quad j \ge N_2(\epsilon).$$

Let $N_3(\epsilon) = \max(N_1(\epsilon), N_2(\epsilon)+N)$. Then as $1-(\tau+h)\epsilon > 1/2$, for $j \ge N_3(\epsilon)$ we have

$$x_h(j+1) \le 2x_h(j-N) + 2(\tau+h)x_h(j-N)^{\beta+\epsilon}.$$

Define $x_h^*(n) = x_h(n(N+1))$ for $n \ge -1$. Therefore for $n \ge N_3(\epsilon)$ we have

$$\begin{aligned} x_h^*(j+1) &\le x_h((j+1)(N+1)) \le 2x_h(j(N+1)) + 2(\tau+h)x_h(j(N+1))^{\beta+\epsilon} \\ &= 2x_h^*(j) + 2(\tau+h)x_h^*(j)^{\beta+\epsilon}. \end{aligned}$$

Thus

$$\log x_h^*(j+1) \le \log 2(\tau+h) + (\beta+\epsilon)\log x_h^*(j) + \log\left(1 + \frac{x_h^*(j)}{(\tau+h)x_h^*(j)^{\beta+\epsilon}}\right).$$

Thus we have, with $u(n) = \log x_h^*(n)$, and all $n > N_5(\epsilon)$, the inequality

$$u(n+1) \le (\beta+2\epsilon)u(n).$$

Thus there exists $K(\epsilon) > 0$ such that $u(n) \le K(\epsilon)(\beta+2\epsilon)^n$ for $n \ge N_5(\epsilon)$. Thus

$$\frac{1}{n}\log u(n) \le \frac{1}{n}\log K(\epsilon) + \log(\beta+2\epsilon).$$

Therefore

$$\limsup_{n\to\infty} \frac{\log_2 x_h(n(N+1))}{n(N+1)h} = \limsup_{n\to\infty} \frac{\log_2 x_h^*(n)}{n(N+1)h} \le \frac{\log(\beta+2\epsilon)}{(N+1)h} = \frac{\log(\beta+2\epsilon)}{\tau+h}.$$

Letting $\epsilon \downarrow 0$, we arrive at

$$\limsup_{n\to\infty} \frac{\log_2 x_h(n(N+1))}{n(N+1)h} \le \frac{\log(\beta)}{\tau+h}. \tag{8.3.18}$$

Suppose $j \ge N+1$. Then there exists $n = n(j) \ge 1$ such that $n(N+1) \le j < (n+1)(N+1)$. Since $(x_h(n))_{n\ge 0}$ is increasing, we have

$$\begin{aligned} \frac{\log_2 x_h(j)}{jh} &\le \frac{\log_2 x_h((n+1)(N+1)))}{jh} \le \frac{\log_2 x_h((n+1)(N+1)))}{n(N+1)h} \\ &= \frac{\log_2 x_h((n+1)(N+1)))}{(n+1)(N+1)h}\cdot\frac{n+1}{n}. \end{aligned}$$

By (8.3.18), we have

$$\limsup_{j\to\infty} \frac{\log_2 x_h(j)}{jh} \le \frac{\log\beta}{\tau+h},$$

which gives the desired upper limit.

Since $f(x) \geq 0$ we have

$$x_h(n+1) \geq x_h(n) + hg(x_h(n-N)) \geq hg(x_h(n-N))$$

and since $x_h(n) \to \infty$ as $n \to \infty$ and $\log g(x)/\log x \to \beta$ as $x \to \infty$, it follows that for every $\epsilon < \beta$ there exists $N_6(\epsilon)$ such that $hg(x_h(n-N)) \geq x_h(n-N)^{\beta-\epsilon} > e$ for $n \geq N_5(\epsilon)$. Hence for $n \geq N_6(\epsilon)$ we have

$$x_h(n+1) \geq x_h(n-N))^{\beta-\epsilon}.$$

Therefore with $u(n) = \log x_h(n)$, we have that

$$u(n+1) = \log x_h(n+1) \geq (\beta-\epsilon)u(n-N).$$

Therefore, there exists $k(\epsilon) > 0$ such that $u(n) \geq k(\epsilon)(\beta-\epsilon)^{n/(N+1)}$ for $n \geq N_6(\epsilon)$. Therefore

$$\frac{1}{n}\log u(n) \geq \frac{1}{n}\log k(\epsilon) + \frac{1}{N+1}\log(\beta-\epsilon).$$

Hence

$$\liminf_{n\to\infty} \frac{\log_2 x_h(n)}{nh} \geq \frac{\log(\beta-\epsilon)}{(N+1)h} = \frac{\log(\beta-\epsilon)}{\tau+h}.$$

Letting $\epsilon \downarrow 0$, we get

$$\liminf_{n\to\infty} \frac{\log_2 x_h(n)}{nh} \geq \frac{\log\beta}{\tau+h}.$$

and so combining this with the other limit we get

$$\lim_{n\to\infty} \frac{\log_2 x_h(n)}{nh} = \frac{\log\beta}{\tau+h},$$

as required.

The proof that (8.2.7) follows from (8.2.6) is identical in all regards to the proof of Theorem 8.2.3 that (8.2.5) follows from (8.2.4), and is therefore omitted.

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Appendix A

Proof of Remark 3.2.2

We wish to show that if $\log_2 f$ is a smoothly regularly varying function with index $\eta > 0$ and f satisfies (2.2.1), then (3.2.2) holds. Define $\zeta_2(x) := \log_2 f(x)$ for $x > 0$. Note $e^{\zeta_2(x)} = \log f(x) = \phi(x) \to \infty$ as $x \to \infty$, forcing $\zeta_2(x) \to \infty$ as $x \to \infty$. Now $\zeta_2'(x) = f'(x)/(f(x)\log f(x))$, thus

$$\frac{f'(x)}{f(x)} = \zeta_2'(x) \log f(x).$$

Therefore

$$\log\left(\frac{f'(x)}{f(x)}\right) = \log \zeta_2'(x) + \log_2 f(x).$$

Differentiating both sides we obtain

$$\frac{f''(x)}{f'(x)} - \frac{f'(x)}{f(x)} = \frac{\zeta_2''(x)}{\zeta_2'(x)} + \frac{f'(x)}{f(x)\log f(x)}.$$

Therefore since $f'(x)/f(x) = \zeta_2'(x)\phi(x)$,

$$\begin{aligned}\frac{f''(x)/f'(x)}{f'(x)/f(x)} &= 1 + \frac{\zeta_2''(x)/\zeta_2'(x)}{f'(x)/f(x)} + \frac{1}{\phi(x)} \\ &= 1 + \frac{x\zeta_2''(x)}{\zeta_2'(x)} \cdot \frac{\zeta_2(x)}{x\zeta_2'(x)} \cdot \frac{1}{\phi(x)} \cdot \frac{1}{\zeta_2(x)} + \frac{1}{\phi(x)}.\end{aligned}$$

Since ζ_2 is smoothly regularly varying function with index η, $x\zeta_2''(x)/\zeta_2(x) \to \eta - 1$ and $\zeta_2(x)/(x\zeta_2'(x)) \to 1/\eta$ as $x \to \infty$. Also $\phi(x) \to \infty$. Thus

$$\lim_{x\to\infty} \frac{f(x)f''(x)}{(f'(x))^2} = \lim_{x\to\infty} \frac{f''(x)/f'(x)}{f'(x)/f(x)} = 1,$$

which verifies (3.2.2).

In general if $\log_n f$, $n \geq 2$ is a smoothly regularly varying function with index $\eta > 0$ and f satisfies (2.2.1), (3.2.2) holds. Now define $\zeta_n(x) := \log_n f(x)$ for $x > 0$. Again $e^{\zeta_n(x)} \to \infty$ as $x \to \infty$, forcing $\zeta_n(x) \to \infty$ as $x \to \infty$. Now

$$\zeta_n'(x) = \frac{1}{\log_{n-1} f(x) \log_{n-2} f(x) \ldots \log f(x)} \cdot \frac{f'(x)}{f(x)}.$$

Thus

$$\log\left(\frac{f'(x)}{f(x)}\right) = \log \zeta_n'(x) + \sum_{j=1}^{n-1} \log_{j+1} f(x).$$

Differentiating both sides we obtain

$$\frac{f''(x)}{f'(x)} - \frac{f'(x)}{f(x)} = \frac{\zeta_n''(x)}{\zeta_n'(x)} + \sum_{j=1}^{n-1} \frac{d}{dx} \log_{j+1} f(x),$$

and so

$$\frac{f''(x)/f'(x)}{f'(x)/f(x)} = 1 + \frac{\zeta_n''(x)/\zeta_n'(x)}{f'(x)/f(x)} + \frac{\sum_{j=1}^{n-1} \frac{d}{dx} \log_{j+1} f(x)}{\zeta_n'(x) \prod_{j=1}^{n-1} \log_j f(x)}.$$

Now for $j \geq 1$,

$$\begin{aligned}
\frac{d}{dx} \log_{j+1} f(x) &= \frac{1}{\prod_{k=1}^{j} \log_k f(x)} \cdot \frac{f'(x)}{f(x)} \\
&= \frac{\zeta_n'(x) \prod_{k=1}^{n-1} \log_k f(x)}{\prod_{k=1}^{j} \log_k f(x)} \\
&= \begin{cases} \zeta_n'(x), & \text{if } j = n-1, \\ \zeta_n'(x) \prod_{k=j+1}^{n-1} \log_k f(x), & \text{if } j < n-1. \end{cases}
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{f''(x)/f'(x)}{f'(x)/f(x)} = 1 &+ \frac{\zeta_n''(x)/\zeta_n'(x)}{\zeta_n'(x) \prod_{j=1}^{n-1} \log_j f(x)} \\
&+ \frac{\sum_{j=1}^{n-2} \zeta_n'(x) \prod_{k=j+1}^{n-1} \log_k f(x)}{\zeta_n'(x) \prod_{k=1}^{n-1} \log_k f(x)} + \frac{\zeta_n'(x)}{\zeta_n'(x) \prod_{k=1}^{n-1} \log_k f(x)},
\end{aligned}$$

which gives

$$\frac{f''(x)/f'(x)}{f'(x)/f(x)} = 1 + \frac{\zeta_n''(x)/\zeta_n'(x)}{\zeta_n'(x) \prod_{j=1}^{n-1} \log_j f(x)} + \sum_{j=1}^{n-2} \frac{1}{\prod_{k=1}^{j} \log_k f(x)} + \frac{1}{\prod_{k=1}^{n-1} \log_k f(x)}.$$

This resolves to

$$\frac{f''(x)/f'(x)}{f'(x)/f(x)} = 1 + \frac{\zeta_n''(x)}{(\zeta_n'(x))^2 \prod_{j=1}^{n-1} \log_j f(x)} + \sum_{j=1}^{n-1} \frac{1}{\prod_{k=1}^{j} \log_k f(x)}$$

Since ζ_n is smoothly regularly varying function with index η, $x\zeta_n''(x)/\zeta_n'(x) \to \eta - 1$ and $\zeta_n(x)/(x\zeta_n'(x)) \to 1/\eta$ as $x \to \infty$. Also $\phi(x) \to \infty$. Thus

$$\lim_{x\to\infty} \frac{\zeta_n''(x)}{(\zeta_n'(x))^2} \frac{1}{\prod_{j=1}^{n-1} \log_j f(x)} = \lim_{x\to\infty} \frac{x\zeta_n''(x)}{\zeta_n'(x)} \cdot \frac{\zeta_n(x)}{x\zeta_n'(x)} \cdot \frac{1}{\prod_{j=1}^{n-1} \log_j f(x)} \cdot \frac{1}{\zeta_n(x)} = 0$$

and so since

$$\lim_{x\to\infty} \sum_{j=1}^{n-1} \frac{1}{\prod_{k=1}^{j} \log_k f(x)} = 0,$$

we have

$$\lim_{x\to\infty} \frac{f''(x)/f'(x)}{f'(x)/f(x)} = 1,$$

which implies (3.2.2).

Appendix B

Proof of Examples 5.3.3 and 5.3.4

Example 5.3.3 We look to prove (5.3.8) where

$$f(x) = (2+x)\log(2+x) - (2+x)^{\alpha}, \quad x \geq 0,$$

$$g(x) = (x+2)^{\alpha e^{\tau}}, \quad x \geq 0,$$

$$f_*(x) = (2+x)\log(2+x), \quad x \geq 0,$$

and F is defined by

$$F(x) = \int_{e-2}^{x} \frac{1}{f(u)}\,du, \quad x > 0.$$

A closed form formula is not available for F or F^{-1}. We establish (5.3.8) by first determining very precise asymptotic information about F (and therefore about F^{-1}). Since $F(x)/F_*(x) \to 1$ as $x \to \infty$, it seems reasonable to write $F(x) = F_*(x) + F(x) - F_*(x)$, and then to determine the asymptotic behaviour of $(F - F_*)(x)$ as $x \to \infty$.

Towards this end for $x > 0$, we note that we have

$$\begin{aligned} F(x) &= F(x) - F_*(x) + \log(\log(x+2)) \\ &= \log(\log(x+2)) + \int_{e-2}^{x} \left\{ \frac{1}{f(u)} - \frac{1}{f_*(u)} \right\} du \\ &= \log(\log(x+2)) + \int_{e-2}^{x} \frac{1}{(2+u)^{2-\alpha}\log^2(2+u) - (2+u)\log(2+u)}\,du. \end{aligned}$$

Since

$$\begin{aligned} &(2+u)^{2-\alpha}\log^2(2+u) - (2+u)\log(2+u) \\ &\qquad = (2+u)^{2-\alpha}\log^2(2+u)\left(1 - \frac{1}{(2+u)^{1-\alpha}\log(2+u)}\right), \end{aligned}$$

for $\alpha \in [0,1]$, the integrand is asymptotic to

$$\frac{1}{u^{2-\alpha}\log^2(u)} \quad \text{as } u \to \infty$$

and therefore the integral converges to a finite value as $x \to \infty$. Define

$$I_{\alpha} := \int_{e-2}^{\infty} \frac{1}{(2+u)^{2-\alpha}\log^2(2+u) - (2+u)\log(2+u)}\,du.$$

Then with

$$\epsilon(x) = -\int_x^\infty \frac{1}{(2+u)^{2-\alpha}\log^2(2+u) - (2+u)\log(2+u)}\,du,$$

we have $\epsilon(x) \to 0$ as $x \to \infty$ and

$$F(x) = \log(\log(x+2)) + I_\alpha + \epsilon(x).$$

Now

$$\begin{aligned} F(x) + \tau &= \log(\log(x+2)) + I_\alpha + \tau + \epsilon(x) = \log(\log(x+2)) + \log(e^{I_\alpha+\tau+\epsilon(x)}) \\ &= \log(e^{I_\alpha+\tau+\epsilon(x)}\log(x+2)). \end{aligned}$$

Hence

$$e^{F(x)+\tau} = e^{I_\alpha+\tau+\epsilon(x)}\log(x+2). \tag{B.0.1}$$

Similarly we obtain

$$F(x) = \log(e^{I_\alpha+\epsilon(x)}\log(x+2)).$$

Therefore

$$e^x = e^{I_\alpha+\epsilon(F^{-1}(x))}\log(F^{-1}(x)+2),$$

which implies

$$e^{F(x)+\tau} = e^{I_\alpha+\epsilon(F^{-1}(F(x)+\tau))}\log(F^{-1}(F(x)+\tau)+2),$$

and using (B.0.1) we have

$$e^{I_\alpha+\epsilon(F^{-1}(F(x)+\tau))}\log(F^{-1}(F(x)+\tau)+2) = e^{I_\alpha+\tau}e^{\epsilon(x)}\log(x+2).$$

Define $\epsilon_1(x) = \epsilon(F^{-1}(F(x)+\tau))$. Since $F^{-1}(F(x)+\tau)) \to \infty$ as $x \to \infty$, we have that $\epsilon_1(x) \to 0$ as $x \to \infty$. Therefore we have

$$e^{I_\alpha}e^{\epsilon_1(x)}\log(F^{-1}(F(x)+\tau)+2) = e^{I_\alpha+\tau}e^{\epsilon(x)}\log(x+2).$$

Hence

$$\log(F^{-1}(F(x)+\tau)+2) = e^{\tau}e^{\epsilon(x)-\epsilon_1(x)}\log(x+2).$$

Since $\epsilon(x) \to 0$ and $\epsilon_1(x) \to 0$ as $x \to \infty$, we get

$$\lim_{x\to\infty} \frac{\log(F^{-1}(F(x)+\tau))}{\log x} = e^{\tau}.$$

Since f is regularly varying at infinity with index 1, it follows that

$$\lim_{x\to\infty} \frac{\log(f(F^{-1}(F(x)+\tau)))}{\log x} = e^{\tau}. \tag{B.0.2}$$

Therefore as $g(x) = (2+x)^{\alpha e^{\tau}}$, we have $\log g(x) = \alpha e^{\tau} \log(2+x)$, so by (B.0.2)

$$\begin{aligned}\lim_{x\to\infty} \frac{\log\left(\frac{g(x)}{f(F^{-1}(F(x)+\tau))}\right)}{\log x} &= \lim_{x\to\infty} \frac{\log g(x)}{\log x} - \frac{\log f(F^{-1}(F(x)+\tau))}{\log x} \\ &= \alpha e^{\tau} - e^{\tau} < 0,\end{aligned}$$

because $\alpha < 1$. Therefore we have

$$\lim_{x\to\infty} \frac{g(x)}{f(F^{-1}(F(x)+\tau))} = 0.$$

In the same way that we derive (B.0.2), we can show for any $\tau_1 > 0$ that

$$\lim_{x\to\infty} \frac{\log(f(F^{-1}(F(x)+\tau_1)))}{\log x} = e^{\tau_1}.$$

Since $\alpha < 1$, there exists $\tau_1 < \tau$ (specifically $\tau_1 \in (\log(\alpha e^{\tau}), \tau)$) Therefore as above we have

$$\begin{aligned}\lim_{x\to\infty} \frac{\log\left(\frac{g(x)}{f(F^{-1}(F(x)+\tau_1))}\right)}{\log x} &= \lim_{x\to\infty} \frac{\log g(x)}{\log x} - \frac{\log f(F^{-1}(F(x)+\tau_1))}{\log x} \\ &= \alpha e^{\tau} - e^{\tau_1} < 0,\end{aligned}$$

because $\tau_1 > \log(\alpha e^{\tau})$. Therefore we have

$$\lim_{x\to\infty} \frac{g(x)}{f(F^{-1}(F(x)+\tau_1))} = 0,$$

as required.

Finally, in the case when $\alpha e^{\tau} > 1$ we have that

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \lim_{x\to\infty} \frac{(2+x)^{\alpha e^{\tau}}}{(2+x)\log(2+x) - (2+x)^{\alpha}} = +\infty.$$

Example 5.3.4 We look to prove (5.3.10) where

$$f(x) = (A+x)\log(A+x)\log_2(A+x) - (A+x)\log_2(A+x), \quad x \geq 0,$$

$$g(x) = \exp(\log^{e^{\tau}}(x+A))\log(\log^{e^{\tau}}(x+A)), \quad x \geq 0,$$

$$f_*(x) = (A+x)\log(A+x)\log_2(A+x),$$

and F is defined by

$$F(x) = \int_A^x \frac{1}{f(u)}\, du, \quad x > 0.$$

A closed form formula is not available for F or F^{-1}. We establish (5.3.10) by first determining very precise asymptotic information about F (and therefore about F^{-1}). Since $F(x)/F_*(x) \to 1$ as $x \to \infty$, it seems reasonable to write $F(x) = F_*(x) + F(x) - F_*(x)$, and then to determine the asymptotic behaviour of $(F - F_*)(x)$ as $x \to \infty$.

Towards this end for $x > 0$, we note that we have

$$\begin{aligned} F(x) &= F(x) - F_*(x) + \log_3(x+A) \\ &= \log_3(x+A) + \int_A^x \left\{ \frac{1}{f(u)} - \frac{1}{f_*(u)} \right\} du \\ &= \log_3(x+2) + \int_A^x \frac{1}{(A+u)\log_2(A+u)\log(A+u)[\log(A+u)-1]}\, du. \end{aligned}$$

Therefore the integral converges to a finite value as $x \to \infty$. Define

$$I = \int_A^\infty \frac{1}{(A+u)\log_2(A+u)\log(A+u)[\log(A+u)-1]}\, du.$$

Then with

$$\epsilon(x) = \int_x^\infty \frac{1}{(A+u)\log_2(A+u)\log(A+u)[\log(A+u)-1]}\, du, \tag{B.0.3}$$

we have $\epsilon(x) > 0$ for $x > 0$, $\epsilon(x) \to 0$ as $x \to \infty$ and

$$F(x) = \log_3(x+A) + I - \epsilon(x).$$

Now

$$\begin{aligned} F(x) + \tau &= \log_3(x+A) + I + \tau - \epsilon(x) = \log(\log_2(x+A)) + \log(e^{I+\tau-\epsilon(x)}) \\ &= \log(e^{I+\tau-\epsilon(x)} \log[\log(x+A)]) = \log(\log[\log^{e^{I+\tau-\epsilon(x)}}(x+A)]) \\ &= \log_2[\log^{e^{I+\tau-\epsilon(x)}}(x+A)]. \end{aligned}$$

Thus we have

$$e^{F(x)+\tau} = \log\left(\log^{e^{I+\tau-\epsilon(x)}}(x+A)\right). \tag{B.0.4}$$

Similarly we obtain

$$F(x) = \log(e^{I-\epsilon(x)}\log_2(x+A)) = \log_2[\log^{e^{I-\epsilon(x)}}(x+A)].$$

Therefore

$$e^{F(x)} = \log[\log^{e^{I-\epsilon(x)}}(x+A)],$$

and so

$$e^{x} = \log[\log^{e^{I-\epsilon(F^{-1}(x))}}(F^{-1}(x)+A)].$$

Define $\epsilon_1(x) = \epsilon(F^{-1}(F(x)+\tau))$. Since $F^{-1}(F(x)+\tau)) \to \infty$ as $x \to \infty$, we have that $\epsilon_1(x) \to 0$ as $x \to \infty$. Using this definition we also obtain

$$e^{F(x)+\tau} = \log[\log^{e^{I-\epsilon_1(x)}}(F^{-1}(F(x)+\tau)+A)]. \tag{B.0.5}$$

Combining (B.0.4) and (B.0.5) we have

$$\log^{e^{I+\tau-\epsilon(x)}}(x+A) = \log^{e^{I-\epsilon_1(x)}}(F^{-1}(F(x)+\tau)+A).$$

Hence

$$\log(F^{-1}(F(x)+\tau)+A) = (\log(x+A))^{\frac{e^{I+\tau-\epsilon(x)}}{e^{I-\epsilon_1(x)}}} = (\log(x+A))^{e^{\tau+\epsilon_1(x)-\epsilon(x)}}.$$

Define $\theta(x) = e^{\epsilon_1(x)-\epsilon(x)}$. This implies

$$\log(F^{-1}(F(x)+\tau)+A) = (\log(x+A))^{e^{\tau}\theta(x)}. \tag{B.0.6}$$

Hence we also have

$$\log_2(F^{-1}(F(x)+\tau)+A) = \log\left((\log(x+A))^{e^{\tau}\theta(x)}\right) = e^{\tau}\theta(x)\log_2(x+A), \tag{B.0.7}$$

and

$$F^{-1}(F(x)+\tau)+A = \exp\left(\{\log(x+A)\}^{e^{\tau}\theta(x)}\right). \tag{B.0.8}$$

Next, we note that $f(x) = (A+x)\log_2(A+x)[\log(A+x)-1]$, so

$$\begin{aligned}
&f(F^{-1}(F(x)+\tau)) \\
&= (A+F^{-1}(F(x)+\tau))\log_2(A+F^{-1}(F(x)+\tau))\left(\log(A+F^{-1}(F(x)+\tau))-1\right) \\
&= \exp\left(\{\log(x+A)\}^{e^{\tau}\theta(x)}\right)\log_2(A+F^{-1}(F(x)+\tau))\left((\log(x+A))^{e^{\tau}\theta(x)}-1\right) \\
&= \exp\left(\{\log(x+A)\}^{e^{\tau}\theta(x)}\right)\cdot e^{\tau}\theta(x)\log_2(x+A)\cdot\left((\log(x+A))^{e^{\tau}\theta(x)}-1\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{x\to\infty} \frac{g(x)}{f(F^{-1}(F(x)+\tau))} \\
&= \lim_{x\to\infty} \frac{\exp(\log^{e^\tau}(x+A))\log(\log^{e^\tau}(x+A))}{\exp\left(\{\log(x+A)\}^{e^\tau\theta(x)}\right)\cdot e^\tau \log_2(x+A)\cdot(\log(x+A))^{e^\tau\theta(x)}} \\
&= \lim_{x\to\infty} \frac{\exp(\log^{e^\tau}(x+A)-\log^{e^\tau\theta(x)}(x+A))}{(\log(x+A))^{e^\tau\theta(x)}} \\
&= \lim_{x\to\infty} \frac{\exp(\log^{e^\tau}(x+A)-\log^{e^\tau\theta(x)}(x+A))}{\exp(\log\left(\log^{e^\tau\theta(x)}(x+A)\right))} \\
&= \lim_{x\to\infty} \exp\left(\log^{e^\tau}(x+A)-\log^{e^\tau\theta(x)}(x+A)-\log\left(\log^{e^\tau\theta(x)}(x+A)\right)\right) \\
&= \exp\left(\lim_{x\to\infty}\left\{\log^{e^\tau}(x+A)-\log^{e^\tau\theta(x)}(x+A)-\log\left(\log^{e^\tau\theta(x)}(x+A)\right)\right\}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{x\to\infty} \frac{g(x)}{f(F^{-1}(F(x)+\tau))} \\
&\quad = \exp\left(\lim_{x\to\infty}\left\{\log^{e^\tau}(x+A)-\log^{e^\tau\theta(x)}(x+A)-\log\left(\log^{e^\tau\theta(x)}(x+A)\right)\right\}\right). \qquad \text{(B.0.9)}
\end{aligned}$$

Now recall that $\epsilon_1(x)=\epsilon(F^{-1}(F(x)+\tau))$. We obtain asymptotic estimates for $\epsilon(x)$ and $\epsilon_1(x)$ as $x\to\infty$. By the definition of ϵ i.e., (B.0.3), we have

$$\epsilon(x) \sim \int_x^\infty \frac{1}{(A+u)\log^2(A+u)\log_2(A+u)}\,du = \int_{\log(x+A)}^\infty \frac{1}{w^2\log w}\,dw.$$

Now, we have that

$$\lim_{y\to\infty} \frac{\int_y^\infty \frac{1}{w^2\log w}\,dw}{\frac{1}{y\log y}} = \lim_{y\to\infty}\frac{\frac{1}{y^2\log y}}{\frac{1}{y^2\log^2 y}(\log y+1)} = 1.$$

Therefore

$$\lim_{x\to\infty}\frac{\epsilon(x)}{\frac{1}{\log(x+A)\log_2(x+A)}} = 1. \qquad \text{(B.0.10)}$$

Since $\epsilon_1(x)=\epsilon(F^{-1}(F(x)+\tau))$, we have

$$\lim_{x\to\infty}\frac{\epsilon_1(x)}{\frac{1}{\log(F^{-1}(F(x)+\tau)+A)\log_2(F^{-1}(F(x)+\tau)+A)}} = 1.$$

Inserting (B.0.7) and (B.0.6) and using the fact that $\theta(x)\to 1$ as $x\to\infty$, we get

$$\lim_{x\to\infty}\frac{\epsilon_1(x)}{\frac{1}{(\log(x+A))^{e^\tau\theta(x)}e^\tau\log_2(x+A)}} = 1.$$

Since $\theta(x) \to 1$ as $x \to \infty$ and $e^\tau > 1$, we therefore have

$$\lim_{x\to\infty} \frac{\epsilon_1(x)}{\epsilon(x)} = \lim_{x\to\infty} \frac{\frac{1}{(\log(x+A))^{e^\tau\theta(x)} e^\tau}}{\frac{1}{\log(x+A)}} = 0. \tag{B.0.11}$$

Now, let $a > 0$ be fixed and suppose that $h(x) = a^x$ for $x > 0$. Then $h'(x) = a^x \log(a)$. Then for every $u, v > 0$ there exists an $\xi(a,u,v)$ between u and v such that

$$a^u - a^v = a^{\xi(a,u,v)} \log(a)(u - v).$$

Applying this is the case where $a = \log(x + A)$, $u = e^\tau$ and $v = e^\tau\theta(x)$, we see that there exists an $\xi(x)$ between e^τ and $e^\tau\theta(x)$ such that

$$(\log(x+A))^{e^\tau} - (\log(x+A))^{e^\tau\theta(x)} = \log(x+A)^{\xi(x)} \log_2(x+A)(e^\tau - e^\tau\theta(x)).$$

Note that $\xi(x) \to e^\tau$ as $x \to \infty$

$$\begin{aligned}&(\log(x+A))^{e^\tau} - (\log(x+A))^{e^\tau\theta(x)}\\ &= e^\tau \log(x+A)^{\xi(x)} \log_2(x+A)\epsilon(x)\frac{(1-e^{\epsilon_1(x)-\epsilon(x)})}{\epsilon(x)-\epsilon_1(x)} \cdot \frac{\epsilon(x)-\epsilon_1(x)}{\epsilon(x)}.\end{aligned}$$

By (B.0.11) and the fact that $(1 - e^y)/y \to -1$ as $y \to 0$, we have

$$(\log(x+A))^{e^\tau} - (\log(x+A))^{e^\tau\theta(x)} = e^\tau \log(x+A)^{\xi(x)} \log_2(x+A)\epsilon(x)\eta_1(x)$$

where $\eta_1(x) \to 1$ as $x \to \infty$. Now we write

$$\begin{aligned}&(\log(x+A))^{e^\tau} - (\log(x+A))^{e^\tau\theta(x)}\\ &\qquad = e^\tau \log(x+A)^{\xi(x)} \frac{1}{\log(x+A)} \frac{\epsilon(x)}{\frac{1}{\log(x+A)\log_2(x+A)}}\eta_1(x),\end{aligned}$$

so by (B.0.10), we have that $\eta_2(x) \to 1$ as $x \to \infty$ and

$$(\log(x+A))^{e^\tau} - (\log(x+A))^{e^\tau\theta(x)} = e^\tau \log(x+A)^{\xi(x)-1}\eta_2(x).$$

Thus

$$\begin{aligned}&(\log(x+A))^{e^\tau} - (\log(x+A))^{e^\tau\theta(x)} - e^\tau\theta(x)\log_2(x+A)\\ &\qquad = e^\tau \log(x+A)^{\xi(x)-1}\eta_2(x) - e^\tau\theta(x)\log_2(x+A).\end{aligned}$$

Since $\xi(x) - 1 \to e^\tau - 1 > 0$ as $x \to \infty$, and $\eta_2(x) \to 1$ as $x \to \infty$ we have that

$$\lim_{x\to\infty} \left\{(\log(x+A))^{e^\tau} - (\log(x+A))^{e^\tau\theta(x)} - e^\tau\theta(x)\log_2(x+A)\right\} = +\infty.$$

Using this limit and (B.0.9), it follows that g and f obey (5.3.10) as claimed.